Roots of Trinomials from the Viewpoint of Amoeba Theory

The behavior of the modulis of roots of univariate trinomials $Z^{s+t} + pZ^t + q \in \mathbb{C}[Z]$ for fixed support $A := \{s+t, t, 0\} \subset \mathbb{N}$ with respect to the choice of coefficients $p, q \in \mathbb{C}$ is a classical 19th century problem. Although algebraically described by P. Bohl in 1908, the geometry and topology of the corresponding configuration space $\mathbb{C}^A$ is unknown. We provide such a description yielded by a reinterpretation of this problem in terms of amoeba theory.

Given an Laurent polynomial $f \in \mathbb{C}[\mathbf{Z}^{\pm 1}] = \mathbb{C}[Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}]$ the *amoeba* $\mathcal{A}(f)$ (introduced by Gel′fand, Kapranov, and Zelevinsky '94) is the image of its variety $\mathcal{V}(f)$ under the Log-map

$$\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n,\ (|z_1| \cdot e^{i \cdot \phi_1}, \ldots, |z_n| \cdot e^{i \cdot \phi_n}) \mapsto (\log|z_1|, \ldots, \log|z_n|)\,,$$

where $\mathcal{V}(f)$ is considered as a subset of the algebraic torus $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$.

Amoebas provide a natural approach to tropical geometry and occur in numerous other fields of mathematics – e.g. complex analysis and the topology of real algebraic curves.