

ON THE NUMBER OF CONNECTED COMPONENTS OF THE PARABOLIC CURVE

BENOÎT BERTRAND AND ERWAN BRUGALLÉ

ABSTRACT. We construct a polynomial of degree d in two variables whose Hessian curve has $(d-2)^2$ connected components using Viro patchworking. In particular, this implies the existence of a smooth real algebraic surface of degree d in $\mathbb{R}P^3$ whose parabolic curve is smooth and has $d(d-2)^2$ connected components.

*“Salut à toi Che Guevara”
Bérurier Noir*

1. INTRODUCTION

The Hessian of a polynomial $P(X_0, \dots, X_n)$ is the determinant of the matrix $(\frac{\partial^2 P}{\partial X_i \partial X_j})_{0 \leq i, j \leq n}$. If P is of degree d , then this determinant, denoted by $\text{Hess}(P)$, is generically a polynomial of degree $(n+1)(d-2)$. In this note, we are interested in the real solutions of the system $\mathcal{P}(P) = \{P = 0\} \cap \{\text{Hess}(P) = 0\}$ when P is a generic polynomial with real coefficients.

In the case when $n = 2$ and $P(X, Y, Z)$ is homogeneous, it is well known that the set $\mathcal{P}(P)$ is the set of real flexes of the curve with equation $P(X, Y, Z) = 0$. In [Kle76] (see also [Ron98], [Sch04], and [Vir88]), Klein proved that the number of real flexes of a smooth real algebraic curve of degree d cannot exceed $d(d-2)$, and that this bound is sharp.

In the case when $n = 3$ and $P(X, Y, Z, T)$ is homogeneous, not much is known about the curve $\mathcal{P}(P)$, called the *parabolic curve* of the surface with equation $P = 0$. If P is of degree d , then according to Harnack inequality the curve $\mathcal{P}(P)$ cannot have more than $2d(d-2)(5d-12) + 2$ connected components. Arnold's problem [Arn04, 2001-2] on the topology of the parabolic curve asks in particular for the maximal number of components of $\mathcal{P}(P)$ or at least for its asymptotic (See also problem 2000-2). Ortiz-Rodriguez constructed in [OR03] smooth real algebraic surfaces of any degree $d \geq 3$ whose parabolic curve is smooth and has $\frac{d(d-1)(d-2)}{2}$ connected components.

Her construction uses auxiliary parabolic curves of graphs of polynomials (i.e. $P(X, Y, Z, 1) = Z - Q(X, Y)$). In this case, the curve $\mathcal{P}(P)$ is the locus of the graph where the Gaussian curvature vanishes and its projection to the plane (X, Y) has equation $\text{Hess}(Q) = 0$ (note that Q and $\text{Hess}(Q)$ are not necessarily homogeneous). If Q is of degree d , then $\text{Hess}(Q)$ is (generically) of degree $2(d-2)$ and defines a curve with at most $(2d-5)(d-3) + 1$ compact connected components in \mathbb{R}^2 . The maximal number of compact connected components of such a curve in this special case is the

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subject of problem [Arn04, 2001-1] (See also problem 2000-1). In [OR03], Ortiz-Rodriguez constructed real polynomials $Q(X, Y)$ of degree $d \geq 3$ whose Hessian define smooth real curves with $\frac{(d-1)(d-2)}{2}$ compact connected components in \mathbb{R}^2 . In small degrees, this construction has been slightly improved in [ORS07].

Note that if $Q(X, Y)$ is a polynomial in two variables, then

$$\text{Hess}(Q) = \frac{\partial^2 Q}{\partial X^2} \frac{\partial^2 Q}{\partial Y^2} - \left(\frac{\partial^2 Q}{\partial X \partial Y} \right)^2$$

In this note, we prove the following result which improves the previously known asymptotic by a factor 2.

Theorem 1.1. *For any $d \geq 4$, there exists a real polynomial $Q_d(X, Y)$ of degree d such that the curve with equation $\text{Hess}(Q_d) = 0$ is smooth and has $(d-2)^2$ compact connected components in \mathbb{R}^2 .*

Theorem 1.1 is proved in section 3. The main tool is Viro's Patchworking theorem to glue Hessian curves (see section 2).

Corollary 1.2. *For any $d \geq 4$, there exists a smooth real algebraic surface in $\mathbb{R}P^3$ of degree d whose parabolic curve is smooth and has $d(d-2)^2$ connected components.*

Proof. As observed in [OR03, Theorem 5], if the real curve $\text{Hess}(Q)(X, Y) = 0$ has k compact connected components in \mathbb{R}^2 , then for ε small enough, the parabolic curve of the surface with equation $R(Z) - \varepsilon Q(X, Y) = 0$, where $R(Z)$ is a real polynomial of degree d with d distinct real roots, has dk connected components. \square

2. GLUING OF HESSIANS

Let $Q_t(X, Y) = \sum a_{i,j}(t)X^iY^j$ be a polynomial whose coefficients are real polynomials in one variable t . Such a polynomial has two natural Newton polytopes depending on whether $Q_t(X, Y)$ is considered as a polynomial in the variables X and Y or as a polynomial in X, Y and t . We denote by $\Delta_2(Q_t)$ the former Newton polytope, and by $\Delta_3(Q_t)$ the latter. There exists a convex piecewise linear function $\nu : \Delta_2(Q_t) \rightarrow \mathbb{R}$ whose graph is the union of the bottom faces of $\Delta_3(Q_t)$. The linearity domains of the function ν induce a subdivision τ_ν of $\Delta_2(Q_t)$. Note that since real numbers are constant real polynomials, this construction makes sense also for polynomials in $\mathbb{R}[X, Y]$, in this case the function ν is constant and the subdivision τ is the trivial one.

Let Δ' be a cell of τ_ν . The restriction of ν to Δ' is given by a linear function $L : (i, j) \mapsto \alpha i + \beta j + \gamma$ which does not coincide with ν on any polygon of τ_ν strictly containing Δ' . If Δ' is of dimension $k \leq 2$ there is a $2 - k$ dimensional family of such functions but the following construction does not depend on the choice of the function as long as $\Delta_3(Q_t) \setminus \nu(\Delta')$ is strictly above the graph of L . We define the Δ' -truncation of $Q_t(X, Y)$ as the polynomial $Q^{\Delta'}(X, Y)$ in $\mathbb{R}[X, Y]$ given by substituting $t = 0$ in the polynomial $t^{-\gamma}Q_t(t^{-\alpha}X, t^{-\beta}Y)$.

Viro's Patchworking Theorem asserts that if all the polynomials $Q^{\Delta'}(X, Y)$ are non singular in $(\mathbb{R}^*)^2$ when Δ' goes through all the cells of τ_ν , then for a small enough real number t , the real algebraic curve with equation $Q_t(X, Y) = 0$ is a gluing of the real algebraic curves with equation $Q^{\Delta'}(X, Y) = 0$ when Δ' goes through all the 2-dimensional polygons of τ_ν . In particular any compact oval in $(\mathbb{R}^*)^2$ of a curve defined by $Q^{\Delta'}(X, Y)$ leads to an oval of the curve defined by $Q_t(X, Y)$ when

t is small enough. For a more precise statement of Viro's Patchworking Theorem, we refer to [Vir84], [Vir89], or [Ris92]. See also [Mik04] for a tropical approach.

The key observation of this paper is that when gluing the polynomials $Q^{\Delta'}(X, Y)$ one also glues their Hessians. We formalize this in the following proposition.

Consider as above a polynomial $Q_t(X, Y)$ whose coefficients are real polynomials. Let us denote by $\tilde{\nu}$ the convex piecewise linear function constructed as above out of the Hessian $\text{Hess}(Q_t)$ of $Q_t(X, Y)$ with respect to the variables X and Y . If Δ' is a polygon of τ , we denote by Δ'_H the Newton polygon of the polynomial $\text{Hess}(Q^{\Delta'})$.

Proposition 2.1. *If Δ' is a cell of τ_ν lying in the region $\{(x, y) \mid x \geq 2 \text{ and } y \geq 2\}$, then Δ'_H is a cell of the subdivision $\tau_{\tilde{\nu}}$ and $\text{Hess}(Q_t)^{\Delta'_H} = \text{Hess}(Q^{\Delta'})$.*

Proof. It is a standard fact ([GKZ94] p. 193) that the Newton polytope of the product of two polynomials corresponds to the Minkowski sum of the Newton polytopes of the factors. Let Γ_1 and Γ_2 be two identical (up to translation) polytopes, and let $\Gamma_1 \oplus \Gamma_2$ be their Minkowski sum. Consider the natural map $\phi : \Gamma_1 \times \Gamma_2 \rightarrow \Gamma_1 \oplus \Gamma_2$. The polytope $\Gamma_1 \oplus \Gamma_2$ is (up to translation) twice the polytope Γ_i and there are natural bijections ι_i between the faces (of any dimension) of $\Gamma_1 \oplus \Gamma_2$ and the faces of Γ_i . It is not difficult to see that for any face F the preimage $\phi^{-1}(F)$ is exactly $\iota_1(F) \times \iota_2(F)$.

Given a vector \vec{u} in \mathbb{R}^3 , we denote by $tr_{\vec{u}}$ the translation along \vec{u} . We denote also respectively by p_1, p_2 and p_3 the polynomials $\frac{\partial^2 Q_t}{\partial X^2}$, $\frac{\partial^2 Q_t}{\partial Y^2}$ and $\frac{\partial^2 Q_t}{\partial X \partial Y}$. Hence the polynomial $\text{Hess}(Q_t)$ is the difference of the two products $p_1 p_2$ and p_3^2 .

A face of $\Delta_3(Q_t)$ lying in the region $\{(x, y, z) \mid x \geq 2 \text{ and } y \geq 2\}$ is also a face of $tr_{(2,0,0)}(\Delta_3(p_1))$, $tr_{(0,2,0)}(\Delta_3(p_2))$ and $tr_{(1,1,0)}(\Delta_3(p_3))$. Since these three polytopes are contained in $\Delta_3(Q_t)$, the result follows immediately from the above discussion about Minkowski sum of two identical polytopes and the fact that coefficients of $p_1 p_2$ and p_3^2 corresponding to a vertex v of $\Delta_3(Q_t)$ are different as long as v has a nonzero first or second coordinate. \square

3. CONSTRUCTION

Here we apply Proposition 2.1 to glue Hessian curves. We first construct pieces we need for patchworking.

Lemma 3.1. *For any $i \geq 2$ and $j \geq 2$, the curves with equation $\text{Hess}(X^i Y^j(1 + Y)) = 0$, $\text{Hess}(X^i Y^j(X + Y)) = 0$, and $\text{Hess}(X^i Y^j(X + Y^2)) = 0$ do not have any real points in $(\mathbb{R}^*)^2$.*

Proof. Up to division by powers of x and y these polynomial are of degree 2 with negative discriminant. \square

By symmetry, the curve with equation $\text{Hess}(X^i Y^j(1 + X)) = 0$ does not have any real points in $(\mathbb{R}^*)^2$.

Lemma 3.2. *For any $i \geq 2$ and $j \geq 2$, the real point set in $(\mathbb{R}^*)^2$ of the curves with equation $\text{Hess}(X^i Y^j(X + Y + Y^2)) = 0$, $\text{Hess}(X^i Y^j(XY + X + Y^2)) = 0$, and $\text{Hess}(X^i Y^j(1 + X + Y)) = 0$ consists of one compact smooth oval.*

Proof. According to Proposition 2.1 and Lemma 3.1, these three curves can only have compact connected component in $(\mathbb{R}^*)^2$. Up to division by powers of X and Y , the discriminant with respect to the variable X of these polynomials are degree

2 polynomials in Y with positive discriminant. They thus have exactly two distinct real roots which attest the existence of exactly one oval for each Hessian curve. \square

To prove Theorem 1.1 we apply Viro's patchworking theorem to a polynomial $Q_t(X, Y)$ whose truncation on the polygons of τ_ν are the polynomials $X^i Y^j (X + Y + Y^2)$, $X^i Y^j (XY + X + Y^2)$ and $X^k Y^2 (1 + X + Y)$ for $2 \leq i, j \leq d - 2$ and $2 \leq k \leq d - 1$. Proposition 2.1 and Lemma 3.2 insure that, for sufficiently small positive t , the Hessian curve of $Q_t(X, Y)$ has at least $(d - 2)^2$ smooth compact connected components.

REFERENCES

- [Arn04] V. I. Arnold. *Arnold's Problems*. Springer Verlag, Berlin, 2004.
- [GKZ94] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1994.
- [Kle76] F. Klein. Eine neue Relation zwischen den Singularitäten einer algebraischen Curve. *Math. Ann.*, 10(2):199–209, 1876.
- [Mik04] G. Mikhalkin. Decomposition into pairs-of-pants for complex algebraic hypersurfaces. *Topology*, 43:1035–106, 2004.
- [OR03] A. Ortiz-Rodríguez. Quelques aspects sur la géométrie des surfaces algébriques réelles. *Bull. Sci. Math.*, 127:149–177, 2003.
- [ORS07] A. Ortiz-Rodríguez and F. Sottile. Real hessian curves. *Boletín de la Sociedad Matemática Mexicana*, 13:157–166, 2007.
- [Ris92] J. J. Risler. Construction d'hypersurfaces réelles (d'après Viro). *Séminaire Bourbaki*, 763, 1992. (French).
- [Ron98] F. Ronga. Klein's paper on real flexes vindicated. In W. Pawlucki B. Jakubczyk and J. Stasica, editors, *Singularities Symposium - Lojasiewicz 70*, volume 44. Banach Center Publications, 1998.
- [Sch04] F. Schuh. An equation of reality for real and imaginary plane curves with higher singularities. *Proc. section of sciences of the Royal Academy of Amsterdam*, 6:764–773, 1903–1904.
- [Vir84] O. Ya. Viro. Gluing of plane real algebraic curves and constructions of curves of degrees 6 and 7. In *Topology (Leningrad, 1982)*, volume 1060 of *Lecture Notes in Math.*, pages 187–200. Springer, Berlin, 1984.
- [Vir88] O. Ya. Viro. Some integral calculus based on Euler characteristic. In *Lecture Notes in Math.*, volume 1346, pages 127–138. Springer Verlag, 1988.
- [Vir89] O. Ya. Viro. Real plane algebraic curves: constructions with controlled topology. *Leningrad Math. J.*, 1(5):1059–1134, 1989.

INSTITUT MATHÉMATIQUE DE TOULOUSE, I.U.T. DE TARBES, 1 RUE LAUTRÉAMONT, BP 1624, 65016 TARBES

E-mail address: benoit.bertrand@iut-tarbes.fr

UNIVERSITÉ PIERRE ET MARIE CURIE, PARIS 6, INSTITUT MATHÉMATIQUES DE JUSSIEU 175 RUE DU CHEVALERET, 75 013 PARIS, FRANCE

E-mail address: brugalle@math.jussieu.fr