ENUMERATION OF REAL CONICS AND MAXIMAL CONFIGURATIONS

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ABSTRACT. We use floor decompositions of tropical curves to prove that any enumerative problem concerning conics passing through projective-linear subspaces in $\mathbb{R}P^n$ is maximal. That is, there exist generic configurations of real linear spaces such that all complex conics passing through these constraints are actually real.

1. INTRODUCTION

A rational curve of degree d in $\mathbb{C}P^n$ is parameterized by a polynomial map

$$\begin{array}{cccc} \phi: & \mathbb{C}P^1 & \longrightarrow & \mathbb{C}P^n \\ & [t,u] & \longmapsto & [P_0(t,u):\ldots:P_n(t,u)] \end{array}$$

where the $P_i(t,u)$'s are homogeneous polynomials in two variables of degree d with no common factors. Since $Aut(\mathbb{C}P^1)$ has dimension 3 and all the $P_i(t,u)$'s are defined up to a common multiplicative constant, the dimension of the space of rational curves of degree d in $\mathbb{C}P^n$ is

$$(n+1)(d+1)-4=(n+1)d+(n-3).$$

Consequently, if we are looking for rational curves in $\mathbb{C}P^n$ satisfying exactly this number of independent conditions, we can reasonably expect the number of solution to be finite. For example, if L is a linear subspace of codimension $j\geq 1$ in $\mathbb{C}P^n$, the condition "to intersect L" imposes exactly $j-1$ independent conditions on rational curves in $\mathbb{C}P^n$. Hence, if we choose a generic configuration $\omega=\{L_1,\ldots,L_\gamma\}$ of linear subspaces of $\mathbb{C}P^n$, with $l_j=\operatorname{codim} L_j\geq 1$, such that

$$\sum_{j=1}^{\gamma}(l_j-1)=(n+1)d+(n-3) \qquad (1)$$

we expect a finite number of rational curves of degree d in $\mathbb{C}P^n$ intersecting all the linear subspaces in ω. The term "generic" means that these linear subspaces have to be chosen so that they impose altogether independent conditions.

It turns out that this number of rational curves, that we denote by $N_{d,n}(l_1,\ldots,l_\gamma)$, is indeed finite and doesn't depend on the configuration ω we have chosen, but only on n, d, $l_1,\ldots,l_\gamma$. The numbers $N_{d,n}(l_1,\ldots,l_\gamma)$ are known as *Gromov-Witten invariants* of the projective space $\mathbb{C}P^n$. For example, since there exists a unique line passing through two distinct points in $\mathbb{C}P^n$, we have

$$N_{1,n}(n,n)=1 \quad \forall n\geq 2.$$

For a more detailed introduction to Gromov-Witten theory, we refer the interested reader to the excellent book [KV06]. In this paper, it is convenient to extend the definition of the numbers $N_{d,n}(l_1,\ldots,l_\gamma)$ to any set of γ numbers in $\mathbb{Z}$ by

$$N_{d,n}(l_1,\ldots,l_\gamma)=0 \quad \text{if} \quad \sum_{j=1}^{\gamma}(l_j-1)\neq(n+1)d+(n-3) \quad \text{or} \quad \exists j,\ l_j\leq 0 \text{ or } l_j\geq n+1.$$

All linear spaces in our generic configuration ω can be chosen to be real. In this case, it makes sense to enumerate real rational curves in $\mathbb{C}P^n$ (i.e. rational curves which are invariant under the complex conjugation of $\mathbb{C}P^n$) of degree d intersecting our configuration of real linear subspaces. Unlike in the enumeration of

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complex curves, the number of real solutions, denoted by $N^{\mathbb{R}}_{d,n}(l_1,\dots,l_\gamma,\omega)$, now depends on the chosen configuration ω of linear spaces. Clearly, we have the inequality

$$N^{\mathbb{R}}_{d,n}(l_1,\dots,l_\gamma,\omega)\le N_{d,n}(l_1,\dots,l_\gamma)\quad \forall\omega.$$

However, it is unknown in general if there exists a real configuration ω such that all complex solutions are real. For example, can the 92 complex conics passing through 8 general lines in $\mathbb{R}P^3$ be real? More generally, it is an important and difficult question to ask how many solutions of an enumerative problem can be real (see [Ful84, §7.2]). When all complex solutions can be real, we say that this enumerative problem is *maximal*.

To stress how difficult these questions are, let us summarize the very few things known in 2011 about the maximality of the enumerative problems defined above. Since the corresponding Gromov-Witten invariant is equal to 1, it is trivial that the problem is maximal in the two following cases:

- $d=1$, $l_j=n$ for some j;
- $n=2$, $d=2$, and $l_1=l_2=l_3=l_4=l_5=2$.

It is also easy to see that the problem is maximal in the case $n=2$ and $d=3$ (and so $l_1=\ldots=l_8=2$). The first systematic non-trivial result was obtained by Sottile who proved in [Sot97] that the problem is maximal as soon as $d=1$ (the so-called problems of "Schubert-type". Actually problems of Schubert-type involving linear subspaces of any dimension turn out to be maximal by [Vak06]). More recently, with the help of tropical geometry, it was proved in [BM07] that the problem above is maximal for $d=2$ and $n=3$. Up to our knowledge, nothing more was known before our investigation.

Our main result is that the enumerative problems discussed above are maximal when $d=2$. Theorem 1.1 is a direct consequence of Theorem 2.5, Lemma 2.6, and Proposition 4.5.

Theorem 1.1. *For $n\ge 2$, $l_1\ge 1$, …, $l_\gamma\ge 1$ satisfying*

$$\sum_1^\gamma (l_j-1)=3n-1,$$

there exists a generic configuration $\omega=\{L_1,\dots,L_\gamma\}$ of real linear subspaces of $\mathbb{C}P^n$ such that $\operatorname{codim} L_j=l_j$ *and*

$$N^{\mathbb{R}}_{2,n}(l_1,\dots,l_\gamma,\omega)=N_{2,n}(l_1,\dots,l_\gamma).$$

To prove Theorem 1.1, we use *floor decomposition of tropical curves.* In his pioneering work [Mik05], Mikhalkin reduced the enumeration of complex and real algebraic curves in $(\mathbb{C}^*)^2$ to the enumeration of some piecewise linear graphs in $\mathbb{R}^2$ called plane tropical curves. Shortly after these results were extended in [Mik] and [NS06] to the computation of genus 0 Gromov-Witten invariants of projective spaces of arbitrary dimension. By stretching configurations of constraints along some specific direction, Brugallé and Mikhalkin replaced in [BM] the enumeration of tropical curves by a purely combinatorial study of their floor decompositions. As an application, they exhibited a generic configuration of 8 real lines in $\mathbb{R}P^3$ with 92 real conics passing through them.

In this paper, we refine the technique used in [BM07] in the case of $\mathbb{C}P^3$ to systematically study the case $d=2$. Along the way, we will give a proof of Sottile's Theorem different from the original one.

The question of existence of non-trivial lower bounds for the numbers $N^{\mathbb{R}}_{2,n}(l_1,\dots,l_\gamma,\omega)$ is also a very important and difficult problem about which not so much is known. The combination of Welschinger invariants (see [Wel05a] and [Wel05b]) and tropical geometry allowed to exhibit such non-trivial lower bounds in the case of rational curves passing through points in $\mathbb{R}P^2$ or $\mathbb{R}P^3$, i.e. for $n=2$ or 3 and $l_i=n$ $\forall i$ (see [Wel05a], [Mik05], and [IKS04] for the case $n=2$, and [Wel05b], and [BM] for the case $n=3$). In the case of enumeration of lines (and more generally in the enumeration of real linear spaces), the existence of some non-trivial lower bounds has been proved by Gabrielov and Eremenko in [EG02]. Up to our knowledge, the exact determination of the minimal value of $N^{\mathbb{R}}_{d,n}(l_1,\dots,l_\gamma,\omega)$ (when non-trivial) is known so far only in the cases $n=2$, $d=3$ ([DK00]) and $d=4$ ([Rey]), and in the cases $d=1$ and $l_i=2$ for all i ([EG02]).

One could also study maximality of more general real enumerative problems, for example by prescribing tangency conditions with constraints. We refer the interested reader to [RTV97], [Ber08], [Sot], and [BBM] for some partial answers in this direction.

We give in section 2 all tropical definitions needed to prove Theorem 1.1. In particular, we set-up tropical enumerative problems studied in this paper. Next, we explain in section 3 the main ideas of the floor decomposition technique to solve these tropical enumerative problems, before focusing on the easier particular cases of enumeration of lines and conics. Theorem 1.1 is finally proved in section 4.

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2. Tropical geometry

In this section, we briefly review the tropical background needed in this paper. For more details, we refer, for example, to [Mik06], [RGST05], [IMS07], and [BPS08].

2.1. **Rational tropical curves.** Given a finite graph C (i.e. C has a finite number of edges and vertices) we denote by $\text{Vert}^\infty(C)$ (resp. $\text{Vert}^0(C)$) the set of its vertices which are (resp. are not) 1-valent, and by $\text{Edge}^\infty(C)$ (resp. $\text{Edge}^0(C)$) the set of its edges which are (resp. are not) adjacent to a 1-valent vertex. Throughout the text, we will always assume that the considered graphs **do not have any 2-valent vertices**.

Definition 2.1. *A rational tropical curve C is a finite compact connected tree equipped with a complete inner metric on $C \setminus Vert^\infty(C)$.*

By definition, the 1-valent vertices of C are at infinite distance from all the other points of C. Elements of $\text{Edge}^\infty(C)$ are called the *ends* of C. An edge in $\text{Edge}^\infty(C)$ (resp. $\text{Edge}^0(C)$) is said to be *unbounded* (resp. *bounded*).

Example 1. An example of rational tropical curve C is depicted in Figure 1. This curve has 5 unbounded edges, and 2 bounded edges of finite length a and b. The length of an edge of C is written close to this edge in Figure 1.

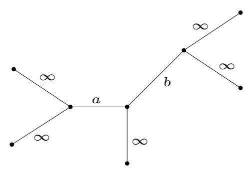


Figure 1. A rational tropical curve

Given e an edge of a tropical curve C, we choose a point p in the interior of e and a unit vector u_e of the tangent line to C at p. Of course, the vector u_e depends on the choice of p and is well-defined only up to multiplication by -1, but this will not matter in the following. We will sometimes need u_e to have a prescribed direction, and we will then precise this direction. The standard inclusion of $\mathbb{Z}^n$ in $\mathbb{R}^n$ induces a standard inclusion of $\mathbb{Z}^n$ in the tangent space of $\mathbb{R}^n$ at any point of $\mathbb{R}^n$.

Definition 2.2. *Let C be a rational tropical curve. A continuous map $f : C \setminus Vert^\infty(C) \to \mathbb{R}^n$ is a tropical morphism if*

- *for any edge e of C, the restriction $f_{|e}$ is a smooth map with $df(u_e) = w_{f,e}u_{f,e}$ where $u_{f,e} \in \mathbb{Z}^n$ is a primitive vector, and $w_{f,e}$ is a non-negative integer;*
- *for any vertex v in $Vert^0(C)$ whose adjacent edges are $e_1, \ldots, e_k$, one has the balancing condition*

$$\sum_{i=1}^{k} w_{f,e_i} u_{f,e_i} = 0$$

where u_{e_i} is chosen so that it points away from v.

The integer $w_{f,e}$ is called the *weight of the edge e with respect to f*. When no confusion is possible, we will speak about the weight of an edge, without referring to the morphism f. By abuse, we will denote $f : C \to \mathbb{R}^n$ instead of $f : C \setminus \text{Vert}^\infty(C) \to \mathbb{R}^n$. If $w_{f,e} = 0$, we say that the morphism f *contracts* the edge e. The morphism f is called *minimal* if it does not contract any edge.

Given $u = (u_1, \ldots, u_n)$ a vector in $\mathbb{R}^n$, we define

$$d_u = \max\{0, u_1, \ldots, u_n\}.$$

The *degree* of a tropical morphism $f : C \to \mathbb{R}^n$ is defined by

$$\sum_{e \in \text{Edge}^\infty(C)} w_{f,e} d_{u_{f,e}}$$

where $u_{f,e}$ is chosen so that it points to its adjacent 1-valent vertex.

We define the following vectors in $\mathbb{R}^n$: $U_1 = (-1, 0, \ldots, 0)$, $U_2 = (0, -1, 0, \ldots, 0)$, …, $U_n = (0, \ldots, 0, -1)$, and $U_{n+1} = (1, \ldots, 1)$. A tropical morphism $f : C \to \mathbb{R}^n$ of degree d is said to be *transverse at infinity* if C has exactly $(n+1)d$ non-contracted ends. Note that in this case, for any $i = 1, \ldots, n+1$ the curve C has exactly d edges $e \in \text{Edge}^\infty(C)$ with $u_{f,e} = U_i$, where $u_{f,e}$ is chosen so that it points to its adjacent 1-valent vertex.

Example 2. In Figure 2 are depicted a tropical conic in $\mathbb{R}^2$ and a tropical conic in $\mathbb{R}^3$. For each unbounded edge e, the vector $u_{f,e}$ pointing to infinity is written close to e.

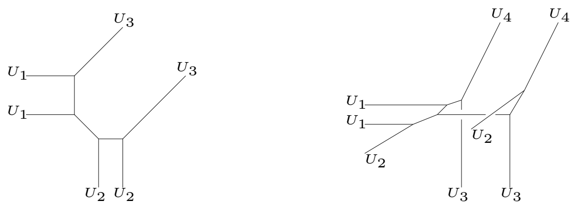


FIGURE 2. A tropical conic in $\mathbb{R}^2$ and in $\mathbb{R}^3$

Two tropical morphisms $f_1 : C_1 \to \mathbb{R}^n$ and $f_2 : C_2 \to \mathbb{R}^n$ are said to be *isomorphic* if there exists an isomorphism of metric graphs $\phi : C_1 \to C_2$ such that $f_1 = f_2 \circ \phi$. In this text, we consider tropical curves and tropical morphisms up to isomorphism.

Two tropical morphisms $h : C_1 \to \mathbb{R}^n$ and $h' : C_2 \to \mathbb{R}^n$ are said to be of the same *combinatorial type* if there exists a homeomorphism of graphs $\phi : C_1 \to C_2$ (i.e. we forget about the metric on C_1 and C_2) such that $h = h' \circ \phi$, and $w_{h,e} = w_{h',\phi(e)}$ for all $e \in \text{Edge}(C_1)$.

In this text, we need the notion of *reducible* tropical morphism. Given C_1 and C_2 two tropical curves, p_1 and p_2 two points respectively on C_1 (resp. C_2), the topological gluing $C_1 \cup_{(p_1,p_2)} C_2$ of C_1 and C_2 at p_1 and p_2 inherits naturally a structure of tropical curve from C_1 and C_2. The curve C_i can be seen as a subset of C, and the point $p_1 = p_2$ is called the *node* of C.

Definition 2.3. *A minimal tropical morphism $f : C \to \mathbb{R}^n$ is said to be reducible if there exist two minimal tropical morphisms $f_1 : C_1 \to \mathbb{R}^n$ and $f_2 : C_2 \to \mathbb{R}^n$, and a point $p_i \in C_i$, such that C is the gluing of C_1 and C_2 at the point p_1 and p_2, and $f_{|C_i} = f_i$.*

We denote such a reducible tropical morphism $f = f_1 \cup_p f_2 : C_1 \cup_p C_2 :\to \mathbb{R}^n$. If $f : C \to \mathbb{R}^n$ is a reducible tropical morphism of degree d, then $f_i : C_i \to \mathbb{R}^n$ is a tropical morphism of degree d_i and $d_1 + d_2 = d$. In particular, if $d = 2$ then $d_1 = d_2 = 1$.

2.2. **Tropical linear spaces.** Defining tropical linear spaces of $\mathbb{R}^n$ in full generality would require much more material than needed in the rest of the paper. Moreover this would force us to make the distinction between *realizable* and *non-realizable* tropical linear spaces, notion that we want to keep out of the scope of this note. Instead, we define a restricted class of tropical linear spaces of $\mathbb{R}^n$, that we call *complete tropical linear spaces*. For a general definition and study of tropical linear spaces, we refer to [Spe08].

Given $1 \leq i < j \leq n+1$, we denote by $E_{i,j}$ the convex polyhedron of $\mathbb{R}^n$ obtained by taking all non-negative real linear combinations of all the vectors U_k but U_i and U_j, and we define

$$H_0^n = \cup_{1\leq i<j\leq n+1} E_{i,j}.$$

Definition 2.4. *A tropical hyperplane of $\mathbb{R}^n$ is the translation of H_0^n along any vector of $\mathbb{R}^n$.*

A complete tropical linear space of dimension j is the intersection of $n-j$ tropical hyperplane in general position. The ambient space $\mathbb{R}^n$ is a complete tropical linear space of dimension n.

One could avoid the genericity assumption in Definition 2.4 by considering tropical (or stable) intersections of tropical hyperplanes in $\mathbb{R}^n$. We refer to [Mik06] or [RGST05] for more details.

A tropical linear space of dimension j is a finite polyhedral complex of pure dimension j.

Example 3. A tropical plane and a tropical line in $\mathbb{R}^3$ are depicted in Figure 3.

FIGURE 3. A tropical plane and a tropical line in $\mathbb{R}^3$

2.3. **Tropical enumerative geometry.** Now we have defined tropical rational curves and complete tropical linear spaces in $\mathbb{R}^n$, we can play the same game as in section 1. Namely, let us fix some integers $d \geq 1$, $n \geq 2$, $\gamma \geq 2$, $l_1 \geq 1$, ..., $l_\gamma \geq 1$ subject to equality (1), and let us choose a configuration $\omega = \{L_1, \ldots, L_\gamma\}$ of complete tropical linear spaces in $\mathbb{R}^n$ such that $\operatorname{codim} L_j = l_j$ for $j = 1, \ldots, \gamma$. Then we define $\mathbb{T}\mathcal{C}(\omega)$ as the set of all minimal rational morphisms $f : C \to \mathbb{R}^n$ of degree d such that $f(C)$ intersects all tropical linear spaces in ω. This game is related to section 1 by the following fundamental Theorem.

Theorem 2.5 (Correspondence Theorem, [Mik05], [Mik], [NS06])**.** *If ω is generic, then the set $\mathbb{T}\mathcal{C}(\omega)$ is finite and composed of tropical morphisms transverse to infinity. Moreover, to each element f in $\mathbb{T}\mathcal{C}(\omega)$, one can associate a positive integer number $\mu(f)$, called the multiplicity of f, which depends only on f and ω such that*

$$N_{d,n}(l_1, \ldots, l_\gamma) = \sum_{f\in\mathbb{T}\mathcal{C}(\omega)} \mu(f).$$

Proof. As explained in [BBM, Section 6] the multiplicity of f reduces to local computations, which are done in [NS06]. □

There is a combinatorial definition of the integer $\mu(f)$ just in terms of the tropical morphism f and the configuration ω. However we won't need it in this paper, so instead of giving the precise definition of $\mu(f)$, let us just explain its geometrical meaning.

Theorem 2.5 is obtained by degenerating the standard complex structure on $(\mathbb{C}^*)^n$ via the following self-diffeomorphism of $(\mathbb{C}^*)^n$

$$\begin{array}{cccc} H_t : & (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^n \\ & (z_i) & \longmapsto & \left(|z_i|^{\frac{1}{\log t}} \frac{z_i}{|z_i|}\right) \end{array}$$

Namely, for any tropical complete linear space L_j of codimension l_j in ω, there exists a family $(L_{t,j})_{t>0}$ of complex linear spaces in $(\mathbb{C}^*)^n$ of codimension l_j such that the sets $Log \circ H_t(L_{t,j})$ converges to L_j when $t \to \infty$, in the Haussdorf metric on compact subsets of $(\mathbb{R}^*)^n$. The map Log is defined by $Log(z_i) = (\log |z_i|)$. Hence for each t, we associate a configuration $\omega_t = \{L_{t,1}, \ldots, L_{t,\gamma}\}$ of linear subspaces of $(\mathbb{C}^*)^n$. For t big enough, the configuration ω_t is generic if ω is generic, so the complex rational curves of degree d passing through all the linear spaces in ω_t form a finite set $\mathcal{C}(\omega_t)$. It turns out, and this is the core of Theorem 2.5, that the set $Log \circ H_t(\mathcal{C}(\omega_t))$ converges to the set $\mathbb{T}\mathcal{C}(\omega)$, and that for any tropical morphism $f \in \mathbb{T}\mathcal{C}(\omega)$ there exist exactly $\mu(f)$ complex curves in $\mathcal{C}(\omega_t)$ whose image under $Log \circ H_t$ converge to $f(C)$.

Suppose now that the linear spaces $L_{t,j}$ are chosen to be all real (this is always possible). In particular, all curves in $\mathcal{C}(\omega_t)$ are either real or come in pairs of complex conjugate curves. A very important property of the map H_t is that it commutes with the standard complex conjugation in $(\mathbb{C}^*)^n$. As a consequence, both curves in a pair of complex conjugated curves in $\mathcal{C}(\omega_t)$ have the same image under $Log \circ H_t$. In particular we have the following Lemma, where $[\mu(f)]_2$ denotes the value modulo 2 of the integer $\mu(f)$.

Lemma 2.6. *If ω is generic, then there exists a generic configuration Ω of real linear spaces in $\mathbb{R}P^n$ such that there exist at least $\sum_{f\in\mathbb{T}\mathcal{C}(\omega)}[\mu(f)]_2$ real rational curves of degree d in $\mathbb{R}P^n$ intersecting all linear spaces in Ω.*

Theorem 1.1 is a consequence of Theorem 2.5 and Lemma 2.6: we exhibit generic configurations ω such that the set $\mathbb{T}\mathcal{C}(\omega)$ contains exactly $N_{2,n}(l_1, \ldots, l_\gamma)$ distinct tropical curves. Hence, all of them must have multiplicity 1, which implies the maximality of the corresponding enumerative problem by Lemma 2.6. The main tool to exhibit such configurations ω is the *floor decomposition* technique.

2.4. **Enumeration of tropical reducible conics.** Here we state some easy facts about a small variation of the problem exposed in section 2.3. Namely, we enumerate tropical reducible conics passing through a generic collection of complete tropical linear spaces.

Next Lemma is standard, see for example [BBM] or [GKM09].

Lemma 2.7. *Let α be a combinatorial type of reducible morphisms $f_1 \cup_p f_2 : C_1 \cup_p C_2 \to \mathbb{R}^n$ of degree 2. Then the space of all reducible tropical morphisms with combinatorial type α is naturally a convex polyhedron of dimension at most $3n-2$, with equality if and only if C_i is trivalent and p_i is not a vertex of C_i for $i = 1, 2$.*

Let us fix some integers $l_0 \geq 0$ and $l_1^1, \ldots, l_{\gamma_1}^1, l_1^2, \ldots, l_{\gamma_2}^2 \geq 1$ such that

$$l_0 + \sum_{i=1,2} \sum_{j=1}^{\gamma_i} (l_j^i - 1) = 3n - 2$$

and let us choose L_0 a tropical complete linear space in $\mathbb{R}^n$ of codimension l_0 and two configurations ω^1 and ω^2 of complete tropical linear spaces in $\mathbb{R}^n$ such that $\omega^i = \{L_1^i, \ldots, L_{\gamma_i}^i\}$ with $\operatorname{codim} L_j^i = l_j^i$. Then we define $\mathbb{T}\mathcal{C}_{red}(L_0, \omega^1, \omega^2)$ as the set of all reducible rational morphisms $f = f_1 \cup_p f_2 : C_1 \cup_p C_2 \to \mathbb{R}^n$ of degree 2 such that $f_i(C_i)$ intersects all tropical linear spaces in ω^i for $i = 1, 2$ and $f(p) \in L_0$. Note that if $\mathbb{T}\mathcal{C}_{red}(L_0, \omega^1, \omega^2) \neq \emptyset$, we necessarily have

$$\sum_{j=1}^{\gamma_i} (l_j^i - 1) \leq 2n - 2 \quad \text{for} \quad i = 1, 2.$$

Next Lemma is a straightforward application of Lemma 2.7 and standard techniques in tropical enumerative geometry, see for example [BBM], [NS06], or [GKM09].

Lemma 2.8. *For a generic choice of L_0, ω^1, and ω^2, the set $\mathbb{T}\mathcal{C}_{red}(L_0, \omega^1, \omega^2)$ is finite, and composed of tropical morphisms transverse to infinity.*

Note that we can pose the same problem in complex geometry, and that we can easily give the answer in terms of the numbers $N_{1,n}$. Namely, let L_0 a linear space in $\mathbb{C}P^n$ of codimension l_0 and two configurations ω^1 and ω^2 of linear spaces in $\mathbb{C}P^n$ such that $\omega^i = \{L_1^i, \ldots, L_{\gamma_j}^i\}$ with $\operatorname{codim} L_j^i = l_j^i$. Then we define $N_{2,n}^{red}(l_0, \{l_1^1, \ldots, l_{\gamma_1}^1\}, \{l_1^2, \ldots, l_{\gamma_1}^2\})$ as the number of reducible conics $C_1 \cup_p C_2$ in $\mathbb{C}P^n$ such that C_i intersects all spaces in ω^i for $i = 1, 2$ and $p \in L_0$.

Lemma 2.9. *With the hypothesis above, we have*

$$N^{red}_{2,n}(l_0,\{l^1_1,\dots,l^1_{\gamma_1}\},\{l^2_1,\dots,l^2_{\gamma_2}\}) = \prod_{i=1,2} N_{1,n}\left(2n-1-\sum_{j=1}^{\gamma_i}(l^i_j-1), l^i_1,\dots,l^i_{\gamma_i}\right).$$

Proof. Let V_i be the algebraic variety in $\mathbb{C}P^n$ given by the union of all lines passing through all linear spaces in ω^i. By definition, the number $N^{red}_{2,n}(l_0,\{l^1_1,\dots,l^1_{\gamma_1}\},\{l^2_1,\dots,l^2_{\gamma_1}\})$ is equal to the number of intersection points in $V_1\cap V_2\cap L_0$, i.e. is equal to the product of the degree of V_1 and V_2. Since the configuration of linear spaces is generic, V_i has dimension $2n-1-\sum_{j=1}^{\gamma_i}(l^i_j-1)$. The degree of V_i is its number of intersection points with a generic linear space in $\mathbb{C}P^n$ of complementary dimension, and this number is by definition precisely $N_{1,n}\left(2n-1-\sum_{j=1}^{\gamma_i}(l^i_j-1), l^i_1,\dots,l^i_{\gamma_i}\right)$. □

3. Floor decomposition of tropical curves

Here we explain how to enumerate complex curves of degree 1 and 2 with the help of the floor decomposition technique. This technique works for any degree (see [BM07], [BM08], [BM]) but the exposition of the method in its full generality would require a quite heavy formalism which in our opinion would harm to the well understanding of this text. Hence we just give the main idea of the method before focusing on the degree 1 and 2 cases. Note that floor decomposition technique has strong connections with the Caporaso and Harris method (see [CH98]), extended later by Vakil (see [Vak00b], [Vak00a]), and with the neck-stretching method in symplectic field theory (see [EGH00]).

We denote by $\pi:\mathbb{R}^n\to\mathbb{R}^{n-1}$ the linear projection forgetting the last coordinate. Given a minimal tropical morphism $f:C\to\mathbb{R}^n$, the morphism $\pi\circ f:C\to\mathbb{R}^{n-1}$ is not minimal in general. However, there exists a unique tropical curve C' equipped with a map $\rho:C\to C'$ and a unique minimal tropical morphism $f':C'\to\mathbb{R}^{n-1}$ such that $f=f'\circ\rho$. We say that f' is *induced* by $\pi\circ f$, and that f is a *lifting* of f'.

3.1. **General method.** The starting idea of floor decomposition is to compute the numbers $N_{d,n}(l_1,\dots,l_\gamma)$ by induction on the dimension n. As easy at it sounds, this approach does not work straightforwardly and one has to work carefully: it is easy to compute that through one point p and two tropical lines L_1 and L_2 in $\mathbb{R}^3$ passes exactly 1 tropical line L (see example 4). However, there exists infinitely many tropical lines in $\mathbb{R}^2$ passing trough $\pi(p)$, $\pi(L_1)$, and $\pi(L_2)$, and without knowing L, it is not clear at all which one of these planar lines is $\pi(L)$.

To make the induction work, we first stretch the configuration ω is the direction $U_n=(0,\dots,0,-1)$. Then the tropical curves we are counting break in several *floors* for which we can apply induction.

Example 4. Let us explain how to use the floor decomposition technique in a simple case. Let use choose a point p and two tropical lines L_1 and L_2 in $\mathbb{R}^3$ such that L_1 (resp. L_2) consists of only one edge with direction $(0,1,0)$ (resp. $(1,0,0)$) contained in the horizontal plane with equation $z=a_1$ (resp. $z=a_2$). If the third coordinate of p is much more bigger than a_1 which in its turn is much more bigger than a_2, then the unique tropical line L in $\mathbb{R}^3$ passing through p, L_1, and L_2 is depicted in Figure 4, and $\pi(L)$ is the unique tropical line in $\mathbb{R}^2$ passing through $\pi(p)$ and $\pi(L_1)\cap\pi(L_2)$.

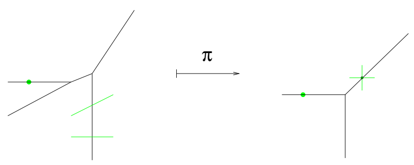
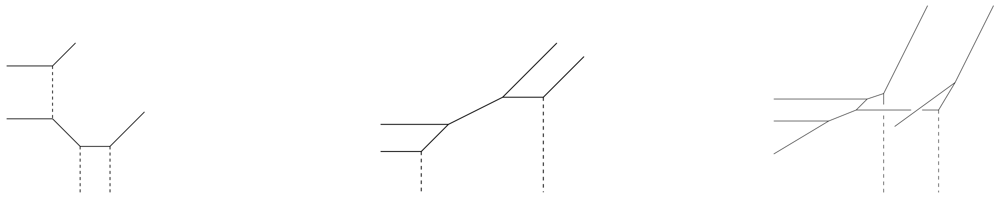


FIGURE 4. Floor decomposition technique to compute $N_{1,3}(3,2,2)=1$

Definition 3.1. *Let C be a rational tropical curve and $f:C\to\mathbb{R}^n$ a tropical morphism. An elevator of f is an edge e of C with $u_{f,e}=(0,\dots,0,\pm1)$. A floor of f is a connected component of C minus all its elevators.*

Note that if f is a morphism of degree d in $\mathbb{R}^n$ and $\mathcal{F}$ is a floor of f, then C induces a structure of tropical curve on $\mathcal{F}$ and $\pi \circ f_{|\mathcal{F}} : \mathcal{F} \to \mathbb{R}^{n-1}$ is a tropical morphism of degree $1 \leq d' \leq d$. The integer d' is called the *degree* of $\mathcal{F}$.

Example 5. Examples of planar and spatial conics are depicted in Figure 5. Elevators are depicted in dotted lines.



a) A planar conic with two floors b) A planar conic with one floor c) A spatial conic with one floor

FIGURE 5.

Definition 3.1 extends to any tropical varieties in $\mathbb{R}^n$, however we keep on restricting ourselves to complete tropical linear spaces.

Definition 3.2. *Let L be a complete tropical linear space in $\mathbb{R}^n$ of codimension j. The wall (resp. floor) of L is the union of all faces of L of codimension j which contain (resp. do not contain) the direction* $(0, \ldots, 0, 1)$.

Note that if L is a complete tropical linear space of codimension j in $\mathbb{R}^n$ with wall W and floor F, then $\pi(W)$ is a complete tropical linear space of codimension j in $\mathbb{R}^{n-1}$, and $\pi(F) = \pi(L)$ is a complete tropical linear space of codimension $j-1$ in $\mathbb{R}^{n-1}$.

Example 6. A tropical plane L in $\mathbb{R}^3$ together with its wall W are depicted in Figure 6. Clearly, $\pi(L) = \mathbb{R}^2$ and $\pi(W)$ is a tropical line in $\mathbb{R}^2$.

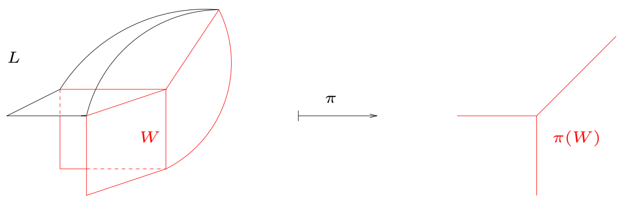


FIGURE 6. $\pi(L) = \mathbb{R}^2$ and $\pi(W)$ is a tropical line

Let us fix some integers $d \geq 1$, $n \geq 2$, $\gamma \geq 2$, $l_1 \geq 1$, ..., $l_\gamma \geq 1$ subject to equality (1), and let us choose a generic configuration $\omega = \{L_1, \ldots, L_\gamma\}$ of complete tropical linear spaces in $\mathbb{R}^n$ such that $\operatorname{codim} L_j = l_j$ for $j = 1, \ldots, \gamma$. As before, $\mathbb{T}\mathcal{C}(\omega)$ is the set of all rational tropical morphisms $f : C \to \mathbb{R}^n$ of degree d such that $f(C)$ intersects all tropical linear spaces in ω.

Definition 3.3. *An element L of ω is called a vertical (resp. horizontal) constraint for $f \in \mathbb{T}\mathcal{C}(\omega)$ if $f(C) \cap L$ lies in the wall (resp. floor) of L.*

Let us denote by $\operatorname{Vert}(L_j)$ the set of vertices of the complete tropical linear space L_j, and let us fix a hypercube $\mathcal{H}_{n-1}$ in $\mathbb{R}^{n-1}$ such that the cylinder $\mathcal{H}_{n-1} \times \mathbb{R}$ contains the set $\cup_{j=1}^{\gamma} \operatorname{Vert}(L_j)$. Given two points $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{R}^n$, we define $|v - w|_n = |v_n - w_n|$. Finally, we define $R_{\mathcal{H}}$ to be the length of the edges of $\mathcal{H}_{n-1}$, and

$$R(\omega) = \min_{j \neq k,\ v \in \operatorname{Vert}(L_j),\ w \in \operatorname{Vert}(L_k)} |v - w|_n.$$

The following observation is the key point of the technique.

Proposition 3.4 (Brugallé-Mikhalkin [BM07], [BM08], [BM]). *There exists a real number $D(n,d)$, depending only in n and d, such that if $R(\omega) \geq R_{\mathcal{H}}D(n,d)$ then for each morphism $f : C \to \mathbb{R}^n$ in $\mathbb{TC}(\omega)$ and for each floor $\mathcal{F}$ of f, $f(\mathcal{F})$ meets one and exactly one horizontal constraints.*

Definition 3.5. *If $R(\omega) \geq R_{\mathcal{H}}D(n,d)$, we say that ω is a (d,n)-decomposing configuration.*

Note that imposing to a configuration $\omega^{\mathbb{T}}$ to be (d,n)-decomposing is the same than imposing conditions on the relative position of the vertices of elements of $\omega^{\mathbb{T}}$. In particular, it makes sense to say that a configuration $\omega^{\mathbb{T}}$ is (d,n)-decomposing even if its elements do not satisfy equality (1).

The choice of the preferred direction U_n provides a natural partial order among the tropical linear spaces.

Definition 3.6. *Let L and L' in $\mathbb{R}^n$ be two complete tropical linear spaces. We say that L is higher than L', and denote $L \gg L'$, if any vertex of L has greater last coordinate than all vertices of L'.*

Note that $\mathbb{R}^n$ is greater than any other tropical linear space. Before explaining in detail the case of lines and conics, we need to introduce a notation. Given a generic configuration $\{L_1, \ldots, L_\gamma\}$ of complete tropical linear spaces in $\mathbb{R}^n$, $k \in \{1, \ldots, \gamma\}$, and $A \subset \{1, \ldots, \gamma\}$, we define the following complete tropical linear spaces in $\mathbb{R}^{n-1}$

$$L'_k = \pi(W_k), \quad \widehat{L}'_k = \pi(L_k), \quad \text{and} \quad \widetilde{L}'_A = \bigcap_{j \in A} \pi(L_j)$$

where W_j is the wall of L_j. We also denote by $\widetilde{L}'_k$ the complete tropical linear space $\widetilde{L}'_{\{1,\ldots,k\}}$.

3.2. **The case $d = 1$.** Suppose that $d = 1$, so that $\sum_{j=1}^{\gamma}(l_j - 1) = 2n - 2$. We choose a $(1,n)$-decomposing configuration $\omega = \{L_1, \ldots, L_\gamma\}$ of complete tropical linear spaces in $\mathbb{R}^n$, $n \geq 3$, such that L_j has codimension l_j and L_{j+1} is higher than L_j for $j = 1, \ldots, \gamma - 1$. We denote by W_k the wall of the constraint L_k. We denote by $\mathbb{TC}(\omega)^{(k)}$ the subset of tropical morphisms in $\mathbb{TC}(\omega)$ whose floor meets the horizontal constraint L_k (remember that in degree one, the floor is unique). According to Proposition 3.4, we have

$$\mathbb{TC}(\omega) = \bigsqcup_{k=1}^{\gamma} \mathbb{TC}(\omega)^{(k)}.$$

Given k in $\{1, \ldots, \gamma\}$, we define $\omega'^{(k)} = \{\widetilde{L}'_{k-1}, \widehat{L}'_k, L'_{k+1}, \ldots, L'_\gamma\}$. Since ω is generic, the tropical linear space L'_k has codimension l_k, $\widehat{L}'_k$ has codimension $l_k - 1$, and $\widetilde{L}'_{k-1}$ has codimension $\sum_{j=1}^{k-1}(l_j - 1)$ if non-empty. If $f : C \to \mathbb{R}^n$ is an element of $\mathbb{TC}(\omega)^{(k)}$, then the tropical morphism $\pi \circ f : C \to \mathbb{R}^n$ induces obviously an element of $\mathbb{TC}(\omega'^{(k)})$. Moreover, any tropical morphism f' in $\mathbb{TC}(\omega'^{(k)})$ can be lifted in a unique way as an element f of $\mathbb{TC}(\omega)^{(k)}$: the only unknown is the location of the elevator of f, which is given by the unique intersection point of $f'(C')$ with $\widetilde{L}'_{k-1}$ (see Examples 4 and 7). Hence, there exists a natural bijection between the two sets $\mathbb{TC}(\omega)^{(k)}$ and $\mathbb{TC}(\omega'^{(k)})$. Moreover this bijection respects multiplicity of tropical curves. In other words, we have the following Proposition

Proposition 3.7 (Brugallé-Mikhalkin [BM07], [BM08], [BM]). *For any k in $\{1, \ldots, \gamma\}$, we have*

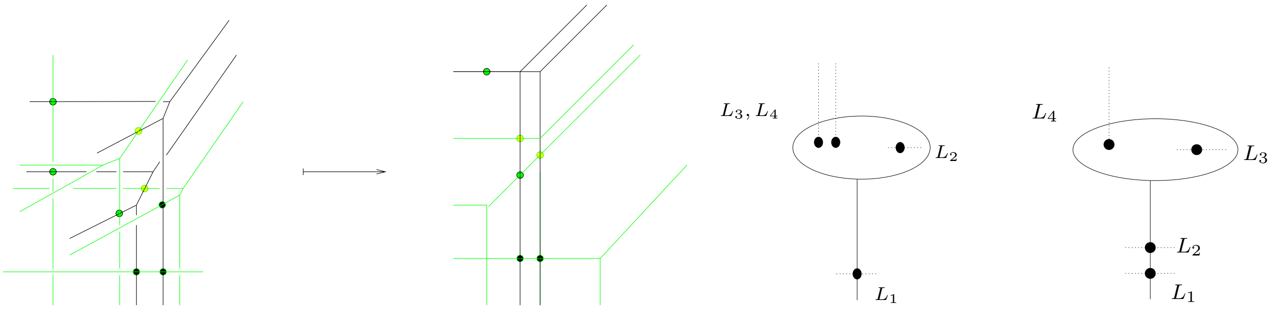
$$\sum_{f \in \mathbb{TC}(\omega)^{(k)}} \mu(f) = \sum_{f' \in \mathbb{TC}(\omega'^{(k)})} \mu(f').$$

Note that $\mathbb{TC}(\omega'^{(k)}) = \emptyset$ if $k = 1$ or $\sum_{j=1}^{k-1}(l_j - 1) > n$. Proposition 3.7 allows one to compute all the numbers $N_{1,n}$ out of the numbers $N_{1,n-1}$. Since it is trivial that $N_{1,2}(2) = 1$, all the numbers $N_{1,n}$ can be computed using Proposition 3.7.

Example 7. Let $\omega = \{L_1, L_2, L_3, L_4\}$ be a $(1,3)$-decomposing configuration of 4 lines in $\mathbb{R}^3$ with $L_{j+1} \gg L_j$. The set $\mathbb{TC}(\omega)^{(k)}$ is non-empty only for $k = 2, 3$, and the corresponding projected configurations in $\mathbb{R}^2$ are $\omega'^{(2)} = \{\pi(L_1), \pi(L_2), \pi(W_3), \pi(W_4)\}$ and $\omega'^{(3)} = \{\pi(L_1) \cap \pi(L_2), \pi(L_3), \pi(W_4)\}$ (see Figure 7a). Since there exists only one line passing through two points in the plane, we get that $N_{1,3}(2,2,2,2) = 1 + 1 = 2$.

To depict tropical curves passing through a decomposing configuration, we use the following convention: a floor (resp. elevator) of the curve is represented by an ellipse (resp. a vertical edge); a constraint intersecting a floor (resp. elevator) is depicted by a dotted segment intersecting the corresponding ellipse (resp. vertical edge); a constraint is represented by a horizontal (resp. vertical) segment if it is a horizontal (resp. vertical) constraint for the curve.

For example, the two tropical lines passing through the four lines $L_1, \ldots, L_4$ are represented by the diagrams depicted in Figures 7b and c.



a) b) Line in $\mathbb{T}\mathcal{C}(\omega)^{(2)}$ c) Line in $\mathbb{T}\mathcal{C}(\omega)^{(3)}$

FIGURE 7. Floor decomposition technique to compute $N_{1,3}(2,2,2,2) = 2$

Corollary 3.8. *Given $l_1, \ldots, l_\gamma \geq 1$ such that $\sum_{j=1}^{\gamma}(l_j - 1) = 2n - 2$, we have*

$$N_{1,n}(l_1, \ldots, l_\gamma) = \sum_{k=2}^{\gamma} N_{1,n-1}\left(\sum_{i=1}^{k-1}(l_i - 1), l_k - 1, l_{k+1}, \ldots, l_\gamma\right).$$

Example 8. Let us compute the numbers $C(n,l) = N_{1,n}(l, l_1, \ldots, l_{2n-1-l})$ with $2 \leq l \leq n$ and $l_1 = \ldots = l_{2n-1-l} = 2$. According to Corollary 3.8, for any $n \geq 2$ and $3 \leq l \leq n$ we get

$$C(n,l) = C(n-1, l-1) + C(n, l+1). \tag{2}$$

Hence we can extend the definition of the numbers $C(n,l)$ for all pairs (n,l) with $n \geq 1$ according to relation (2), which is a Pascal type relation. The sequence

$$A(n,l) = \binom{2n-l-1}{n-1} - \binom{2n-l-1}{n}$$

also satisfies relation (2), and we have

$$\forall n \geq 1 \quad C(n,0) = A(n,0) = 0 \quad \text{and} \quad C(n,n) = A(n,n) = 1$$

so these two sequences must be equal on the set $\{(n,l) \in \mathbb{Z}^2 \mid n \geq 1\}$. Hence we get

$$\forall n \geq 2, \quad \forall l \in \{2, \ldots, n\}, \quad C(n,l) = \binom{2n-l-1}{n-1} - \binom{2n-l-1}{n}. \tag{3}$$

In particular, we find again the Catalan numbers

$$C(n,2) = C(n,1) = \frac{1}{n}\binom{2n-2}{n-1}.$$

Note that Corollary 3.8 and the relation (3) almost immediatly imply that

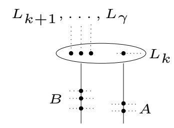
$$\forall n \geq 2, \quad \forall k, l \in \{2, \ldots, n\}, \quad N_{1,n}(k, l, l_1, \ldots, l_{2n-k-l}) = \binom{2n-l-k}{n-k} - \binom{2n-l-k}{n}$$

where $l_1 = \ldots = l_{2n-k-l} = 2$.

3.3. **The case** $d = 2$. Let us suppose that $d = 2$, so that $\sum_{j=1}^{\gamma}(l_j - 1) = 3n - 1$. We choose a $(2, n)$-decomposing configuration $\omega = \{L_1, \ldots, L_\gamma\}$ of complete tropical linear spaces in $\mathbb{R}^n$, $n \geq 3$, such that L_j has codimension $l_j \geq 2$ and L_{j+1} is higher than L_j for $j = 1, \ldots, \gamma - 1$. As in section 3.2, we denote by W_k the wall of the constraint L_k. Given f an element of $\mathbb{T}\mathcal{C}(\omega)$, either f has one floor of degree 2, or it has 2 floors of degree 1.

Let us first deal with the case of a conic with one floor of degree 2. Let k be an integer in $\{1, \ldots, \gamma\}$, and $A \sqcup B$ a partition of $\{1, \ldots, k-1\}$ into two sets. We denote by $\mathbb{T}\mathcal{C}(\omega)^{(k,A,B)}$ the set of all tropical morphisms in $\mathbb{T}\mathcal{C}(\omega)$ with one floor $\mathcal{F}$ of degree 2 such that

- the floor $\mathcal{F}$ meets the horizontal constraint L_k;
- one of the two elevators of f meets all the constraints L_j with $j \in A$, while the other elevator meets all the constraints L_j with $j \in B$.



According to Proposition 3.4, the set of all tropical morphisms in $\mathbb{T}\mathcal{C}(\omega)$ with a single floor is equal to

$$\bigsqcup_{k=1}^{\gamma} \quad \bigsqcup_{A \sqcup B = \{1, \ldots, k-1\}} \mathbb{T}\mathcal{C}(\omega)^{(k,A,B)}.$$

We define $\omega'^{(k,A,B)} = \{\widetilde{L}'_A, \widetilde{L}'_B, \widehat{L}'_k, L'_{k+1}, \ldots, L'_\gamma\}$. The complete tropical linear space L'_k has codimension l_k, the space $\widehat{L}'_k$ has codimension $l_k - 1$, and $\widetilde{L}'_A$ (resp. $\widetilde{L}'_B$) has codimension $\sum_{j \in A}(l_j - 1)$ (resp. $\sum_{j \in B}(l_j - 1)$) if non-empty. As in section 3.2, there is natural map

$$\phi_{(k,A,B)} : \mathbb{T}\mathcal{C}(\omega)^{(k,A,B)} \to \mathbb{T}\mathcal{C}(\omega'^{(k,A,B)}).$$

Contrary to section 3.2, the map $\phi_{(k,A,B)}$ is injective and respects the multiplicity if and only if

$$\operatorname{codim} \widetilde{L}'_A \geq 2, \quad \operatorname{codim} \widetilde{L}'_B \geq 2, \quad \text{and} \quad \operatorname{codim} \widehat{L}'_k \geq 2.$$

In general, given $f' \in \mathbb{T}\mathcal{C}(\omega'^{(k,A,B)})$ we have

$$\sum_{f \in \phi^{-1}_{(k,A,B)}(f')} \mu(f) = 2^{m_{(k,A,B)}} \mu(f').$$

where $m_{(k,A,B)}$ is the number of spaces in $\{\widetilde{L}'_A, \widetilde{L}'_B, \widehat{L}'_k\}$ of codimension 1. The factor $2^{m_{(k,A,B)}}$ is just the manifestation of the fact that a conic in the projective space intersect a hyperplane into two points (counted with multiplicity). Altogether we hence have the following Proposition.

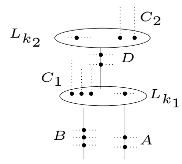
Proposition 3.9 (Brugallé-Mikhalkin [BM07], [BM08], [BM]). *Given any k, A, and B as above, we have*

$$\sum_{f \in \mathbb{T}\mathcal{C}(\omega)^{(k,A,B)}} \mu(f) = 2^{m_{(k,A,B)}} \sum_{f' \in \mathbb{T}\mathcal{C}(\omega'^{(k,A,B)})} \mu(f').$$

We treat now the case of tropical morphisms f in $\mathbb{T}\mathcal{C}(\omega)$ with two floors of degree 1. Let $k_1 < k_2$ be two integers in $\{1, \ldots, \gamma\}$, $A \sqcup B$ be a partition of $\{1, \ldots, k_1 - 1\}$, D a subset of $\{k_1 + 1, \ldots, k_2 - 1\}$, and $C_1 \sqcup C_2$ be a partition of $\{k_1 + 1, \ldots, \gamma\} \setminus (\{k_2\} \cup D)$.

We denote by $\mathbb{T}\mathcal{C}(\omega)^{(k_1,k_2,A,B,C_1,C_2,D)}$ the set of tropical morphisms in $\mathbb{T}\mathcal{C}(\omega)$ with two floors $\mathcal{F}_1$ and $\mathcal{F}_2$ of degree 1 such that

- the floor $\mathcal{F}_i$ meets the constraint L_{k_i}, and all the constraints L_j with $j \in C_i$;
- one of the two unbounded elevators of f meets all the constraints L_j with $j \in A$, while the other unbounded elevator meets all the constraints L_j with $j \in B$;
- the bounded elevator of f meets all the constraints L_j with $j \in D$.



According to Proposition 3.4, the set of all tropical morphisms in $\mathbb{T}\mathcal{C}(\omega)$ with two floors is equal to

$$\bigsqcup_{1\le k_1<k_2\le\gamma}\ \bigsqcup_{A,B,C_1,C_2,D} \mathbb{T}\mathcal{C}(\omega)^{(k_1,k_2,A,B,C_1,C_2,D)}.$$

Given $i=1,2$, we define the following integers

$$l'^i_j = l_j \text{ if } j\in C_i \qquad l'_A=\sum_{j\in A}(l_j-1) \qquad l'^1_0 = 2n-\sum_{j\in C_1}(l'_j-1)-l'_A-l'_B-l'_{k_1}$$

$$l'_{k_i}=l_{k_i}-1 \qquad l'_B=\sum_{j\in B}(l_j-1) \qquad l'^2_0=2n-2-\sum_{j\in C_2}(l'_j-1)-l'_{k_2}$$

Finally, we denote by γ_i the cardinal of the set C_i.

Proposition 3.10 (Brugallé-Mikhalkin [BM07], [BM08], [BM]). *Given k_1,k_2,A,B,C_1,C_2, and D as above, we have*

$$\sum_{f\in\mathbb{T}\mathcal{C}(\omega)^{(k_1,k_2,A,B,C_1,C_2,D)}} \mu(f) = N_{1,n-1}(l'^1_{i_1},\ldots,l'^1_{i_{\gamma_1}},l'_A,l'_B,l'_{k_1},l'^1_0)N_{1,n-1}(l'^2_{j_1},\ldots,l'^2_{j_{\gamma_2}},l'_{k_2},l'^2_0).$$

Let us give a heuristic of the proof of Proposition 3.10. We define $\omega'^1=\{\widetilde{L}'_A,\widetilde{L}'_B,\widehat{L}'_{k_1},L'_j : j\in C_1\}$, $\omega'^2=\{\widehat{L}'_{k_2},L'_j : j\in C_2\}$, and $\omega^{(k_1,k_2,A,B,C_1,C_2,D)}=(\widetilde{L}'_D,\omega'^1,\omega'^2)$. As in section 3.2, there is a natural and bijective map

$$\phi:\mathbb{T}\mathcal{C}(\omega)^{(k_1,k_2,A,B,C_1,C_2,D)}\to\mathbb{T}\mathcal{C}_{red}(\omega^{(k_1,k_2,A,B,C_1,C_2,D)})$$

and Proposition 3.10 now follows from Lemma 2.9.

Once again, since it is trivial that $N_{2,1}(2)=1$ and $N_{2,2}(5)=1$, all the numbers $N_{n,2}$ can be computed inductively using Propositions 3.7, 3.9, and 3.10.

Example 9. (see [BM07]) In Figure 8, we depict all possible floor decompositions for tropical conics in $\mathbb{R}^3$ passing through a $(2,3)$-decomposing configuration of 8 tropical lines. In each case, we precise the number of (k_1,k_2,A,B,C_1,C_2,D) or (k,A,B) with a non-empty corresponding set of tropical morphisms, and the sum of the multiplicities of the corresponding curves in $\mathbb{T}\mathcal{C}(\omega)^{(k_1,k_2,A,B,C_1,C_2,D)}$ or $\mathbb{T}\mathcal{C}(\omega)^{(k,A,B)}$ for each such choice.

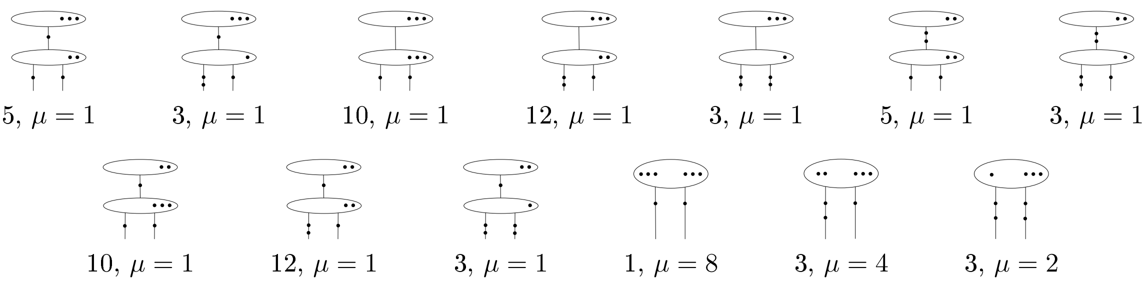


FIGURE 8. Floor decompositions of the 92 tropical conics passing through 8 lines in $\mathbb{R}^3$

4. Maximal configurations for conics

4.1. **Well-ordered totally decomposing configurations.** To prove Theorem 1.1, we exhibit *maximal* configurations, that is configurations ω of complete tropical linear spaces such that the cardinal of the set $|\mathbb{T}\mathcal{C}(\omega)|$ is equal to the corresponding Gromov-Witten invariant.

In this section, we define *well-ordered totally decomposing configurations*. The rest of the paper will be devoted to the proof that any such configuration is maximal when dealing with conics.

Definition 4.1. *Let $\omega=\{L_1,\ldots,L_\gamma\}$ be a (d,n)-decomposing configuration of complete tropical linear spaces in $\mathbb{R}^n$, and let W_i be the wall of L_i. We say that ω is a (d,n)-totally decomposing configuration if it satisfies one of the two following conditions*

- *$n = 2$;*
- *for any subset $\Gamma \subset \{1, \dots, \gamma\}$ the configuration $\{\pi(L_i),\ \pi(W_j)\ :\ i \in \Gamma, j \notin \Gamma\}$ is $(d, n-1)$-totally decomposing.*

Since a hyperplane in $\mathbb{R}^n$ has a single vertex, the existence of totally decomposing configurations of tropical hyperplanes is straightforward. Now suppose that we want to construct a totally decomposing configuration $\omega = \{L_1, \dots, L_\gamma\}$ with $\operatorname{codim} L_i = l_i$. We start with a totally decomposing configuration of tropical hyperplanes $\{H_1, \dots, H_\gamma\}$, and we construct L_i by intersecting l_i copies of H_i translated along very small vectors.

Remark. Let $\{L_1, \dots, L_\gamma\}$ be a generic totally decomposing configuration of complete tropical linear spaces, $\Gamma \subset \{1, \dots, \gamma\}$, and ω' the configuration $\{\pi(L_i),\ \pi(W_j)\ :\ i \in \Gamma, j \notin \Gamma\}$. Then, it follows directly from Definition 4.1 that given any elements $\mathcal{L}_1, \dots, \mathcal{L}_k$ of ω', the floor of the complete tropical linear spaces $\cap_{i=1}^k \mathcal{L}_k$ is contained in the floor of the lowest space among $\mathcal{L}_1, \dots, \mathcal{L}_k$ (see Figure 9).

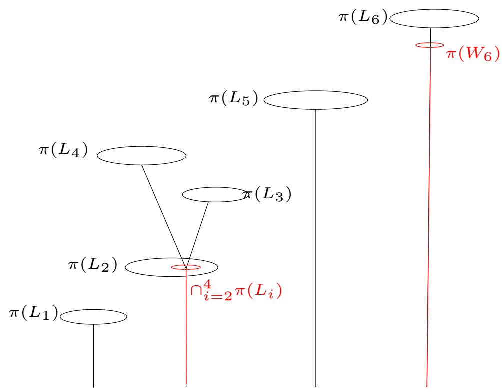


FIGURE 9. $\omega = (L_1, \dots, L_6)$

Next lemma provides an alternate proof of Sottile's Theorem about maximality of real enumerative problems concerning lines in projective spaces.

Lemma 4.2. *Let ω be a $(1, n)$-totally decomposing configuration of complete tropical linear spaces in $\mathbb{R}^n$ subject to equality (1) with $d = 1$. Then ω is maximal.*

Proof. We prove the lemma by induction on n. Clearly, the lemma is true for $n = 2$. Suppose now that $n \geq 3$ and that the lemma is true in dimension $n - 1$. In what follows, we use the notation of section 3.2. Any projected configuration $\omega'^{(k)}$ in $\mathbb{R}^{n-1}$ is $(1, n-1)$-totally decomposing, and thus maximal by the induction hypothesis. Hence any tropical morphism f in $\mathbb{T}\mathcal{C}(\omega'^{(k)})$ has multiplicity 1. Since the two sets $\mathbb{T}\mathcal{C}(\omega)^{(k)}$ and $\mathbb{T}\mathcal{C}(\omega'^{(k)})$ have the same cardinal, we deduce from Proposition 3.7 that

$$\sum_{f \in \mathbb{T}\mathcal{C}(\omega)} \mu(f) = \sum_{k=2}^{\gamma} |\mathbb{T}\mathcal{C}(\omega)^{(k)}| = |\mathbb{T}\mathcal{C}(\omega)|.$$

In other words, the configuration ω is maximal. □

It turns out that $(2, n)$-totally decomposing configurations are not necessarily maximal.

Definition 4.3. *Let $\omega = \{L_1, \dots, L_\gamma\}$ be a (d, n)-totally decomposing configuration of complete tropical linear spaces in $\mathbb{R}^n$, and let W_i be the wall of L_i. We say that ω is a well-ordered (d, n)-totally decomposing configuration if it satisfies one of the two following conditions*

- *$n = 2$;*
- *for any subset $\Gamma \subset \{1, \dots, \gamma\}$ the configuration $\{\pi(L_i),\ \pi(W_j)\ :\ i \in \Gamma, j \notin \Gamma\}$ is a well-ordered $(d, n-1)$-totally decomposing configuration; moreover for any i and j such that $L_i \gg L_j$, we have $\pi(L_i) \gg \pi(L_j)$ and $\pi(W_i) \gg \pi(L_j)$ if $\operatorname{codim} \pi(L_j) \geq 1$, $\pi(L_i) \gg \pi(W_j)$ and $\pi(W_i) \gg \pi(W_j)$.*

Again it is trivial that well-ordered totally decomposing configurations of hyperplanes exist, from which it follows that there exists a well ordered totally decomposing configurations $\omega = \{L_1, \dots, L_\gamma\}$ with $\operatorname{codim} L_i = l_i$ for any fixed positive integers $l_1, \dots, l_\gamma$.

Remark. Let $\{L_1, \dots, L_\gamma\}$ be a generic well-ordered totally decomposing configuration of complete tropical linear spaces, $\Gamma \subset \{1, \dots, \gamma\}$, and ω' the configuration $\{\pi(L_i),\ \pi(W_j)\ :\ i \in \Gamma, j \notin \Gamma\}$. Then, it follows directly from Definition 4.3 that given any elements $\mathcal{L}_1, \dots, \mathcal{L}_k$ of ω', the configuration obtained out of ω' by replacing $\mathcal{L}_1, \dots, \mathcal{L}_k$ by $\cap_{i=1}^k \mathcal{L}_k$ is still a well-ordered totally decomposing configuration (see Figure 9, where $L_6 \gg \dots \gg L_1$).

We will prove in Proposition 4.5 that a well-ordered $(2, n)$-totally decomposing configuration is maximal. We will treat the case of tropical conics with two floors of degree one using the total decomposition hypothesis more or less as in Lemma 4.2 (see Proposition 4.7), and we will treat the case of tropical conics with one floor of degree 2 using the well-order hypothesis.

Before proving Proposition 4.5 in its full generality, let us first illustrate in a simple example how the well-order hypothesis solves the maximality problem for the case of tropical conics with one floor of degree 2.

4.2. **Conics through 8 lines in $\mathbb{R}^3$.** Maximality of the real enumerative problem concerning conics passing through 8 spatial lines in general position in $\mathbb{R}P^3$ was first announced by Brugallé and Mikhalkin in [BM07].

Let us fix a well-ordered $(2, 3)$-totally decomposing configuration $\omega = \{L_1, \dots, L_8\}$ of 8 tropical lines in $\mathbb{R}^3$. We suppose that $L_8 \gg L_7 \gg \dots \gg L_1$, and we denote by W_i the wall of L_i. Recall that notations have been defined in section 3.3.

Lemma 4.4. *If the triple (k, A, B) is such that $\mathbb{T}\mathcal{C}(\omega)^{(k,A,B)}$ is non-empty, then it contains exactly $2^{m_{(k,A,B)}}$ tropical morphisms, all of them of multiplicity 1.*

Proof. Let us fix such a triple (k, A, B). It is easy to see that the three higher lines L_6, L_7 and L_8 are vertical constraints for elements of $\mathbb{T}\mathcal{C}(\omega)^{(k,A,B)}$ (see Figure 8). In particular, the configuration $\omega'^{(k,A,B)}$ in $\mathbb{R}^2$ contains the 3 points $w_8 = \pi(W_8)$, $w_7 = \pi(W_7)$, and $w_6 = \pi(W_6)$. Since ω is a well-ordered $(2, 3)$-totally decomposing configuration, the configuration $\omega'^{(k,A,B)}$ is $(2, 2)$-decomposing and the points w_8, w_7, and w_6 are its highest elements. Moreover, $N_{2,2}(2, 2, 2, 2, 2) = 1$ so $\omega'^{(k,A,B)}$ is maximal. Hence the set $\mathbb{T}\mathcal{C}(\omega'^{(k,A,B)})$ reduces to a unique plane tropical conic $f : C \to \mathbb{R}^2$ (passing through the three points w_8, w_7, w_6, and two other ones) which intersects the $m_{(k,A,B)} = 1$, 2 or 3 tropical lines in $\omega'^{(k,A,B)}$.

It remains to show that each of these $m_{(k,A,B)}$ tropical lines intersects the tropical conic $f(C)$ in two distinct points, which would imply the lemma by Proposition 3.9. According to what we discussed above about the configuration $\omega'^{(k,A,B)}$, the tropical conic f has two floors of degree 1 passing through the points w_8 and w_6, while the bounded elevator of f passes through w_7 (see figure 10). Since any tropical line in $\omega'^{(k,A,B)}$ is lower than w_6, it must intersect $f(C)$ along its unbounded elevators. In particular, it intersects $f(C)$ in two distinct points. □

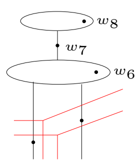


FIGURE 10. Floor decomposition of a planar conic

It is immediate that the set $\mathbb{T}\mathcal{C}(\omega)^{(k_1,k_2,A,B,C_1,C_2,D)}$ is either empty of composed of a unique tropical morphism of multiplicity 1 (see Figure 8). So the set $\mathbb{T}\mathcal{C}(\omega)$ is composed of tropical morphisms of multiplicity 1. Hence according to Lemma 2.6, there exists a generic configuration of 8 lines in $\mathbb{R}P^3$ such that exactly 92 real conics intersect these 8 lines.

4.3. **General case.** Theorem 1.1 is a direct consequence of Proposition 4.5, Theorem 2.5, and Lemma 2.6. Next proposition relies on Proposition 4.7 which is proved in next section.

Proposition 4.5. *Let ω be a well-ordered $(2,n)$-totally decomposing configuration of complete tropical linear spaces in $\mathbb{R}^n$ subject to equality (1). Then ω is maximal.*

Proof. Our goal is to prove that $\sum_{f\in\mathbb{T}\mathcal{C}(\omega)}\mu(f)=|\mathbb{T}\mathcal{C}(\omega)|$. This is obviously true when $n=2$ since the left hand side is 1. Suppose that $n\geq 3$ and the equality is true in lower dimensions. Since ω is $(2,n)$-decomposing, we have

$$\sum_{f\in\mathbb{T}\mathcal{C}(\omega)}\mu(f)=\sum_{(k,A,B)}\left(\sum_{f\in\mathbb{T}\mathcal{C}(\omega)^{(k,A,B)}}\mu(f)\right)+\sum_{(k_1,k_2,A,B,C_1,C_2,D)}\left(\sum_{f\in\mathbb{T}\mathcal{C}(\omega)^{(k_1,k_2,A,B,C_1,C_2,D)}}\mu(f)\right)$$

where the sums are taken following section 3.3.

Suppose that $\mathbb{T}\mathcal{C}(\omega)^{(k,A,B)}$ is non-empty. By induction, $\omega'^{(k,A,B)}$ is a well-ordered $(2,n-1)$-totally decomposing configuration and so is maximal. Moreover, the map $\phi_{(k,A,B)}:\mathbb{T}\mathcal{C}(\omega)^{(k,A,B)}\to\mathbb{T}\mathcal{C}(\omega'^{(k,A,B)})$ is $2^{m(k,A,B)}$ to 1, counted with multiplicities. Hence, it remains to show that any conic in $\mathbb{T}\mathcal{C}(\omega'^{(k,A,B)})$ intersects each of the $m(k,A,B)$ hyperplanes of $\omega'^{(k,A,B)}$ in 2 distinct points, which would imply, by Proposition 3.9, that

$$|\mathbb{T}\mathcal{C}(\omega)^{(k,A,B)}|=2^{m(k,A,B)}|\mathbb{T}\mathcal{C}(\omega'^{(k,A,B)})|=2^{m(k,A,B)}\sum_{f'\in\mathbb{T}\mathcal{C}(\omega'^{(k,A,B)})}\mu(f')=\sum_{f\in\mathbb{T}\mathcal{C}(\omega)^{(k,A,B)}}\mu(f).$$

We have $\omega'^{(k,A,B)}=\{\widetilde{L}'_A,\widetilde{L}'_B,\widehat{L}'_k,L'_{k+1},\ldots,L'_\gamma\}$, so a tropical hyperplane H in $\omega'^{(k,A,B)}$ is either $\widehat{L}'_k$, $\widetilde{L}'_A$ or $\widetilde{L}'_B$. Since $\omega'^{(k,A,B)}$ is a well-ordered configuration, $\widehat{L}'_k$, $\widetilde{L}'_A$ and $\widetilde{L}'_B$ are its 3 lowest elements. In particular, H is higher than at most two other elements of $\omega'^{(k,A,B)}$. The configuration $\omega'^{(k,A,B)}$ is $(2,n-1)$-decomposing, so any unbounded elevator and any floor of a conic in $\mathbb{T}\mathcal{C}(\omega'^{(k,A,B)})$ has to intersect at least one element of $\omega'^{(k,A,B)}$ which is not a hyperplane. In particular, a hyperplane in $\omega'^{(k,A,B)}$ intersects such a conic strictly below the lowest floor, that is along its two unbounded elevators in two distinct points (see figure 11).

FIGURE 11. Floor decompositions in $\mathbb{T}\mathcal{C}(\omega'^{(k,A,B)})$ and hyperplanes.

Suppose that $\mathbb{T}\mathcal{C}(\omega)^{(k_1,k_2,A,B,C_1,C_2,D)}$ is non-empty. According to section 3.3, we have

$$\begin{aligned}|\mathbb{T}\mathcal{C}_{red}(\omega^{(k_1,k_2,A,B,C_1,C_2,D)})| &= |\mathbb{T}\mathcal{C}(\omega)^{(k_1,k_2,A,B,C_1,C_2,D)}|\\ &\leq \textstyle\sum_{f\in\mathbb{T}\mathcal{C}(\omega)^{(k_1,k_2,A,B,C_1,C_2,D)}}\mu(f)\\ &\leq N_{1,n-1}(l'^1_{i_1},\ldots,l'^1_{i_{\gamma_1}},l'_A,l'_B,l'_{k_1},l'^1_0)N_{1,n-1}(l'^2_{j_1},\ldots,l'^2_{j_{\gamma_2}},l'_{k_2},l'^2_0).\end{aligned}\tag{4}$$

According to Proposition 4.7 (see next section) applied to $\omega^{(k_1,k_2,A,B,C_1,C_2,D)}=(\widetilde{L}'_D,\omega'^1,\omega'^2)$, we get that

$$|\mathbb{T}\mathcal{C}_{red}(\omega^{(k_1,k_2,A,B,C_1,C_2,D)})|=N_{1,n-1}(l'^1_{i_1},\ldots,l'^1_{i_{\gamma_1}},l'_A,l'_B,l'_{k_1},l'^1_0)N_{1,n-1}(l'^2_{j_1},\ldots,l'^2_{j_{\gamma_2}},l'_{k_2},l'^2_0).$$

Hence all inequalities in (4) are in fact equalities, and we have

$$|\mathbb{T}\mathcal{C}(\omega)^{(k_1,k_2,A,B,C_1,C_2,D)}|=\sum_{f\in\mathbb{T}\mathcal{C}(\omega)^{(k_1,k_2,A,B,C_1,C_2,D)}}\mu(f)$$

which achieves to prove the proposition. ☐

4.4. **Reducible conics through a well-ordered totally decomposing configuration.** A reducible tropical morphism $f : C \to \mathbb{R}^n$ of degree 2 has either two floors of degree 1, or a unique (reducible) floor of degree 2. The proof of Proposition 3.4 only relies on the finiteness of the set $\mathbb{T}\mathcal{C}(\omega)$ and on vectors $u_{f,e}$ for $e \in \text{Edge}^\infty(C)$. In particular Lemma 2.8 implies that Proposition 3.4 still holds for elements of $\mathbb{T}\mathcal{C}_{red}(L_0, \omega^1, \omega^2)$. In this section we compute the numbers of reducible tropical morphisms of degree 2 passing through a particular configuration of complete tropical linear spaces in $\mathbb{R}^n$.

Let $l_0 \geq 0$ and $l_1^1, \ldots, l_{\gamma_1}^1, l_1^2, \ldots, l_{\gamma_2}^2 \geq 1$ be some integers such that

$$l_0 + \sum_{i=1,2} \sum_{j=1}^{\gamma_i} (l_j^i - 1) = 3n - 2.$$

We choose L_0 a complete linear space in $\mathbb{R}^n$ of codimension l_0 and two configurations ω^1 and ω^2 of complete tropical linear spaces in $\mathbb{R}^n$ such that $\omega^i = \{L_1^i, \ldots, L_{\gamma_i}^i\}$ with $\operatorname{codim} L_j^i = l_j^i$ and $L_j^i \gg L_{j+1}^i$.

Definition 4.6. *The configuration $(L_0, \omega^1, \omega^2)$ is said to be separated if the configuration $\{L_0\} \cup \omega^1 \cup \omega^2$ is a well-ordered $(2, n)$-totally decomposing configuration, and if L_0 and elements of ω^1 are below elements of ω^2.*

Note that we do not make any assumption about the mutual position of L_0 and elements of ω^1, and that the projection to $\mathbb{R}^{n-1}$ of a separated configuration in $\mathbb{R}^n$ is still separated. Given a separated configuration $\{L_0, \omega^1, \omega^2\}$ in $\mathbb{R}^n$, we denote by $\mathbb{T}N_n^{red}(l_0, \{l_1^1, \ldots, l_{\gamma_1}^1\}, \{l_1^2, \ldots, l_{\gamma_2}^2\})$ the cardinal of $\mathbb{T}\mathcal{C}_{red}(L_0, \omega^1, \omega^2)$. We will see in Proposition 4.7 that this cardinal does not depend on L_0, ω^1, and ω^2 as long as $\{L_0, \omega^1, \omega^2\}$ is separated.

Proposition 4.7. *For any generic separated configuration $\{L_0, \omega^1, \omega^2\}$ in $\mathbb{R}^n$, we have*

$$\mathbb{T}N_n^{red}(l_0, \{l_1^1, \ldots, l_{\gamma_1}^1\}, \{l_1^2, \ldots, l_{\gamma_2}^2\}) = \prod_{i=1,2} N_{1,n} \left(2n - 1 - \sum_{j=1}^{\gamma_i} (l_j^i - 1), l_1^i, \ldots, l_{\gamma_i}^i \right).$$

Proof. The case $n = 2$ is straightforward. Let us suppose now that $n \geq 3$ and that the Proposition is true in lower dimensions.

Let $f : C_1 \cup_p C_2 \to \mathbb{R}^n$ be an element of $\mathbb{T}\mathcal{C}_{red}(L_0, \omega^1, \omega^2)$. An elevator of f has to meet at least one constraint, and the elements of ω^2 are above L_0 and the elements of ω^1, so f has two floors of degree 1; moreover the node of C is either on both elevators of C_1 and C_2, or on the floor of C_1 and the elevator of C_2 (see figure 12).

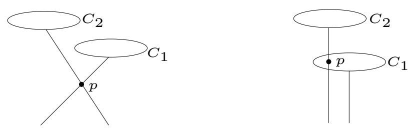


FIGURE 12.

We define k_0 as the smallest integer such that $L_{k_0}^1$ is higher than L_0 if such an element of ω^1 exists, and by $k_0 = 0$ otherwise. We denote by $\mathcal{F}_i$ the floor of C_i, $i = 1, 2$.

We consider the partition

$$\bigsqcup_{\substack{k_0 \leq k_1 \leq \gamma_1 \\ 1 \leq k_2 \leq \gamma_2}} \mathcal{C}_1^{k_1,k_2} \quad \bigsqcup_{1 \leq k_2 \leq \gamma_2} \mathcal{C}_2^{k_2} \quad \bigsqcup_{\substack{1 \leq k_1 \leq k_0 - 1 \\ 1 \leq k_2 \leq \gamma_2}} \mathcal{C}_3^{k_1,k_2}$$

of the set $\mathbb{T}\mathcal{C}_{red}(L_0, \omega^1, \omega^2)$ where

- $\mathcal{C}_1^{k_1,k_2}$ is the set of all elements $f_1 \cup_p f_2 : C_1 \cup_p C_2 \to \mathbb{R}^n$ in $\mathbb{T}\mathcal{C}_{red}(L_0, \omega^1, \omega^2)$ such that p is on both elevators of f_1 and f_2, and the floor $\mathcal{F}_i$ meets the horizontal constraint $L_{k_i}^i$; note that $\mathcal{C}_1^{k_1,k_2} = \emptyset$ if $k_0 = 0$;

- $\mathcal{C}_2^{k_2}$ is the set of all elements $f_1 \cup_p f_2 : C_1 \cup_p C_2 \to \mathbb{R}^n$ in $\mathbb{T}\mathcal{C}_{red}(L_0, \omega^1, \omega^2)$ such that p is on the floor $\mathcal{F}_1$, L_0 is the horizontal constraint of $\mathcal{F}_1$, and the floor $\mathcal{F}_2$ meets the horizontal constraint $L^2_{k_2}$; note that $\mathcal{C}_2^{k_2} = \emptyset$ if $k_0 \le 1$;
- $\mathcal{C}_3^{k_1,k_2}$ is the set of all elements $f_1 \cup_p f_2 : C_1 \cup_p C_2 \to \mathbb{R}^n$ in $\mathbb{T}\mathcal{C}_{red}(L_0, \omega^1, \omega^2)$ such that p is on the floor $\mathcal{F}_1$, and the floor $\mathcal{F}_i$ meets the horizontal constraint $L^i_{k_i}$.

We denote by W^i_j (resp. W_0) the wall of the constraint L^i_j (resp. L_0). We consider the following complete tropical linear spaces in $\mathbb{R}^{n-1}$

$$L'^i_j = \pi(W^i_j), \qquad \widehat{L}'^i_j = \pi(L^i_j), \qquad \widetilde{L}'^{k_1,k_2}_0 = \pi(L_0) \bigcap_{i=1,2} \bigcap_{1\le j\le k_i-1} \pi(L^i_j),$$

$$\widetilde{L}'^1_k = \bigcap_{1\le j\le k-1} \pi(L^1_j), \qquad \widetilde{L}'^{k_2}_0 = \pi(L_0) \bigcap_{1\le j\le k_2-1} \pi(L^2_j), \qquad \mathcal{L}'^{k_2}_0 = \pi(W_0) \bigcap_{1\le j\le k_2-1} \pi(L^2_j)$$

and the following configurations

$$\omega_1^{i,k_i} = \{\widehat{L}'^i_{k_i}, L'^i_{k_i+1}, \dots, L'^i_{\gamma_i}\} \quad \text{for } i=1,2, \quad \omega_2^1 = \{\widetilde{L}'^1_{k_0}, L'^1_{k_0}, \dots, L'^1_{\gamma_1}\}, \quad \omega_3^{1,k_1} = \{\widehat{L}'^1_{k_1}, \widetilde{L}'^1_{k_1}, L'^1_{k_1+1}, \dots, L'^1_{\gamma_1}\}.$$

Given an element f of $\mathcal{C}_1^{k_1,k_2}$, the tropical morphism $\pi\circ f$ induces an element of $\mathbb{T}\mathcal{C}_{red}(\widetilde{L}'^{k_1,k_2}_0, \omega_1^{1,k_1}, \omega_1^{2,k_2})$. Conversely any element $f'_1 \cup_{p'} f'_2 : C'_1 \cup_{p'} C'_2 \to \mathbb{R}^{n-1}$ of $\mathbb{T}\mathcal{C}_{red}(\widetilde{L}'^{k_1,k_2}_0, \omega_1^{1,k_1}, \omega_1^{2,k_2})$ has a unique lift $f_1 \cup_p f_2 : C_1 \cup_p C_2 \to \mathbb{R}^n$ in $\mathcal{C}_1^{k_1,k_2}$; the elevators of f_1 and f_2 correspond to p', and the node p is at the unique intersection point of the elevator of C_i with L_0. Hence, we get that the total number of tropical morphisms f in $\mathcal{C}_1^{k_1,k_2}$ is

$$\mathbb{T}N^{red}_{n-1}\left(l_0 - 1 + \sum_{i=1,2}\sum_{j=1}^{k_i-1}(l^i_j - 1),\ \{l^1_{k_1} - 1, l^1_{k_1+1}, \dots, l^1_{\gamma_1}\},\ \{l^2_{k_2} - 1, l^2_{k_2+1}, \dots, l^2_{\gamma_2}\}\right).$$

Given an element f of $\mathcal{C}_2^{k_2}$, the tropical morphism $\pi\circ f$ induces an element of $\mathbb{T}\mathcal{C}_{red}(\widetilde{L}'^{k_2}_0, \omega_2^1, \omega_1^{2,k_2})$. Conversely any element $f'_1 \cup_{p'} f'_2 : C'_1 \cup_{p'} C'_2 \to \mathbb{R}^{n-1}$ of $\mathbb{T}\mathcal{C}_{red}(\widetilde{L}'^{k_2}_0, \omega_2^1, \omega_1^{2,k_2})$ has a unique lift $f_1 \cup_p f_2 : C_1 \cup_p C_2 \to \mathbb{R}^n$ in $\mathcal{C}_2^{k_2}$; the elevator of f_2 corresponds to the node p', the elevator of f_1 corresponds to the unique intersection point of C'_1 and $\widetilde{L}'^1_{k_0}$, and the node p corresponds to the unique intersection point of the elevator of f_2 and L_0. Hence, we get that the total number of tropical morphisms f in $\mathcal{C}_2^{k_2}$ is

$$\mathbb{T}N^{red}_{n-1}\left(l_0 - 1 + \sum_{j=1}^{k_2-1}(l^2_j - 1),\ \{\sum_{j=1}^{k_0-1}(l^1_j - 1), l^1_{k_0}, \dots, l^1_{\gamma_1}\},\ \{l^2_{k_2} - 1, l^2_{k_2+1}, \dots, l^2_{\gamma_2}\}\right).$$

Given an element f of $\mathcal{C}_3^{k_1,k_2}$, the tropical morphism $\pi\circ f$ induces an element of $\mathbb{T}\mathcal{C}_{red}(\mathcal{L}'^{k_2}_0, \omega_3^{1,k_1}, \omega_1^{2,k_2})$. Conversely any element $f'_1 \cup_{p'} f'_2 : C'_1 \cup_{p'} C'_2 \to \mathbb{R}^{n-1}$ of $\mathbb{T}\mathcal{C}_{red}(\mathcal{L}'^{k_2}_0, \omega_3^{1,k_1}, \omega_1^{2,k_2})$ has a unique lift $f_1 \cup_p f_2 : C_1 \cup_p C_2 \to \mathbb{R}^n$ in $\mathcal{C}_3^{k_1,k_2}$. Hence, we get that the total number of tropical morphisms f in $\mathcal{C}_3^{k_1,k_2}$ is

$$\mathbb{T}N^{red}_{n-1}\left(l_0 + \sum_{j=1}^{k_2-1}(l^2_j - 1),\ \{\sum_{j=1}^{k_1-1}(l^1_j - 1), l^1_{k_1} - 1, l^1_{k_1+1}, \dots, l^1_{\gamma_1}\},\ \{l^2_{k_2} - 1, l^2_{k_2+1}, \dots, l^2_{\gamma_2}\}\right).$$

Altogether with the induction hypothesis, we get that

$$\mathbb{T}N^{red}_n(l_0, \{l^1_1, \dots, l^1_{\gamma_1}\}, \{l^2_1, \dots, l^2_{\gamma_2}\}) = AB$$

where

$$A = \sum_{k_2=1}^{\gamma_2} N_{1,n-1}(2n - 2 - \sum_{j=k_2}^{\gamma_2}(l^2_j - 1),\ l^2_{k_2} - 1, l^2_{k_2+1}, \dots, l^2_{\gamma_2})$$

and

$$\begin{aligned} B &= \textstyle\sum_{k_1=k_0}^{\gamma_1} N_{1,n-1}(2n-2-\sum_{j=k_1}^{\gamma_1}(l_j^1-1),\ l_{k_1}^1-1, l_{k_1+1}^1,\dots,l_{\gamma_1}^1) \\ &\quad +N_{1,n-1}(2n-2-\textstyle\sum_{j=1}^{\gamma_1}(l_j^1-1),\ \sum_{j=1}^{k_0-1}(l_j^1-1),\ l_{k_0}^1,\dots,l_{\gamma_1}^1) \\ &\quad +\textstyle\sum_{k_1=1}^{k_0-1} N_{1,n-1}(2n-1-\sum_{j=1}^{\gamma_1}(l_j^1-1),\ \sum_{j=1}^{k_1-1}(l_j^1-1),\ l_{k_1}^1-1, l_{k_1+1}^1,\dots,l_{\gamma_1}^1)) \end{aligned}$$

Now it follows from Corollary 3.8 that

$$A = N_{1,n}(2n-1-\sum_{j=1}^{\gamma_2}(l_j^2-1), l_1^2,\dots,l_{\gamma_2}^2)$$

and that

$$B = N_{1,n}(2n-1-\sum_{j=1}^{\gamma_1}(l_j^1-1), l_1^1,\dots,l_{\gamma_1}^1)$$

which completes the proof of the Proposition. □

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