Some aspects of tropical geometry

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The goal of this text is to provide a quick overview of tropical geometry: what it is about and why it is useful, through some examples. After having defined basic objects in tropical geometry in Sections 1 and 2, we explain their relations to classical geometry via the notion of *amoebas* of algebraic curves in Section 3 and give some applications of tropical geometry to real and enumerative geometry in Section 4. We end this text with some further remarks in Section 5.

Only a few bibliographical references are given along the text and we tried each time to refer to the most elementary texts. For the interested reader who would like to deepen parts of this note, we refer for example to the general texts [Bru09], [RGST05], [BPS08], [IM], [Vir08], [Gat06], [IMS07], [Mik06] and references therein.

1 Tropical algebra

The first point of view on tropical geometry given here will be algebraic geometry built upon the *tropical semi-field*.

Tropical semi-field

The set of tropical numbers is defined by $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$, which we equip with the following tropical addition and multiplication (written within quotation marks):

$$\text{“}x + y\text{”} = \max(x, y) \qquad \text{“}x \times y\text{”} = x + y.$$

For example, we have the following identities:

$$\text{“}1 + 1\text{”} = 1, \qquad \text{“}1 + 2\text{”} = 2, \qquad \text{“}1 \times 2\text{”} = 3,$$
$$\text{“}1 \times (-2)\text{”} = -1, \qquad \text{“}(5 + 3)^2\text{”} = 10.$$

It follows immediately from the definition that tropical addition is *idempotent*, i.e. “$x + x$” $= x$ for any tropical number x. As a consequence, a tropical number x does not have an inverse for tropical addition except if $x = -\infty$ (the neutral element for “ + ”). However, this lack of additive inverse is the only axiom of a field which is not fulfilled by $(\mathbb{T}, \text{“}+\text{”}, \text{“}\times\text{”})$. This is precisely the definition of a semi-field. Note that any $x \neq -\infty$ has an inverse for tropical multiplication and that “x^{-1}” $= -x$.

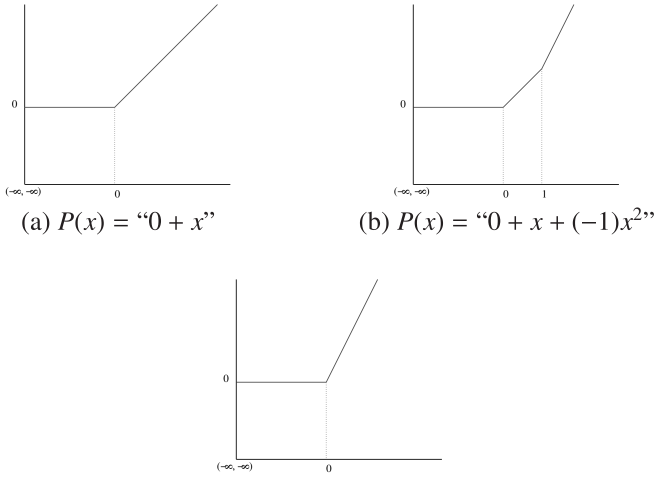
Tropical polynomials

As soon as addition and multiplication are defined, polynomials show up. By definition, a tropical polynomial function is of the form

$$P(x) = \text{“}\sum_{i=0}^{d} a_i x^i\text{”} = \max_i(a_i + ix), \text{ where } a_0, \ldots, a_d, x \in \mathbb{T}.$$

What is a tropical root of a tropical polynomial? Here we face a recurring problem in tropical mathematics: several equivalent definitions in classical mathematics may produce different tropical objects.

If we look at tropical numbers x_0 such that $P(x_0) = -\infty$, the answer doesn't bring so much information about P: since $P(x) \geq a_0$ for any $x \in \mathbb{T}$, there does not exist such a x_0 except in the case where $a_0 = -\infty$ and $x_0 = -\infty$ ...



(a) $P(x) =$ “$0 + x$”

(b) $P(x) =$ “$0 + x + (-1)x^2$”

(c) $P(x) =$ “$0 + x^2$”

Figure 1. Graph of tropical polynomials

To bring out a meaningful notion of a tropical root, let us look instead at the graph of a tropical polynomial function (see Figure 1). A tropical polynomial $P(x)$ is a piecewise affine function and its graph has some *corner points* that will be our tropical roots. They are exactly the tropical numbers where $P(x)$ is not locally given by a monomial.

More precisely, we say that $x_0 \in \mathbb{T}$ is a tropical root of order k of $P(x) =$ “$\sum_{i=0}^{d} a_i x^i$” if there exist two indices i and j such that in a neighbourhood of x_0 in $\mathbb{T}$ we have $P(x) =$ “$a_i x^i + a_j x^j$” and $k = |i - j|$ is minimal. The next proposition shows that this notion of tropical root is equivalent to a more classical definition.

Proposition 1.1. *The tropical semi-field $\mathbb{T}$ is algebraically closed. Moreover, x_0 is a tropical root of order k of a tropical polynomial $P(x)$ if and only if there exists a tropical polynomial $Q(x)$ which does not have x_0 as a tropical root and such that*

$$P(x) = \text{“}(x + x_0)^k Q(x)\text{”} \quad \forall x \in \mathbb{T}.$$

For example we have the following factorisations (see Figure 1):

“$0+x+(-1)x^2$” = “$(-1)(x+0)(x+1)$” and “$0+x^2$” = “$(x+0)^2$”.

2 Tropical curves

Definition

Let us now turn to tropical polynomials in two variables and the tropical curves they define. Since it makes all definitions simpler, we restrict ourselves to tropical curves in $\mathbb{R}^2$ instead of $\mathbb{T}^2$.

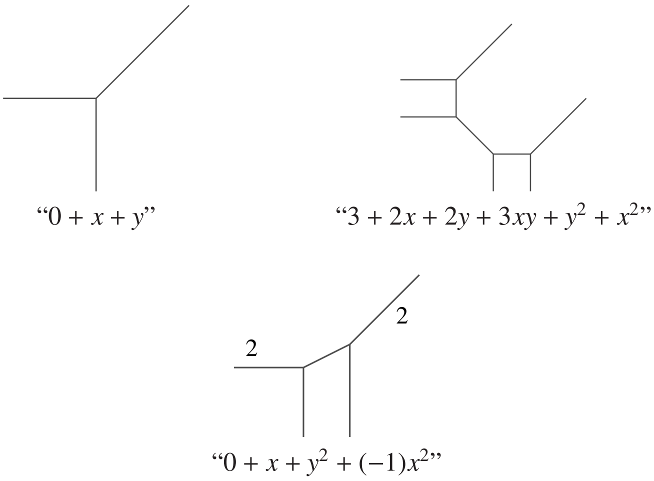


Figure 2. A tropical line and two tropical conics

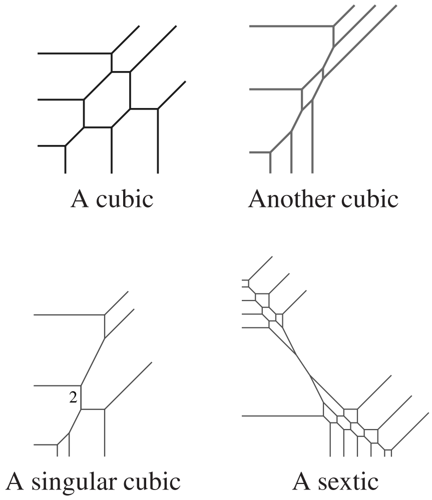


Figure 3. Tropical cubics and sextic

Similarly to the case of univariate polynomials, let us define the *corner set* $V(P) \subset \mathbb{R}^2$ of a tropical polynomial $P(x, y) =$ "$\sum a_{i,j}x^i y^j$" as the set where $P(x, y)$ is not locally given by a monomial. That is to say

$$V(P) = \{(x_0, y_0) \in \mathbb{R}^2 \mid \exists (i, j) \neq (k, l),$$
$$P(x_0, y_0) = \text{“}a_{i,j}x_0^i y_0^j\text{”} = \text{“}a_{k,l}x_0^k y_0^l\text{”}\}.$$

Since $P(x, y)$ is a piecewise affine function, the set $V(P)$ is a piecewise linear graph in $\mathbb{R}^2$ (from now on the word "graph" has to be understood in its graph theoretical sense). Any edge e of $V(P)$ is adjacent to exactly two connected components E_1 and E_2 of $\mathbb{R}^2 \setminus V(P)$. Let us say that the value of $P(x, y)$ is given by the monomial "$a_{i,j}x^i y^j$" on E_1 and by "$a_{k,l}x^k y^l$" on E_2. We define the *weight* of e as

$$w(e) = \gcd(|i - k|, |j - l|).$$

The weight of an edge might be seen as the 2-dimensional analogue of the order of a tropical root. The *tropical curve C defined by $P(x, y)$* is the set $V(P)$ enhanced with this weight function on its edges. In the examples of tropical curves depicted in Figures 2 and 3, we specify the weight of an edge only when it is at least 2.

Balancing condition

There exists an equivalent definition of a plane tropical curve that one can formulate into combinatorial terms. Let Γ be a piecewise linear graph in $\mathbb{R}^2$ equipped with a weight function $w : \mathrm{Edge}(\Gamma) \to \mathbb{Z}_{>0}$ and whose edges admit an integral direction. Given an edge e of Γ adjacent to a vertex v, we may choose $\vec{u}_{v,e}$ the smallest (i.e. primitive) direction in $\mathbb{Z}^2$ of e pointing away from v. The graph Γ is called *balanced* if it satisfies the following *balancing condition* at any vertex v:

$$\sum_{e \text{ adjacent to } v} w(e)\vec{u}_{v,e} = 0.$$

Proposition 2.1 (Mikhalkin)**.** *Tropical curves in $\mathbb{R}^2$ correspond exactly to balanced graphs in $\mathbb{R}^2$.*

For example, the three primitive integral directions in the case of the vertex of a tropical line are $(-1, 0)$, $(0, -1)$ and $(1, 1)$, whose sum is indeed 0.

Bézout's Theorem

Tropical curves share many properties with plane complex algebraic curves, i.e. subsets of $\mathbb{C}^2$ with equation $P(z, w) = 0$ where $P(z, w)$ is a polynomial with complex coefficients. For example, both classes of objects satisfy Bézout's Theorem, the genus formula, etc. Here we focus on the former.

Classically, Bézout's Theorem states that two plane complex algebraic curves of degree d and d' in general position have exactly dd' intersection points.

To prove an analogous statement for tropical curves, we first have to introduce the *multiplicity* of a tropical intersection point. Let C and C' be two tropical curves such that the set $C \cap C'$ does not contain any vertex of C or C'. Hence, a point $p \in C \cap C'$ lies on an edge e of C of weight w and on an edge e' of C' of weight w' (see Figure 4a). Let $\vec{u}_e$ be a primitive integral direction of e and $\vec{u}_{e'}$ be a primitive integral direction of e'. The multiplicity $m(p)$ of p is defined as the Euclidean area of the parallelogram spanned by the two vectors $w\vec{u}_e$ and $w'\vec{u}_{e'}$, i.e.

$$m(p) = ww' |\det(\vec{u}_e, \vec{u}_{e'})|.$$

The following tropical version of Bézout's Theorem has a very elementary proof which requires only very basic mathematical knowledge.

Proposition 2.2 (Sturmfels)**.** *If C and C' are two tropical curves of degree d and d' in general position then $\sum_{p \in C \cap C'} m(p) = dd'$.*

For example, the tropical line and conic in Figure 4b have two intersection points, both of multiplicity 1, whereas the ones in Figure 4c have only one intersection point, of multiplicity 2.

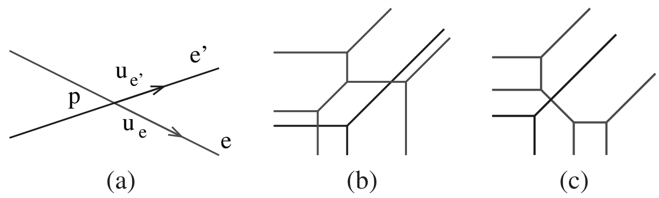


Figure 4. Tropical intersection

3 Link to classical geometry

The similarity between tropical and complex curves mentioned in the previous section is much more than an amusing coincidence. Tropical geometry has very deep connections and relations with classical algebraic geometry.

Maslov dequantization

Let us start by relating the tropical semi-field with a classical semi-field that we all know quite well: $(\mathbb{R}_{\geq 0}, +, \times)$. This process, studied by Maslov and his collaborators since the 90s, is known as the *Maslov dequantization of real numbers*.

Given a positive real number t, the bijection

$$\log_t = \frac{\log}{\log t} : \mathbb{R}_{\geq 0} \to \mathbb{T} = [-\infty; +\infty)$$

induces a semi-field structure on $\mathbb{T}$, where the two operations denoted by "$+_t$" and "$\times_t$" are given by

$$\text{“}x +_t y\text{”} = \log_t(t^x + t^y) \quad \text{and} \quad \text{“}x \times_t y\text{”} = \log_t(t^x t^y) = x + y.$$

We already have the appearance of classical addition as multiplication "$\times_t$" on $\mathbb{T}$. By construction, all the semi-fields $(\mathbb{T}, \text{“}+_t\text{”}, \text{“}\times_t\text{”})$ are isomorphic to $(\mathbb{R}_+, +, \times)$. Moreover, we have the following simple inequalities:

$$\forall t > 0, \ \ \max(x, y) \leq \text{“}x +_t y\text{”} \leq \max(x, y) + \log_t 2.$$

In particular, when t tends to infinity, the law "$+_t$" converges to tropical addition! Hence, the tropical semi-field arises naturally as a degeneration of the classical semi-field $(\mathbb{R}_+, +, \times)$. From an alternative perspective, one can see the classical semi-field $(\mathbb{R}_+, +, \times)$ as a deformation of the tropical semi-field, which justifies the "dequantization" terminology.

Amoebas

This dequantization process also applies to plane complex curves. For this, we need the following map

$$\begin{array}{rccc} \mathrm{Log}_t : & (\mathbb{C}^*)^2 & \longrightarrow & \mathbb{R}^2 \\ & (z, w) & \longmapsto & (\log_t |z|, \log_t |w|). \end{array}$$

Given an algebraic curve in $(\mathbb{C}^*)^2$, its image under the map Log_t is called its *amoeba* (in base t). Let us look more closely at these amoebas with the help of a concrete example, namely the line L with equation $z + w + 1 = 0$ in $(\mathbb{C}^*)^2$. One can compute by hand that the amoeba of L is as depicted in Figure 5a. In particular, we see that it has three asymptotic directions: $(-1, 0)$, $(0, -1)$, and $(1, 1)$.

By the definition of $\log_t$, the amoeba of L in base t is a contraction by a factor $\log t$ of the amoeba of L in base e (see Figures 5b and 5c). Hence when t goes to $+\infty$, the whole amoeba is contracted to the origin and only the three asymptotic directions remain. In other words, what we see at the limit in Figure 5d is a tropical line!

Of course, the same strategy applied to any classical curve will produce a similar picture at the limit: the origin from which the asymptotic directions of the amoeba emerge. To get a more interesting limit, one should look not at amoebas in base t of a single complex curve but at the family of amoebas $(\mathrm{Log}_t(C_t))_{t>0}$, where $(C_t)_{t>0}$ is now *a family* of complex curves. If we do so then the limit becomes much richer. For example, we depict in Figure 6 the shape of the amoeba of the curve with equation $1 - z - w + t^2z^2 - tzw + t^2y^2 = 0$ for t large enough and its limit, which is ... a tropical conic.

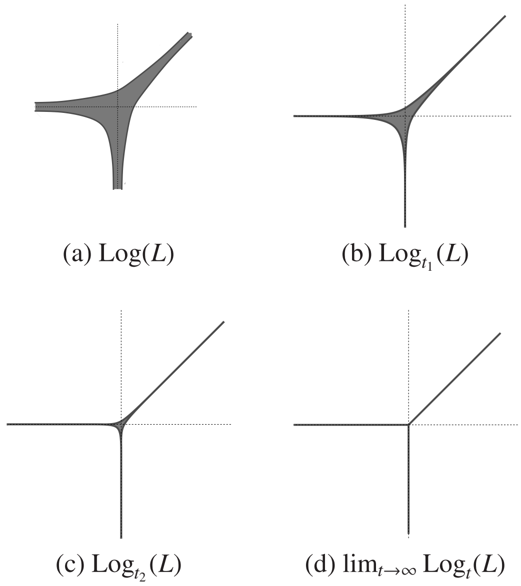
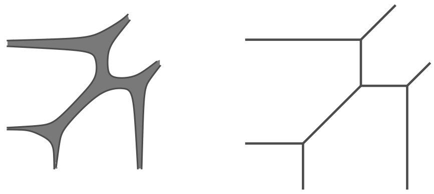


Figure 5. Dequantization of a line ($e < t_1 < t_2$)



$\mathrm{Log}_t(C_t)$ $\qquad$ $\lim_{t\to\infty} \mathrm{Log}_t(C_t)$

Figure 6. $C_t : 1 - z - w + t^2z^2 - tzw + t^2y^2 = 0$

Theorem 3.1 (Mikhalkin, Rullgård)**.** *Let $P_t(z, w) = \sum_{i,j} \alpha_{i,j}(t)$ z^iw^j be a polynomial whose coefficients are functions $\alpha_{i,j} : \mathbb{R} \to \mathbb{C}$ and suppose that $\alpha_{i,j}(t) \sim \gamma_{i,j}t^{a_{i,j}}$ when t goes to $+\infty$ with $\gamma_{i,j} \in \mathbb{C}^*$. If C_t denotes the curve in $(\mathbb{C}^*)^2$ defined by the polynomial $P_t(z, w)$ then the amoeba $\mathrm{Log}_t(C_t)$ converges to the tropical curve defined by the tropical polynomial $P_{trop}(x, y) = \text{“}\sum_{i,j} a_{i,j}x^iy^j\text{”}$.*

It remains for us to explain the relation between amoebas and weights of a tropical curve. Let $P_t(z, w)$ and $P'_t(z, w)$ be two families of complex polynomials, defining two families of complex algebraic curves $(C_t)_{t>0}$ and $(C'_t)_{t>0}$ respectively. As in Theorem 3.1, these two families of polynomials induce two tropical polynomials $P_{trop}(x, y)$ and $P'_{trop}(x, y)$, which in turn define two tropical curves C and C'.

Proposition 3.2 (Mikhalkin)**.** *Let $p \in C \cap C'$ which is a vertex of neither C nor C'. Then the number of intersection points of C_t and C'_t whose image under Log_t converges to p is exactly equal to $m(p)$.*

It is worth remarking that the number of intersection points which converge to *p only depends on C and C'*, that is to say only on the order at infinity of the coefficients of $P_t(z, w)$ and $P'_t(z, w)$.

A combination of Theorem 3.1, Proposition 3.2 and the tropical version of Bézout's Theorem provides a proof of Bézout's Theorem for complex curves. In the next section, we explore two deeper applications of tropical geometry.

4 Examples of application

Combinatorial construction of real algebraic curves

Given a real polynomial $P(z, w)$, it is usually very difficult to compute the "picture" realised by the real algebraic curve with equation $P(z, w) = 0$ in $\mathbb{R}^2$. More generally, the problem of classifying all possible mutual arrangements of the connected components of a real algebraic curve of a fixed degree is a beautiful but extremely difficult question. This is the 16th problem that Hilbert posed in his famous list, and is still widely open. Up till now the complete answer is known only up to degree 7, and the method described below was one of the tools thanks to which Viro completed this classification.

Tropical geometry produces real algebraic curves whose arrangement of connected components can be recovered thanks to some elementary combinatorial rules. Despite its simplicity, this method produces real algebraic curves with very rich topology. For example, Itenberg disproved drastically in this way Ragsdale's conjecture posed one century ago (see [IV96])!

We now describe this method, known as *combinatorial patchworking*. Let $P_t(z, w) = \sum_{i,j} \alpha_{i,j}(t) z^i w^j$ be a polynomial whose coefficients are *real* valued functions $\alpha_{i,j} : \mathbb{R} \to \mathbb{R}$. As in Theorem 3.1, we suppose that $\alpha_{i,j}(t) \sim \gamma_{i,j} t^{a_{i,j}}$ at infinity, and we denote by C the tropical curve defined by $P_{trop}(x, y)$. For simplicity, we assume the following technical condition: all weights of C are equal to 1 and, given any vertex v of C and any two edges e and e' adjacent to v, the Euclidean area of the triangle spanned by $\vec{u}_{v,e}$ and $\vec{u}_{v,e'}$ is the minimum possible, i.e. $|\det(\vec{u}_{v,e}, \vec{u}_{v,e'})| = 1$. Such a tropical curve is said to be *non-singular*. This condition ensures that the complex algebraic curve defined by $P_t(z, w)$ in $(\mathbb{C}^*)^2$ is non-singular when t is large enough. All the tropical curves depicted in Figures 2 and 3 except the ones of Figures 2c and 3c are non-singular.

We first recover the real algebraic curve defined by $P_t(z, w)$ in the quadrant $(\mathbb{R}_{>0})^2$. Given an edge e of C adjacent to two connected components E_1 and E_2 of $\mathbb{R}^2 \setminus C$, we may assume that the value of $P_{trop}(x, y)$ is given by the monomial "$a_{i,j} x^i y^j$" on E_1 and by "$a_{k,l} x^k y^l$" on E_2. Let us erase e if the signs of $\gamma_{i,j}$ and $\gamma_{k,l}$ coincide.

We denote by $\mathbb{R}C$ the piecewise linear curve obtained in $\mathbb{R}^2$ after performing this operation to all edges of C. For example, starting with the tropical curve depicted in Figure 7a, we obtain the curve depicted in Figure 7b by choosing appropriate signs for the coefficients $\gamma_{i,j}$.

Theorem 4.1 (Viro). *For t large enough, the real algebraic curve defined by $P_t(z, w)$ in $(\mathbb{R}_{>0})^2$ is isotopic to $\mathbb{R}C$.*

In other words, the mutual arrangement of the connected components of the real algebraic curve defined by $P_t(z, w)$ in the positive quadrant of $\mathbb{R}^2$ is given by $\mathbb{R}C$. By symmetry, one can of course deduce the real curve defined by $P_t(z, w)$ in the whole plane $\mathbb{R}^2$. For example, the signs we chose to obtain Figure 7b actually produce the real tropical curve depicted in Figure 7c, which attests the existence of a real algebraic sextic in $\mathbb{R}^2$ arranged as in Figure 7d. Such a curve was first constructed by Gudkov in the late 60s. An interesting piece of trivia is that Hilbert claimed in 1900 that such a curve could not exist!

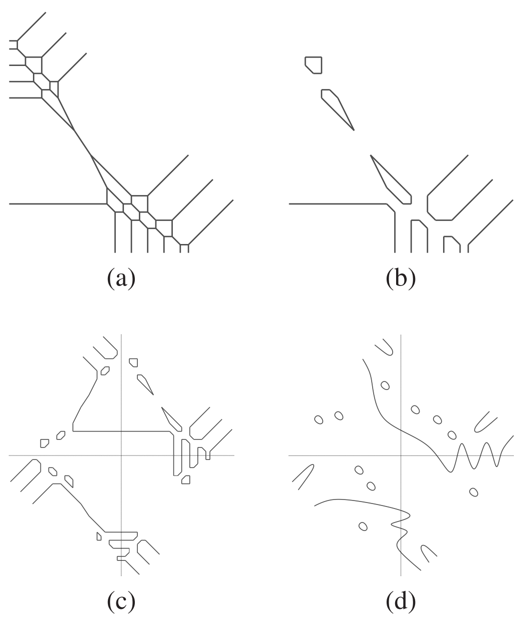


Figure 7. Gudkov curve

Enumerative geometry

Tropical geometry has also turned out to be very fruitful in enumerative geometry, the art of counting curves.

A double point of a complex algebraic curve C is a point where two branches of C intersect. The typical example of such a double point is the intersection point of two lines, and an irreducible complex algebraic curve of degree d in $\mathbb{C}^2$ has at most $\frac{(d-1)(d-2)}{2}$ double points. Given two integers $d \geq 1$ and $0 \leq r \leq \frac{(d-1)(d-2)}{2}$, a simple example of an enumerative problem is the following: how many irreducible complex algebraic curves of degree d with r double points pass through a generic configuration of $\frac{d(d+3)}{2} - r$ points?

Note that this number does not depend on the choice of the generic configuration of points (like the number of roots of a complex polynomial only depends on its degree and not on its coefficients). It is known as a *Gromov-Witten* invariant of the plane and we denote it by $N(d, r)$.

For example, $N(1, 0) = 1$ since there is a unique line passing through 2 points and, more generally, $N(d, 0) = 1$ since the solution curve is given by an invertible system of linear equations on its coefficients. The first non-trivial value is $N(3, 1) = 12$, i.e. there exist 12 cubic curves with one double point passing through 8 points.

There exist several methods to compute those numbers and one of them, suggested by Kontsevich, is via tropical geometry. Indeed, one can reformulate this classical enumerative problem into tropical terms and the answer turns out to be the same as in complex geometry. This is a deep and beautiful theorem by Mikhalkin ([Mik05]).

Theorem 4.2 (Mikhalkin)**.** *The number of irreducible tropical curves, counted with multiplicity, of degree d with r double points passing through a generic configuration of $\frac{d(d+3)}{2} - r$ points in $\mathbb{R}^2$ is equal to the corresponding number of complex curves.*

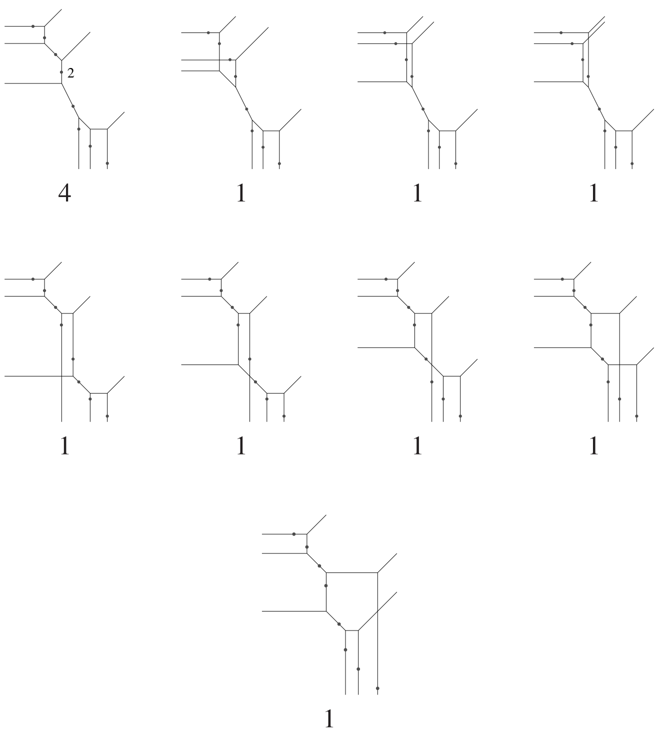


Figure 8. $N(3, 1) = 12$

We do not have space here to define a double point of a tropical curve or to specify the multiplicity of a tropical curve (see instead [Mik05] or [BPS08]). Let us just say that those definitions are completely combinatorial. As an example, we depict in Figure 8 all tropical cubics with one node passing through 8 points and we indicate in each case the multiplicity of the tropical curve. Summing up all those multiplicities, we find again $N(3, 1) = 12$.

It is worth mentioning that tropical geometry is also a very powerful tool in *real enumerative geometry*. Counting real algebraic curves is much more delicate than counting complex algebraic curves. One of the main advantages of tropical geometry is that it solves an enumerating problem by *exhibiting* all the solutions. In particular it allows one to count complex *and* real curves at the same time. For example, tropical geometry is a very useful tool to compute *Welschinger invariants*, a real analogue of Gromov-Witten invariants when r is maximum. For example, we can deduce from those Welschinger invariants and Figure 8 that there *always* exist *at least* eight real cubics with one double point passing through eight points in the real plane.

Theorem 4.2 has been the starting point of many developments in tropical enumerative geometry and its applications. For example, the problem of computing Gromov-Witten and Welschinger invariants can be turned into a purely combinatorial problem via the so-called *floor diagrams* (see [BPS08]).

5 Further in tropical geometry

Tropical varieties of higher (co)dimension

So far we only considered plane tropical curves. What about tropical varieties of higher dimension and codimension?

We have seen three equivalent definitions of a tropical curve:

(1) an algebraic one via tropical polynomials;
(2) a combinatorial one via balanced graphs;
(3) a geometric one via limits of amoebas.

All these three definitions can be generalised to arbitrary dimension. If they remain equivalent for tropical hypersurfaces of $\mathbb{R}^n$ (see [IMS07]), they produce different objects in higher codimension. For example, it is known in classical geometry that any cubic curve of genus 1 in $\mathbb{C}P^3$ is contained in a plane. However, there exist tropical cubics of genus 1 in the 3-space which are not contained in any tropical hyperplane (see [Mik05]). Such a pathological tropical curve cannot be a limit of amoebas of any family of spatial complex cubic curves.

The problem of determining which balanced polyhedral complexes are limits of amoebas is very important in tropical geometry and is still widely open.

Tropical projective spaces

The logarithm transforms multiplications to additions. As a consequence, any operation performed in complex algebraic geometry using only monomial maps translates mutatis mutandis in the tropical setting. In other words, tropical toric varieties can be constructed exactly as in complex geometry. Let us illustrate this with a classical construction: projective spaces.

The projective line $\mathbb{C}P^1$ may be obtained by taking two copies of $\mathbb{C}$, with coordinates z_1 and z_2, and gluing them along $\mathbb{C}^*$ via the identification $z_2 = z_1^{-1}$. Similarly, the projective plane $\mathbb{C}P^2$ can be constructed by taking three copies of $\mathbb{C}^2$, with coordinates (z_1, w_1), (z_2, w_2) and (z_3, w_3), and gluing them along $(\mathbb{C}^*)^2$ via the identifications

$$(z_2, w_2) = (z_1^{-1}, w_1) \text{ and } (z_3, w_3) = (z_1, w_1^{-1}).$$

Since "x^{-1}" $= -x$, the above constructions also yield the projective tropical line $\mathbb{T}P^1$ and plane $\mathbb{T}P^2$. In particular, we see that $\mathbb{T}P^1$ is a segment (Figure 9a) and $\mathbb{T}P^2$ is a triangle (Figure 9b). More generally, the projective space $\mathbb{T}P^n$ is a simplex of dimension n, each of its faces corresponding to a coordinate hyperplane.

For example, the tropical 3-space $\mathbb{T}P^3$ is a tetrahedron (see Figure 9c). Note that tropical projective spaces carry much more than just a topological structure: since all gluing maps are classical linear maps with integer coefficients, each open face of dimension p can be identified to $\mathbb{R}^p$ together with the lattice $\mathbb{Z}^p$ inside.

As usual, the space $\mathbb{R}^2 = (\mathbb{T}^*)^2$ embeds naturally into $\mathbb{T}P^2$ and any tropical curve in $\mathbb{R}^2$ has a closure in $\mathbb{T}P^2$. For example, we depict in Figure 9d the closure in $\mathbb{T}P^2$ of a tropical line in $\mathbb{R}^2$.

Tropical modifications

A new feature of tropical geometry comes out at that point: the polymorphism of tropical objects.

Over the complex numbers, there is no difference between an abstract projective line and a line in $\mathbb{C}P^2$; they are isomorphic. However, the tropical version of these two different situations produces two different objects: a segment in Figure 9a and a tripod in Figure 9d. It seems that we constructed two different tropical projective lines . . .

What does it mean? What is the relation between these two tropical manifestations of the same projective line? For-

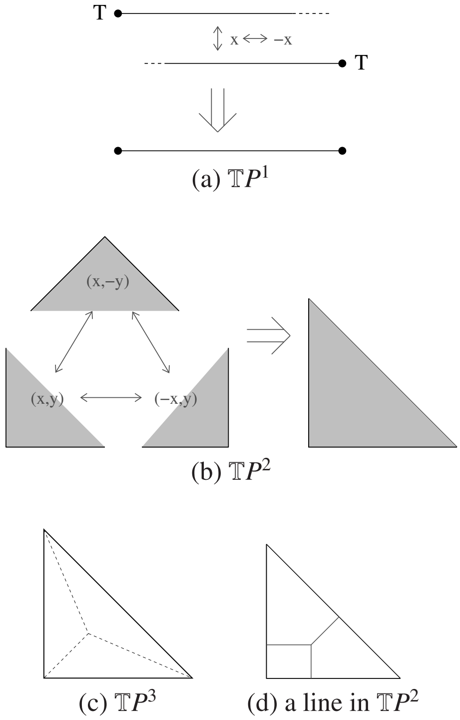


Figure 9. Tropical projective spaces

getting the second coordinate in $\mathbb{T}P^2$, the tropical line is projected to the horizontal side of the triangle. However, this side is nothing else but … the $\mathbb{T}P^1$ of Figure 9a (see Figure 10a). Hence, even if we constructed two different $\mathbb{T}P^1$, they are related by this projection of the tripod to the segment which contracts an edge of the tripod.

This phenomenon is not specific to the one dimensional case. For example, we depict in Figure 10b the tropical projective plane Π obtained by taking the closure in $\mathbb{T}P^3$ of a tropical plane in $\mathbb{R}^3$. Forgetting the third coordinate induces a projection from Π to the triangle $\mathbb{T}P^2$, which contracts 3 of the 6 faces of Π to a tropical line (see Figure 10c).

These two projections are examples of the so-called *tropical modifications*. Any tropical variety has infinitely many models as polyhedral complexes, and all these models are related by a sequence of tropical modifications. So the two tropical projective lines or planes that we constructed above are not two different $\mathbb{T}P^1$ or $\mathbb{T}P^2$ but two different *representatives* of $\mathbb{T}P^1$ and $\mathbb{T}P^2$.

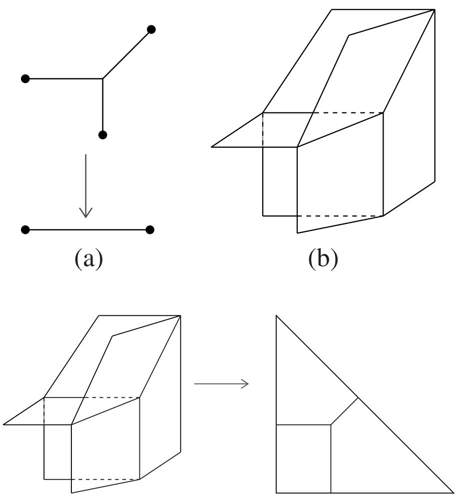


Figure 10. Tropical modifications

What is the significance of those infinitely many tropical representatives of the same variety? Given a family of complex algebraic varieties $X = (X_t)_{t>0}$, there is no canonical way of associating a tropical variety to X. In other words, X has no canonical *tropicalization*. All embeddings of (open subsets of) X as a family in $(\mathbb{C}^*)^n$ will produce as many different balanced polyhedral complexes in $\mathbb{R}^n$. However, all those tropical representatives are related by tropical modifications and the consideration of all those tropical modifications (or rather the inverse limit) is *the* tropicalization of X. This is a topological space, which is homeomorphic to another construction in algebraic geometry: the analytification in the Berkovich sense of X (see [Pay09]).

Hence, given some problem on X, one step in the tropical approach is to choose the simplest tropical representative of the tropicalization of X for which tropical geometry can actually help.

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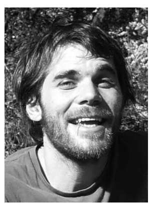

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