REAL PLANE ALGEBRAIC CURVES WITH ASYMPTOTICALLY MAXIMAL NUMBER OF EVEN OVALS

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Abstract

It has been known for a long time that a nonsingular real algebraic curve of degree $2k$ in the projective plane cannot have more than $\frac{7k^2}{4} - \frac{9k}{4} + \frac{3}{2}$ even ovals. We show here that this upper bound is asymptotically sharp, that is to say we construct a family of curves of degree $2k$ such that $\frac{p}{k^2} \to_{k\to\infty} \frac{7}{4}$, where p is the number of even ovals of the curves. We also show that the same kind of result is valid dealing with odd ovals.

1 Introduction

The set $\mathbb{R}A$ of the real points of a nonsingular real algebraic curve in $\mathbb{R}P^2$ is a disjoint union of circles. Each of these circles either disconnects $\mathbb{R}P^2$ or not. In the former case, such a circle is called *an oval* of $\mathbb{R}A$. The part of $\mathbb{R}P^2$ cut along this oval which is homeomorphic to a disk is called *the interior* of the oval. An oval of a curve which contains no other ovals of the curve is called *an empty oval.* All the connected components of a curve of even degree are ovals. For such a curve, the ovals which are contained in an even (resp. odd) number of ovals are called *even* (resp. *odd*) ovals. Given a real plane algebraic curve of degree $2k$, its number of even (resp. odd) ovals will be denoted by p (resp. n). This separation of ovals in two groups is important for many reasons. One of them is the fact that curves with many even ovals can be used to construct real algebraic surfaces with big Betti numbers (see section 6).

What are the maximal possible values for p and n with respect to k? The first step in the study of this problem is due to V. Ragsdale. In 1906, she conjectured in [Rag06] that $p \leq \frac{3k(k-1)}{2} + 1$ and $n \leq \frac{3k(k-1)}{2}$. About 30 years later, I. G. Petrovsky proved in [Pet33] that $p - n \leq \frac{3k(k-1)}{2} + 1$ and $n - p \leq \frac{3k(k-1)}{2}$ (these inequalities were also conjectured by Ragsdale), and formulated a conjecture similar to Ragsdale's one (it seems clear that Petrovsky was not familiar with Ragsdale's work). Combining the first Petrovsky inequality with the Harnack Theorem (which gives an upper bound for the number of connected components of a real plane algebraic curve with respect to its degree) one can obtain the following upper bounds for p and n :

$$p \leq \frac{7k^2}{4} - \frac{9k}{4} + \frac{3}{2} \text{ and } n \leq \frac{7k^2}{4} - \frac{9k}{4} + 1.$$

The first counterexamples to Ragsdale's conjecture for n (but not to Petrovsky's one) were constructed by O. Ya. Viro in the late 70's (see [Vir89]). In 1993, I. Itenberg gave in [Ite93] counterexamples to Ragsdale's and Petrovsky's conjectures. He has constructed for every positive integer k curves of degree $2k$ with $\frac{13k^2}{8} + O(k)$ even ovals and curves of degree $2k$ with $\frac{13k^2}{8} + O(k)$ odd ovals. These lower bounds were successively improved by B. Haas (see [Haa95]) and Itenberg (see [Ite01]). The best lower bound known before the present paper was $\frac{81k^2}{48} + O(k)$ for both p and n. We point out the fact that no counterexamples of Ragsdale's conjectures is known among curves with the maximal number of connected components.

All these constructions are based on a particular case of the so called Viro method (see [Vir84], [Vir]), the combinatorial patchworking. One can note that dealing with non convex triangulations (and so with pseudo-holomorphic curves, see [IS02]), F. Santos ([San]) constructed curves with $\frac{17k^2}{10}+O(k^{\frac{3}{2}})$ even ovals.
It seemed to us that the T-construction is more or less "rigid" and that the general Viro method gives one more flexibility and possibilities to construct real algebraic curves. Then, we resumed the work of Itenberg and Santos in this way, trying to increase the density of even ovals. It turned out that gluing curves whose Newton polygon is not anymore a triangle but a hexagon, it was possible to prove that the upper bounds given by the Harnack theorem and the Petrovsky inequalities are asymptotically sharp.

This is the main result of this article.

Theorem 1.1 *There exists a family $(C_{2k})_{k\geq 0}$ of nonsingular real algebraic curves of degree $2k$ in $\mathbb{R}P^2$ such that*

$$\lim_{k\to\infty}\frac{p}{k^2}=\frac{7}{4}.$$

There exists a family $(C_{2k})_{k\geq 0}$ of nonsingular real algebraic curves of degree $2k$ in $\mathbb{R}P^2$ such that

$$\lim_{k\to\infty}\frac{n}{k^2}=\frac{7}{4}.$$

Proof. The assertion relative to p is a direct consequence of corollary 3.5. The assertion relative to n can be proved choosing the Viro polynomial in section 3 positive along the axis $\{Y=0\}$ and in all empty ovals (hence, the empty ovals of C_k^n are odd). □

We point out that since these families of curves are asymptotically maximal for the sum of Harnack's and Petrovsky's inequalities, they are asymptotically maximal for both of them. In particular, the curves we construct are asymptotically maximal curves.

This article is organized as follows : in section 2, we recall some facts about rational geometrically ruled surfaces. In section 3, we prove the first part of Theorem 1.1. The constructions in this section are based on the classical Viro method. We assume in this section the existence of some special curves in rational geometrically ruled surfaces. The construction of the latter curves are based on the less classical *real rational graphs* theoretical method. Section 4 is devoted to the definition and properties of such graphs, and in section 5, we construct the special curves used in section 3. In section 6, we give some applications of Theorem 1.1 to real algebraic surfaces.

Acknowledgment. I am grateful to Ilia Itenberg for useful discussions and advisements and to Olivier Le Gal for his art in hexagons counting.

2 Rational geometrically ruled surfaces

The n^{th} rational geometrically ruled surface, denoted by Σ_n, is the surface obtained by taking four copies of $\mathbb{C}^2$ with coordinates (x,y), (x_2,y_2), (x_3,y_3) and (x_4,y_4), and by gluing them along $(\mathbb{C}^*)^2$ with the identifications $(x_2,y_2)=(1/x,y/x^n)$, $(x_3,y_3)=(x,1/y)$ and $(x_4,y_4)=(1/x,x^n/y)$. Let us denote by E (resp. B and F) the algebraic curve in Σ_n defined by the equation $\{y_3=0\}$ (resp. $\{y=0\}$ and $\{x=0\}$). The coordinate system (x,y) is called *standard.* The projection $\pi:(x,y)\mapsto x$ on Σ_n defines a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$. We have $B\circ B=n$, $F\circ F=0$ and $B\circ F=1$. The surface Σ_n has a natural real structure induced by the complex conjugation in $\mathbb{C}^2$, and the real part of Σ_n is a torus if n is even and a Klein bottle if n is odd. The restriction of π on $\mathbb{R}\Sigma_n$ defines a pencil of lines denoted by $\mathcal{L}$.

The group $H_2(\Sigma_n, \mathbb{Z})$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and is generated by the classes of B and F. Moreover, one has $E = B - nF$. An algebraic curve on Σ_n is said to be of *bidegree* (k, l) if it realizes the homology class $kB + lF$ in $H_2(\Sigma_n, \mathbb{Z})$. A curve of bidegree $(3, 0)$ is called *a trigonal curve* on Σ_n.

In the rational geometrically ruled surfaces, we study real curves up to *isotopy with respect to* $\mathcal{L}$. Two curves are said to be isotopic with respect to the fibration $\mathcal{L}$ if there exists an isotopy of Σ_n which transforms the first curve to the second one, and which maps each line of $\mathcal{L}$ in another line of $\mathcal{L}$. In this paper, curves in a rational geometrically ruled surface are depicted up to isotopy with respect to $\mathcal{L}$.

3 Construction of real algebraic curves with many even ovals

We will use the Viro method to construct real plane algebraic curves. The unfamiliar readers can refer to [Vir84] and [Vir].
In this section, we will use the following proposition which will be proved in section 5.

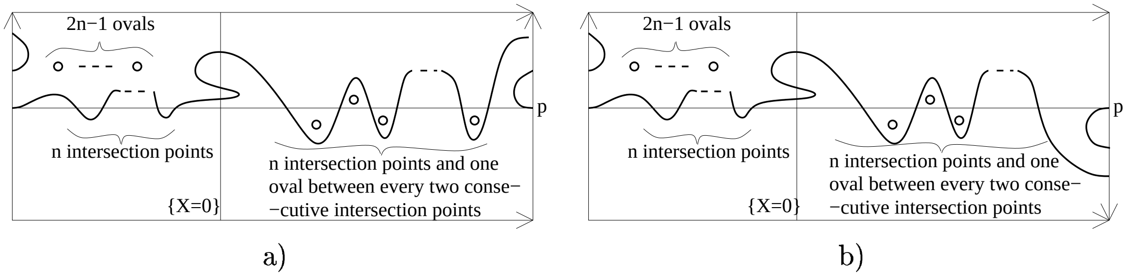

Figure 1:

Proposition 3.1 *For any $n \geq 1$, there exists a maximal real algebraic trigonal curve C_n in Σ_n whose position with respect to $\mathcal{L}$ and the axis $\{Y = 0\}$ and $\{X = 0\}$ is depicted in Figure 1a) if n is even and 1b) if n is odd. Moreover, p is a tangency point of order n of C_n and the axis $\{Y = 0\}$.*

The curve for C_4 is depicted on figure 2.

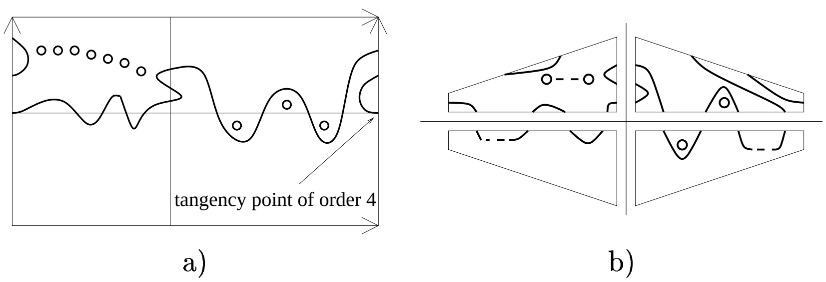

Figure 2:

Let us fix an even integer n.
The Newton polygon of the curve C_n is the quadrangle with vertices $(0, 0)$, $(2n, 0)$, $(2n, 1)$ and $(0, 3)$ and the chart of C_n is depicted on Figure 2b) (we disjointed the 4 symmetric copies of the Newton polygon of C_n for convenience). Moreover, performing the transformation $\widetilde{Y} = \lambda Y$ if necessary, we can assume that the truncation of C_n on $[(0, 3); (2n, 1)]$ is $\alpha Y^3 + \beta Y^2 X^n + \alpha Y X^{2n}$ with α and β two

real numbers. Let us denote by H_n the hexagon obtained by gluing the charts of the polynomials (see Figure 3a))

$$X^{2n}Y^3C_n(X,Y),\ X^{2n}Y^3C_n(\tfrac{1}{X},Y),\ X^{2n}Y^3C_n(\tfrac{1}{X},\tfrac{1}{Y}),\ \text{and}\ X^{2n}Y^3C_n(X,\tfrac{1}{Y})\ .$$

Let us fix an integer k and denote by T_{2k} the triangle with vertices $(0,0)$, $(2k,0)$ and $(0,2k)$. We start a subdivision of T_{2k} in the following way : for each integers l and h, we put two copies of the hexagon H_n centered in the points $(1+2n+4l, 3+8h)$ and $(1+4n+4l, 7+8h)$ if these hexagon are contained in T_{2k}. In this way, we obtain the beginning of a patchwork of a real plane curve of degree $2k$ as depicted on Figure 4 (here were chose n=4 for convenience).

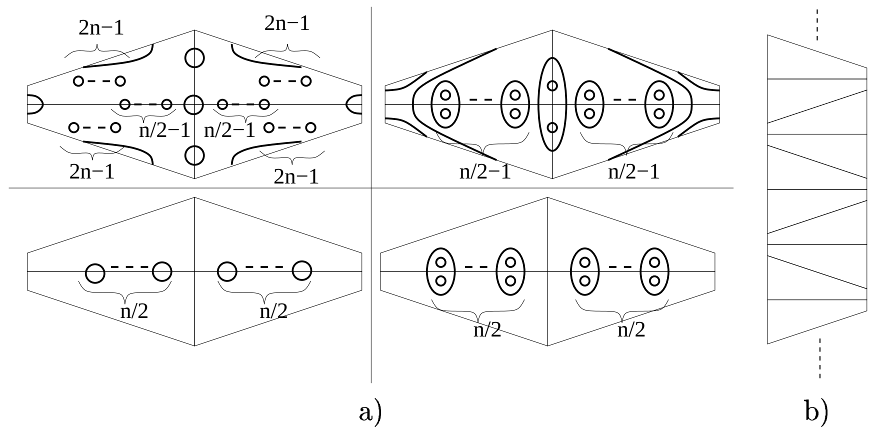

Figure 3:

Lemma 3.2 *There exist convex subdivisions of* T_{2k} *containing every hexagon as a face.*

Proof. The union of the hexagons can be decomposed in vertical strips as depicted on Figure 3b). Given any convex function on the left edge of the strip, one can extend it to a convex function on the whole strip which induces this subdivision. □

Suppose we are given an extension of our beginning of patchwork to the whole T_{2k}, satisfying the hypothesis of the Viro Theorem (see [Vir84]). Then, by Viro's Theorem, we obtain a real algebraic curve of degree $2k$ in $\mathbb{R}P^2$, which we will denote by C_k^n. Next, let us choose such an extension such that the corresponding Viro polynomial is positive along the axis $\{Y=0\}$ and negative in all empty ovals. Thus, the empty ovals of C_k^n are even.

Lemma 3.3 *Each hexagon contributes of at least* $14n-5$ *even ovals to the curve* C_k^n.

Proof. Straightforward. □

Lemma 3.4 *There exist absolute constants* a, b, c, d *and* e *such that the curve* C_k^n *has at least* $\frac{7k^2}{4} + a\frac{k^2}{n} + bnk + ck + dn^2 + en$ *even ovals.*

Proof. According to Lemma 3.3, each hexagon H_n in the patchwork of the curve C_k^n gives at least $14n-5$ even ovals. Then, if the patchwork contains N hexagons H_n, the curve C_k^n will have at least $N(14n-5)$ even ovals. The triangle T'_{2k} with vertices $(6n, 6n)$, $(2k-12n, 6n)$ and $(6n, 2k-12n)$ is contained in the union of the hexagons, so

$$N \geq \frac{Area(T'_{2k})}{Area(H_n)} = \frac{(k-9n)^2}{8n}.$$

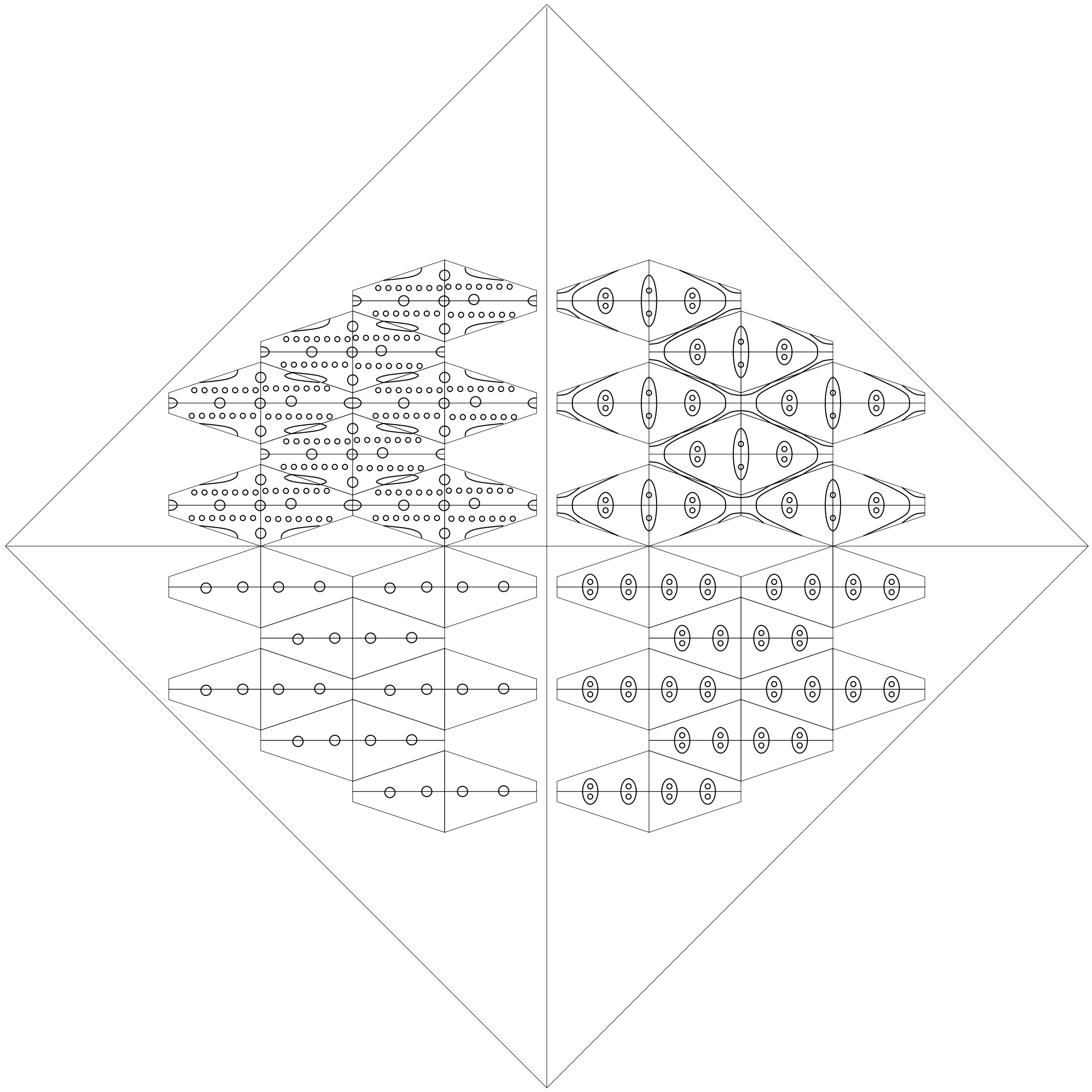

Figure 4:

Hence the number of even ovals of C_k^n is at least $\frac{(k-9n)^2}{8n}(14n-5)$. Developing this quantity, we obtain the lower bound stated in the lemma. □

The same construction can be done with an odd integer n. The curve obtained is also denoted by C_k^n and the lower bound of lemma 3.4 for its number of even oval is still valid.
Now we are able to prove the main theorem of this paper. We denote the integer part of a real r by $[r]$.

Corollary 3.5 *The curve* $C_k^{[\sqrt{k}]}$ *has* $\frac{7k^2}{4}+O(k^{\frac{3}{2}})$ *even ovals.* □

4 Real rational graphs on $\mathbb{C}P^1$

This section deals with the following problem : given a real arrangement of roots of three real polynomials (called *a root scheme* below), do there exist two real polynomials P and Q such that the real roots of P, Q and $P+Q$ realize the given arrangement?
This question can be reformulated in terms of existence of a certain graph on $\mathbb{C}P^1$ (called *a real rational graph* below).
We start with the following fact : to any rational map $f : \mathbb{C}P^1 \to \mathbb{C}P^1$, one can naturally associate a graph on $\mathbb{C}P^1$, namely $f^{-1}(\mathbb{R}P^1)$. This correspondence is used for example by S. Natanzon, B. Shapiro and A. Vainshtein to classify topologically generic real rational maps (see [NSV02] and [SV03]). Another application of these graphs has been exploited by A. Zvonkin ([Zvo]). He used these graphs to study the minimal degree of $P^3 - Q^2$, where P and Q are complex polynomials of degrees $2k$ and $3k$, respectively. Following Zvonkin, Orevkov proposed in [Ore03] a new way to construct real algebraic trigonal curves in rational geometrically ruled surfaces. Using Orevkov's approach, we proved in [Bru] the non-realizability of some $\mathcal{L}$-schemes by real algebraic trigonal curves. All the graphs considered in both these papers are special cases of real rational graphs, called *real trigonal graphs* in [Bru]. However, arguments used in [Ore03] are valid in the more general context of real rational graphs.

Definition 4.1 *A root scheme is a k-uplet* $((l_1, m_1), \ldots, (l_k, m_k)) \in (\{p, q, r\} \times \mathbb{N})^k$ *with* k *a natural number (here,* p, q *and* r *are symbols and do not stand for natural numbers).*
A root scheme $((l_1, m_1), \ldots, (l_k, m_k))$ *is realizable by polynomials of degree* n *if there exist two real polynomials in one variable of degree* n*, with no common roots,* $P(X)$ *and* $Q(X)$ *such that if* $x_1 < x_2 < \ldots < x_k$ *are the real roots of* P, Q *and* $P+Q$*, then* $l_i = p$ *(resp.,* q, r*) if* x_i *is a root of* P *(resp.,* $Q, P+Q$*) and* m_i *is the multiplicity of* x_i*.*
The polynomials P*,* Q *and* $P+Q$ *are said to realize the root scheme* $((l_1, m_1), \ldots, (l_k, m_k))$*.*

In a root scheme, we will abbreviate a sequence S repeated u times by S^u.

From now on, let RS be a root scheme and suppose that RS is realized by P, Q and $P+Q$ of degree n. Put $R(X) = P(X) + Q(X)$ and consider the rational function $f(X) = \frac{R(X)}{Q(X)} = \frac{P(X)}{Q(X)} + 1$. Color and orient $\mathbb{R}P^1$ as depicted in Figure 5a). Let Γ be $f^{-1}(\mathbb{R}P^1)$ with the coloring and the orientation induced by those chosen on $\mathbb{R}P^1$. Then, Γ is a colored and oriented graph on $\mathbb{C}P^1$, invariant under the action of the complex conjugation. The coloring and the orientation of $\Gamma \cap \mathbb{R}P^1$ as well as the multiplicities of the vertices of Γ lying on $\mathbb{R}P^1$ can clearly be extracted from RS.

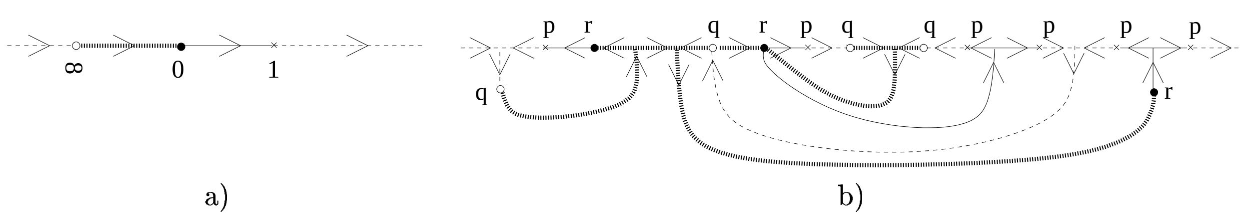

Figure 5:

Definition 4.2 *The colored and oriented graph on* $\mathbb{R}P^1$ *constructed above is called the real graph associated to* RS*.*

Definition 4.3 *Let* Γ *be a graph on* $\mathbb{C}P^1$ *invariant under the action of the complex conjugation and* $\pi : \Gamma \to \mathbb{R}P^1$ *a continuous map. Then the coloring and orientation of* $\mathbb{R}P^1$ *shown in Figure 5a) defines a coloring and an orientation of* Γ *via* π*.*

The graph Γ equipped with this coloring and this orientation is called a real rational graph if

- *any vertex of Γ has an even valence,*
- *any connected component D of $\mathbb{C}P^1 \setminus \Gamma$ is homeomorphic to an open disk,*
- *for any connected component D of $\mathbb{C}P^1 \setminus \Gamma$, the map $\pi_{|\partial D}$ is a covering of $\mathbb{R}P^1$ of degree d_D.*

The sum of the degrees d_D for all connected component D of $\{Im(z) > 0\} \setminus \mathbb{R}P^1$ of is called the degree of Γ.

The importance of real rational graphs is given by the following proposition.

Proposition 4.4 *Let RS be a root scheme and G its real graph. Then RS is realizable by polynomials of degree n if and only if there exists a real rational graph Γ of degree n such that $\Gamma \cap \mathbb{R}P^1 = G$.*

Proof. The proof used in [Ore03] to construct real trigonal curves in Σ_n can be used here in the same way. □

For example, the root scheme $((p,1),(r,1),(q,2),(r,3),(p,1),(q,1),(q,1),(p,1),(p,1),(p,1),(p,1))$ is realizable by polynomials of degree 6 as it is depicted on Figure 5b).

Lemma 4.5 *Let $RS = ((l_1,m_1),\ldots,(l_k,m_k))$ a root scheme such that there exist i and s such that $\forall j \in \{i,\ldots,i+s\}, l_j = l_i$. Define the root scheme $RS' = ((l'_1,m'_1),\ldots,(l'_{k-s},m'_{k-s}))$ by*

- *$(l'_t,m'_t) = (l_t,m_t)$ for $t < i$,*
- *$(l'_i,m'_i) = (l_i, m_i + \ldots + m_{i+s})$,*
- *$(l'_t,m'_t) = (l_{t-s},m_{t-s})$ for $t > i+s$,*

Then RS is realizable by polynomials of degree n if and only if RS' is realizable by polynomials of degree n.

Proof. Straightforward. □

5 Construction of reducible curves with a deep tangency point

Let us define the root schemes RS_n by

- $\left((p,n),\left[(q,1),(r,1)^2,(q,1)\right]^k,\left[(p,1),(r,1)^2,(p,1)\right]^k,(q,1)^n\right)$ if $n = 2k$.
- $\left((p,n),\left[(r,1),(q,1)^2,(r,1)\right]^k,(r,1),(q,1),(p,1),(r,1),\left[(p,1),(r,1)^2,(p,1)\right]^k,(q,1)^n\right)$ if $n = 2k+1$,

Proposition 5.1 *For all n in $\mathbb{N}^*$, the root scheme RS_n is realizable by polynomials of degree $2n$.*

Proof. According to lemma 4.5, one can substitute $(p,1)^n$ instead of (p,n) in RS_n, and according to proposition 4.4, one has just to construct a rational graph on $\mathbb{C}P^1$ with a real part corresponding to the real graph of RS_n. We will prove it by induction on n. All the pictures will represent the half $\{Im(z) \leq 0\}$ of $\mathbb{C}P^1$.

The rational graph corresponding to RS_1 is depicted on Figure 6a).

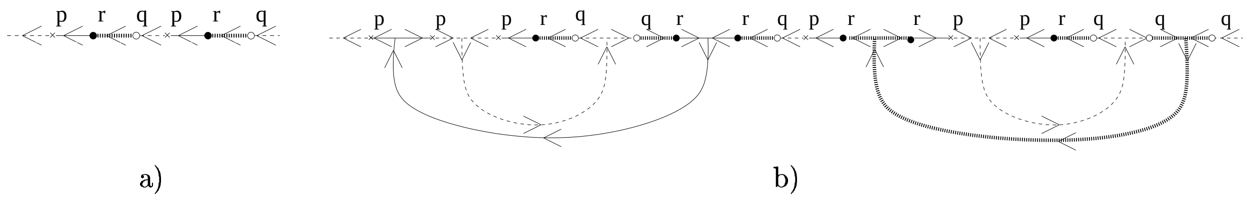

Figure 6:

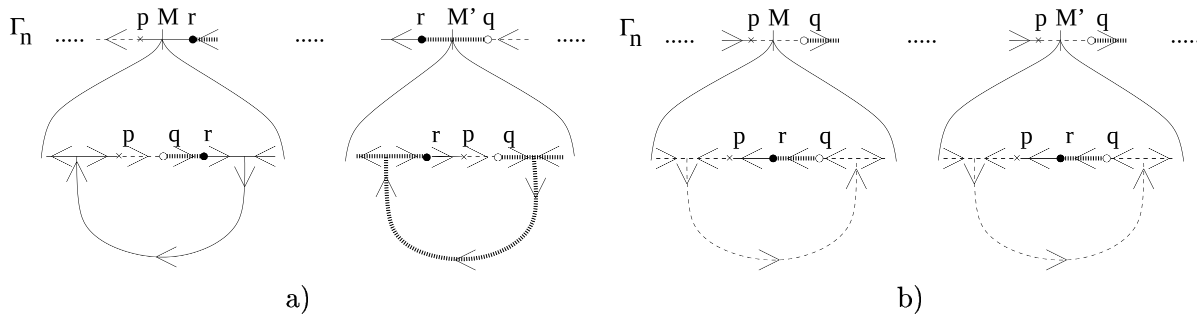

Figure 7:

Suppose that a rational graph Γ_n corresponding to RS_n is constructed. Let us examine $\Gamma_n \cap \mathbb{R}P^1$ from the left to the right.
Consider, first, the case $n = 2k+1$.
Let M be a point on $\Gamma_n \cap \mathbb{R}P^1$ between the n^{th} point corresponding to p in RS_n and the first point corresponding to r in RS_n. Then cut Γ_n at M and glue the piece of graph depicted in Figure 7a).
Let M' be a point on $\Gamma_n \cap \mathbb{R}P^1$ between the last point corresponding to r in RS_n and the $n^{th}+1$ point corresponding to q in RS_n. Then cut Γ_n at M' and glue the piece of graph depicted in Figure 7a).
Consider now the case $n = 2k$.
Let M be a point on $\Gamma_n \cap \mathbb{R}P^1$ between the n^{th} point corresponding to p in RS_n and the first point corresponding to q in RS_n. Then cut Γ_n at M and glue the piece of graph depicted in Figure 7b).
Let M' be a point on $\Gamma_n \cap \mathbb{R}P^1$ between the last point corresponding to p in RS_n and the $n^{th}+1$ point corresponding to q in RS_n. Then cut Γ_n at M' and glue the piece of graph depicted in Figure 7b).

For example, Γ_3 is depicted in Figure 6b). According to proposition 4.4, the rational graphs Γ_n ensure the realizability of the root schemes RS_n by polynomials of degree $2n$. □

Corollary 5.2 *For all n in $\mathbb{N}^*$, there exists three real polynomials $a_1(X)$, $a_2(X)$ and $b(X)$ of degree n such that*

- *all the roots of a_1, a_2, b and $a_1b + a_2$ are real,*
- *all the roots of a_2 and $a_1b + a_2$ are smaller than the roots of b.*

Proof. Let $P(X)$, $Q(X)$ and $R(X) = P(X) + Q(X)$ three polynomials of degree $2n$ realizing the root scheme RS_n. Then

- $Q(X) = \prod_{i=1}^{2n}(X - y_i)$ with $y_1 < y_2 < \ldots < y_{2n}$,
- $P(X) = (X - \alpha)^n \prod_{i=1}^{n}(X - x_i)$ with $\alpha < x_1 < x_2 < \ldots < x_n < y_{n+1}$,

- $R(X) = \prod_{i=1}^{2n}(X - z_i)$ with $z_1 < z_2 < \ldots < z_{2n} < y_{n+1}$.

Let us define $a_2(X) = X^{2n} P\left(-\frac{1}{X} + \alpha\right)$, $A_1(X) = \prod_{i=1}^{n}(X - y_i)$, $a_1 = X^n A_1\left(-\frac{1}{X} + \alpha\right)$, $B(X) = \prod_{i=n+1}^{2n}(X - y_i)$ and $b = X^n B\left(-\frac{1}{X} + \alpha\right)$.
As $a_1 b + a_2 = X^{2n} R\left(-\frac{1}{X} + \alpha\right)$, the corollary follows from the definition of P, Q and R. □

Now we are able to prove proposition 3.1.
Proof of proposition 3.1. We construct here explicitly only curves in Σ_{2k}. The construction of curves in Σ_{2k+1} is done in the same way. Let us fix an even $n \geq 1$ and consider the polynomials $a_1(X)$, $a_2(X)$, and $b(X)$ of degree n constructed in corollary 5.2. Multiplying these three polynomials by -1 and performing a linear change of coordinates if necessary, we can assume that the leading coefficient of b is positive, all the roots of b are positive, and all the roots of a_2 and $a_1 b + a_2$ are negative. Then, the curve $Y(Y - b(X))$ in Σ_n is depicted in Figure 8a).

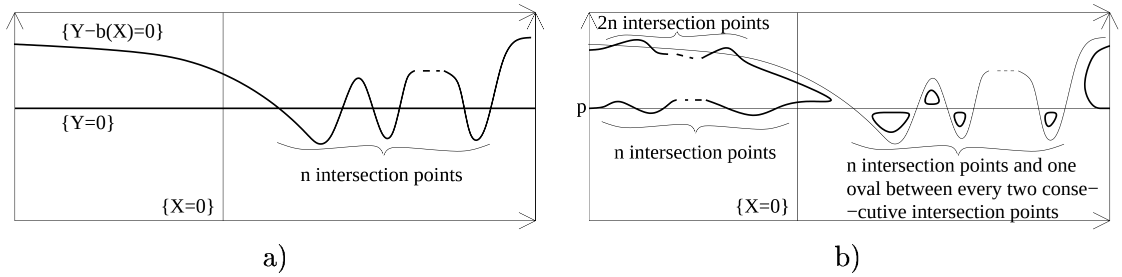

Figure 8:

For t small enough and of suitable sign, the relative positions of the curves $D_n(X, Y) = Y(Y - b(X)) + t(a_1(X)Y + a_2(X))$, Y and $Y - b(X)$ are as depicted in Figure 8b), where p is a tangency point of order n of D_n and Y. Indeed, the definition of $a_1(X)$, $a_2(X)$, and $b(X)$ exactly means that the intersection points of $a_1(X)Y + a_2(X)$ and $Y(Y - b(X))$ have negative abscissa. Perturbing all the double points of $D_n(X, Y)(Y - b(X))$ in order to have the maximal number of ovals and keeping the tangency point of order n with Y, we obtain a curve C_n with n even, satisfying the conditions of proposition 3.1. □

6 Applications to real algebraic surfaces

Here we recall and follow the notations proposed in [Bih]. We consider only $\mathbb{Z}/2\mathbb{Z}$-homology.
Let d, i, k and n be integers, $\beta_i(\mathbb{R}X_d^n)$ the i^{th} Betti number of the real part of a nonsingular real hypersurface of degree d in $\mathbb{C}P^n$, and $\beta_i(\mathbb{R}Y_{2k}^n)$ the i^{th} Betti number of the real part (for some real structure) of a double covering of $\mathbb{C}P^n$ branched along a nonsingular real hypersurface of degree $2k$. An interesting question concerns the asymptotic behavior of *max* $\beta_i(\mathbb{R}X_d^n)$ and *max* $\beta_i(\mathbb{R}Y_{2k}^n)$ when d and k go to infinity. In [Bih], Bihan studied the sequences of real numbers $(\zeta_{i,n})_{i,n\in\mathbb{N}^2}$ and $(\delta_{i,n})_{i,n\in\mathbb{N}^2}$ which are defined by

$$max\ \beta_i(\mathbb{R}X_d^n) \underset{d\to\infty}{\sim} \zeta_{i,n} d^n \qquad \text{and} \qquad max\ \beta_i(\mathbb{R}Y_{2k}^n) \underset{k\to\infty}{\sim} \delta_{i,n} k^n.$$

The exact value of the numbers $\zeta_{i,n}$ and $\delta_{i,n}$ are known only for small n. The following equalities are well known (see [Bih], [Ite01]).

$$\delta_{0,0} = 2,\ \zeta_{0,1} = \delta_{0,1} = \delta_{1,1} = 1,\ \zeta_{0,2} = \zeta_{1,2} = \frac{1}{2},$$

$$\delta_{0,2} \leq \frac{7}{4},\ \delta_{1,2} \leq \frac{7}{2},\ \zeta_{0,3} \leq \frac{5}{12} \text{ and } \zeta_{1,3} \leq \frac{5}{6}.$$

The upper bounds are classical and are obtained using the Harnack and Comessati-Petrovsky-Oleinik inequalities. Lower bounds for $\delta_{0,2}$ and $\delta_{1,2}$ are directly related to the asymptotically maximal number of even ovals of a curve of even degree in $\mathbb{R}P^2$, and before the results of the present paper, the best known lower bounds for these two numbers were, respectively, $\frac{27}{16}$ and $\frac{27}{8}$ (see [Ite01]). In [Bih], Bihan has constructed nonsingular real algebraic surfaces in $\mathbb{R}P^3$ with Betti numbers related to $\delta_{0,2}$ and $\delta_{1,2}$.

Theorem 6.1 (Bihan) *One has* $\frac{\delta_{0,2}}{6} + \frac{1}{12} \leq \zeta_{0,3}$ *and* $\frac{\delta_{1,2}}{6} + \frac{1}{6} \leq \zeta_{1,3}$.

Theorem 1.1 gives as immediate corollaries the exact values of $\delta_{0,2}$ and $\delta_{1,2}$ and improves the known lower bounds for $\zeta_{0,3}$ and $\zeta_{1,3}$.

Proposition 6.2 *One has* $\delta_{0,2} = \frac{7}{4}$ *and* $\delta_{1,2} = \frac{7}{2}$.

Corollary 6.3 *One has* $\frac{9}{24} \leq \zeta_{0,3} \leq \frac{5}{12}$ *and* $\frac{9}{12} \leq \zeta_{1,3} \leq \frac{5}{6}$.

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