FLOOR DECOMPOSITIONS OF TROPICAL CURVES : THE PLANAR CASE

ERWAN BRUGALLÉ AND GRIGORY MIKHALKIN

Abstract. In [BM07] we announced a formula to compute Gromov-Witten and Welschinger invariants of some toric varieties, in terms of combinatorial objects called floor diagrams. We give here detailed proofs in the tropical geometry framework, in the case when the ambient variety is a complex surface, and give some examples of computations using floor diagrams. The focusing on dimension 2 is motivated by the special combinatoric of floor diagrams compared to arbitrary dimension.

We treat a general toric surface case in this dimension: the curve is given by an arbitrary lattice polygon and include computation of Welschinger invariants with pairs of conjugate points. See also [FM] for combinatorial treatment of floor diagrams in the projective case.

1. Introduction

Let Δ be a lattice polygon in $\mathbb{R}^2$, g a non-negative integer, and ω a generic configuration of $Card(\partial\Delta \cap \mathbb{Z}^2) - 1 + g$ points in $(\mathbb{C}^*)^2$. Then, there exists a finite number $N(\Delta, g)$ of complex algebraic curves in $(\mathbb{C}^*)^2$ of genus g and Newton polygon Δ passing through all points in ω. Moreover, $N(\Delta, g)$ doesn't depend on ω as long as it is generic. If the toric surface $Tor(\Delta)$ corresponding to Δ is Fano, then the numbers $N(\Delta, g)$ are known as *Gromov-Witten invariants* of the surface $Tor(\Delta)$. Kontsevich first computed in [KM94] the series $N(\Delta, 0)$ for convex surfaces $Tor(\Delta)$, and Caporaso and Harris computed in [CH98] all $N(\Delta, g)$'s where $Tor(\Delta)$ is Fano or a Hirzebruch surface.

Suppose now that the surface $Tor(\Delta)$ is equipped with a real structure $conj$, i.e. $conj$ is a antiholomorphic involution on $Tor(\Delta)$. For example, one can take the tautological real structure given in $(\mathbb{C}^*)^2$ by the standard complex conjugation. Suppose moreover that ω is a real configuration, i.e. $conj(\omega) = \omega$. Then it is natural to study the set $\mathbb{R}\mathcal{C}(\omega)$ of real algebraic curves in $(\mathbb{C}^*)^2$ of genus g and Newton polygon Δ passing through all points in ω. It is not hard to see that, unlike in the enumeration of complex curves, the cardinal of this set depends heavily on ω. However, Welschinger proved in [Wel05] that when $g = 0$ and $Tor(\Delta)$ is Fano, one can define an invariant. A real nodal curve C in $Tor(\Delta)$ has two types of real nodes, isolated ones (locally given by the equation $x^2 + y^2 = 0$) and non-isolated ones (locally given by the equation $x^2 - y^2 = 0$). Welschinger defined the *mass* $m(C)$ of the curve C as the number of isolated nodes of C, and proved that if $g = 0$ and $Tor(\Delta)$ is Fano, then the number

$$W(\Delta, r) = \sum_{C \in \mathbb{R}\mathcal{C}(\omega)} (-1)^{m(C)}$$

depends only on Δ and the number r of pairs of complex conjugated points in ω.

Tropical geometry is an algebraic geometry over the tropical semi-field $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ where the tropical addition is taking the maximum, and the tropical multiplication is the classical addition. As in the classical setting, given Δ a lattice polygon in $\mathbb{R}^2$, g a non-negative integer, and ω a generic configuration of $Card(\partial\Delta \cap \mathbb{Z}^2) - 1 + g$ points in $(\mathbb{R}^*)^2$, we can enumerate tropical curves in $\mathbb{R}^2$

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of genus g and Newton polygon Δ passing through all points in ω. It was proved in [Mik05] that provided that we count tropical curves with an appropriate multiplicity, then the number of tropical curves does not depend on ω and is equal to $N(\Delta, g)$. Moreover, tropical geometry allows also one to compute quite easily Welschinger invariants $W(\Delta, r)$ of Fano toric surfaces equipped with the tautological real structure (see [Mik05], [Shu06]). This has been the first systematic method to compute Welschinger invariants of these surfaces.

In [BM07], we announced a formula to compute the numbers $N(\Delta, g)$ and $W(\Delta, r)$ easily in terms of combinatorial objects called *floor diagrams*. This diagrams encode degeneracies of tropical curves passing through some special configuration of points. This paper is devoted to explain how floor diagrams can be used to compute the numbers $N(\Delta, g)$ and $W(\Delta, r)$ (Theorems 3.6 and 3.9), and to give some examples of concrete computations (section 6). In [BM07], we announced a more general formula computing Gromov-Witten and Welschinger invariants of some toric varieties of any dimension. However, floor diagrams corresponding to plane curves have a special combinatoric compared with the general case, and deserve some special attention. Details of the proof of the general formula given in [BM07] will appear soon.

In section 2 we remind some convention we use throughout this paper about graphs and lattice polygons. Then, we state in section 3 our main formulas computing the numbers $N(\Delta, g)$ and $W(\Delta, r)$ when Δ is a *h-transverse* polygon. We present tropical enumerative geometry in section 4, and prove our main formulas in section 5. We give some examples of computations using floor diagrams in section 6, and we end this paper with some remarks in section 7.

2. CONVENTION

2.1. **Graphs.** In this paper, graphs are considered as (non necessarily compact) abstract 1 dimensional topological objects. Remind that a *leaf* of a graph is an edge which is non-compact or adjacent to a 1-valent vertex. Given a graph Γ, we denote by

- $\mathrm{Vert}(\Gamma)$ the set of its vertices,
- $\mathrm{End}(\Gamma)$ the set of its 1-valent vertices,
- $\mathrm{Edge}(\Gamma)$ the set of its edges,
- $\mathrm{Edge}^{\infty}(\Gamma)$ the set of its non-compact leaves.

If in addition Γ is oriented so that there are no oriented cycles, then there exists a natural partial ordering on Γ : an element a of Γ is greater than another element b if there exists an oriented path from b to a. In this case, we denote by $\mathrm{Edge}^{+\infty}(\Gamma)$ (resp. $\mathrm{Edge}^{-\infty}(\Gamma)$) the set of edges e in $\mathrm{Edge}^{\infty}(\Gamma)$ such that no vertex of Γ is greater (resp. smaller) than a point of e.

We say that Γ is a *weighted graph* if each edge of Γ is prescribed a natural weight, i.e. we are given a function $\omega : \mathrm{Edge}(\Gamma) \to \mathbb{N}^*$. Weight and orientation allow one to define the *divergence* at the vertices. Namely, for a vertex $v \in \mathrm{Vert}(\Gamma)$ we define the divergence $\mathrm{div}(v)$ to be the sum of the weights of all incoming edges minus the sum of the weights of all outgoing edges.

2.2. **Lattice polygons.** We remind that a *primitive integer vector*, or shortly a primitive vector, is a vector (α, β) in $\mathbb{Z}^2$ whose coordinates are relatively prime. A *lattice polygon* Δ is a convex polygon in $\mathbb{R}^2$ whose vertices are in $\mathbb{Z}^2$. For such a polygon, we define

$$\partial_l \Delta = \{p \in \partial\Delta \mid \forall t > 0,\ p + (-t, 0) \notin \Delta\},$$
$$\partial_r \Delta = \{p \in \partial\Delta \mid \forall t > 0,\ p + (t, 0) \notin \Delta\}.$$

A lattice polygon Δ is said to be *h-transverse* if any primitive vector parallel to an edge of $\partial_l\Delta$ or $\partial_r\Delta$ is of the form $(\alpha, \pm 1)$ with α in $\mathbb{Z}$.

If e is a lattice segment in $\mathbb{R}^2$, we define the *integer length* of e by $l(e) = Card(e \cap \mathbb{Z}^2) - 1$. If Δ is a h-transverse polygon, we define its *left directions* (resp. *right directions*), denoted by $d_l(\Delta)$

(resp. $d_r(\Delta)$), as the unordered list that consists of the elements α repeated $l(e)$ times for all edge vectors $e = \pm l(e)(\alpha, -1)$ of $\partial_l\Delta$ (resp. $\partial_r\Delta$). If Δ has a bottom (resp. top) horizontal edge e then we set $d_-(\Delta) = l(e)$ (resp. $d_+(\Delta) = l(e)$) and $d_-(\Delta) = 0$ (resp. $d_+(\Delta) = 0$) otherwise.

There is a natural 1-1 correspondence between quadruples (d_l, d_r, d_-, d_+) and h-transverse polygons Δ considered up to translation as the polygon can be easily reconstructed from such a quadruple.

We have

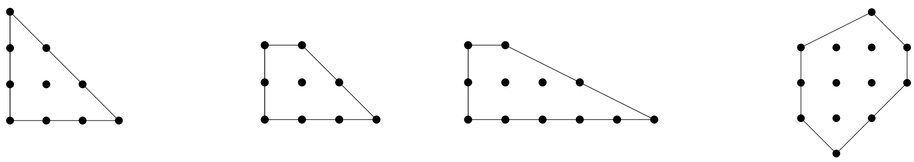
$$Card(d_l(\Delta)) = Card(d_r(\Delta)) = Card(\partial_l\Delta \cap \mathbb{Z}^2) - 1 = Card(\partial_r\Delta \cap \mathbb{Z}^2) - 1 \tag{1}$$

and

$$2Card(d_l(\Delta)) + d_-(\Delta) + d_+(\Delta) = Card(\partial\Delta \cap \mathbb{Z}^2). \tag{2}$$

We call the cardinality $Card(d_l(\Delta))$ the *height* of h-transversal polygon Δ.

Example 2.1. Some h-transverse polygons are depicted in Figure 1. By abuse of notation, we write unordered lists within brackets $\{\}$.



a) $d_l = \{0,0,0\}$, $d_r = \{1,1,1\}$, $d_- = 3$, $d_+ = 0$

b) $d_l = \{0,0\}$, $d_r = \{1,1\}$, $d_- = 3$, $d_+ = 1$

c) $d_l = \{0,0\}$, $d_r = \{2,2\}$, $d_- = 5$, $d_+ = 1$

d) $d_l = \{1,0,0,-2\}$, $d_r = \{-1,-1,0,1\}$, $d_- = 0$, $d_+ = 0$

FIGURE 1. Examples of h-transverse polygons

Remark 2.2. If Δ is a lattice polygon and if v is a primitive integer vector such that for any edge e of Δ we have $|det(v,e)| \leq l(e)$, then Δ is a h-transverse polygon after a suitable change of coordinates in $SL_2(\mathbb{Z})$.

In this paper, we denote by Δ_d the lattice polygon with vertices $(0,0)$, $(d,0)$, and $(0,d)$.

3. Floor diagrams

Here we define the combinatorial objects that can be used to replace the algebraic curves in real and complex enumerative problems. In this section, we fix an h-transverse lattice polygon Δ.

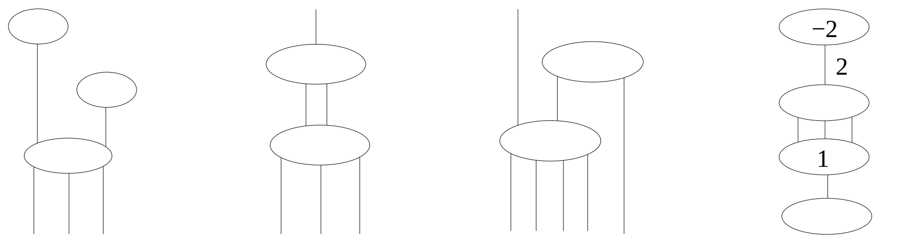
3.1. Enumeration of complex curves.

Definition 3.1. *A (plane) floor diagram $\mathcal{D}$ of genus g and Newton polygon Δ is the data of a connected weighted oriented graph Γ and a map $\theta : Vert(\Gamma) \to \mathbb{Z}$ which satisfy the following conditions*

- *the oriented graph Γ is acyclic,*
- *the first Betti number $b_1(\Gamma)$ equals g,*
- *there are exactly $d_\pm(\Delta)$ edges in $Edge^{\pm\infty}(\Gamma)$, and all of them are of weight 1,*
- *the (unordered) collection of numbers $\theta(v)$, where v goes through vertices of Γ, coincides with $d_l(\Delta)$,*
- *the (unordered) collection of numbers $\theta(v) + div(v)$, where v goes through vertices of Γ, coincides with $d_r(\Delta)$.*

In order to avoid too many notation, we will denote by the same letter $\mathcal{D}$ a floor diagram and its underlying graph Γ. Here are the convention we use to depict floor diagrams : vertices of $\mathcal{D}$ are represented by ellipses. We write $\theta(v)$ inside the ellipse v only if $\theta(v) \neq 0$. Edges of $\mathcal{D}$ are represented by vertical lines, and the orientation is implicitly from down to up. We write the weight of an edge close to it only if this weight is at least 2. In the following, we define $s = Card(\partial\Delta \cap \mathbb{Z}^2) + g - 1$.

Example 3.2. Figure 2 depicts an example of floor diagram for any h-transverse polygon depicted in Figure 1.



a) $g = 0$ b) $g = 1$ c) $g = 0$ d) $g = 2$

FIGURE 2. Examples of floor diagrams whose Newton polygon are depicted in Figure 1

Note that Equations (1) and (2) combined with Euler's formula imply that for any floor diagram $\mathcal{D}$ of genus g and Newton polygon Δ we have

$$Card(\mathrm{Vert}(\mathcal{D})) + Card(\mathrm{Edge}(\mathcal{D})) = s.$$

A map m between two partially ordered sets is said *increasing* if

$$m(i) > m(j) \Longrightarrow i > j$$

Definition 3.3. *A marking of a floor diagram $\mathcal{D}$ of genus g and Newton polygon Δ is an increasing map $m : \{1, \ldots, s\} \to \mathcal{D}$ such that for any edge or vertex x of $\mathcal{D}$, the set $m^{-1}(x)$ consists of exactly one element.*

A floor diagram enhanced with a marking is called a *marked floor diagram* and is said to be marked by m.

Definition 3.4. *Two marked floor diagrams $(\mathcal{D}, m)$ and $(\mathcal{D}', m')$ are called equivalent if there exists a homeomorphism of oriented graphs $\phi : \mathcal{D} \to \mathcal{D}'$ such that $w = w' \circ \phi$, $\theta = \theta' \circ \phi$, and $m = m' \circ \phi$.*

Hence, if $m(i)$ is an edge e of $\mathcal{D}$, only the knowledge of e is important to determine the equivalence class of $(\mathcal{D}, m)$, not the position of $m(i)$ on e. From now on, we consider marked floor diagrams up to equivalence. To any (equivalence class of) marked floor diagram, we assign a sequence of non-negative integers called *multiplicities* : a *complex* multiplicity, and some *r-real* multiplicities.

Definition 3.5. *The complex multiplicity of a marked floor diagram $\mathcal{D}$, denoted by $\mu^{\mathbb{C}}(\mathcal{D})$, is defined as*

$$\mu^{\mathbb{C}}(\mathcal{D}) = \prod_{e \in Edge(\mathcal{D})} w(e)^2$$

Note that the complex multiplicity of a marked floor diagram depends only on the underlying floor diagram. Next theorem is the first of our two main formulas.

Theorem 3.6. *For any h-transverse polygon Δ and any genus g, one has*

$$N(\Delta, g) = \sum \mu^{\mathbb{C}}(\mathcal{D})$$

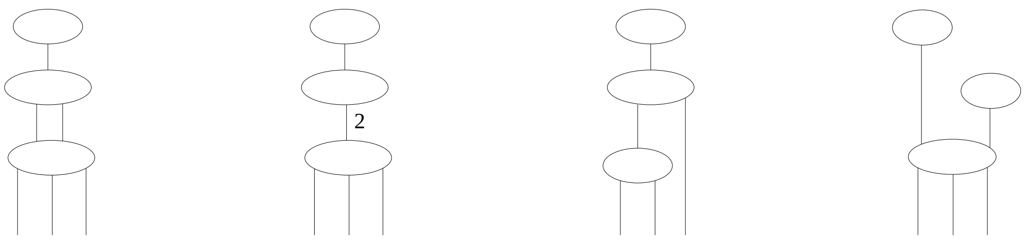
where the sum is taken over all marked floor diagrams of genus g and Newton polygon Δ.

Theorem 3.6 is a corollary of Proposition 5.9 proved in section 5.

Example 3.7. Using marked floor diagrams depicted in Figures 3 and 4 we verify that

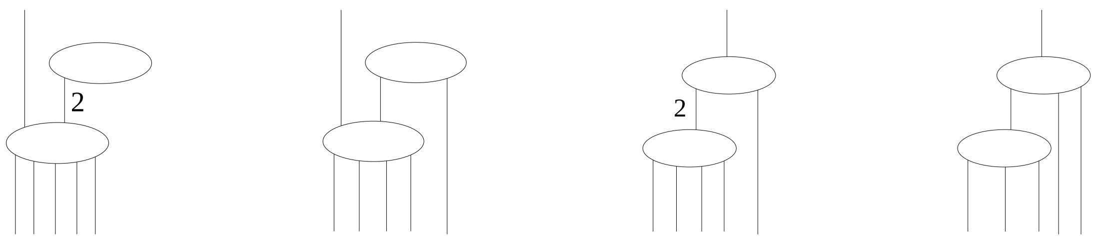
$N(\Delta_3, 1) = 1$ (see Figure 3a), $N(\Delta_3, 0) = 12$ (see Figure 3b,c,d).

$N(\Delta, 0) = 84$ (see Figure 4), where Δ is the polygon depicted in Figure 1c.



a) $\mu^{\mathbb{C}} = 1$, 1 marking b) $\mu^{\mathbb{C}} = 4$, 1 marking c) $\mu^{\mathbb{C}} = 1$, 5 markings d) $\mu^{\mathbb{C}} = 1$, 3 markings

FIGURE 3. Floor diagrams of genus 1 and 0, and Newton polygon Δ_3



a) $\mu^{\mathbb{C}} = 4$, 3 markings b) $\mu^{\mathbb{C}} = 1$, 23 markings c) $\mu^{\mathbb{C}} = 4$, 7 markings d) $\mu^{\mathbb{C}} = 1$, 21 markings

FIGURE 4. Floor diagrams of genus 0 and Newton polygon depicted in Figure 1c

3.2. **Enumeration of real curves.** First of all, we have to define the notion of real marked floor diagrams. Like before, we define $s = Card(\partial\Delta \cap \mathbb{Z}^2) + g - 1$. Choose an integer $r \geq 0$ such that $s - 2r \geq 0$, and $\mathcal{D}$ a floor diagram of genus 0 and Newton polygon Δ marked by a map m.

The set $\{i, i+1\}$ is a called *r-pair* if $i = s - 2k + 1$ with $1 \leq k \leq r$. Denote by $\Im(m, r)$ the union of all the r-pairs $\{i, i+1\}$ where $m(i)$ is not adjacent to $m(i+1)$. Let $\rho_{m,r} : \{1, \ldots, s\} \to \{1, \ldots, s\}$ be the bijection defined by $\rho_{m,r}(i) = i$ if $i \notin \Im(m, r)$, and by $\rho_{m,r}(i) = j$ if $\{i, j\}$ is a r-pair contained in $\Im(m, r)$. Note that $\rho_{m,r}$ is an involution.

We define o_r to be the half of the number of vertices v of $\mathcal{D}$ in $m(\Im(m, r))$ with odd divergence $\mathrm{div}(v)$, and we set $A = \mathrm{Edge}(\mathcal{D}) \setminus m(\{1, \ldots, s - 2r\})$.

Definition 3.8. *A marked floor diagram $(\mathcal{D}, m)$ is called r-real if the two marked floor diagrams $(\mathcal{D}, m)$ and $(\mathcal{D}, m \circ \rho_{m,r})$ are equivalent.*

The r-real multiplicity of a r-real marked floor diagram, denoted by $\mu_r^{\mathbb{R}}(\mathcal{D}, m)$, is defined as

$$\mu_r^{\mathbb{R}}(\mathcal{D}, m) = (-1)^{o_r} \prod_{e \in A} w(e)$$

if all edges of $\mathcal{D}$ of even weight contains a point of $m(\Im(m, r))$, and as

$$\mu_r^{\mathbb{R}}(\mathcal{D}, m) = 0$$

otherwise.

For convenience we set $\mu_r^{\mathbb{R}}(\mathcal{D}, m) = 0$ also in the case when $(\mathcal{D}, m)$ is not r-real.

Note that $\mu_0^{\mathbb{R}}(\mathcal{D}, m) = 1$ or 0 and is equal to $\mu^{\mathbb{C}}(\mathcal{D})$ modulo 2, hence doesn't depend on m. However, $\mu_r^{\mathbb{R}}(\mathcal{D}, m)$ depends on m as soon as $r \geq 1$. Next theorem is the second main formula of this paper.

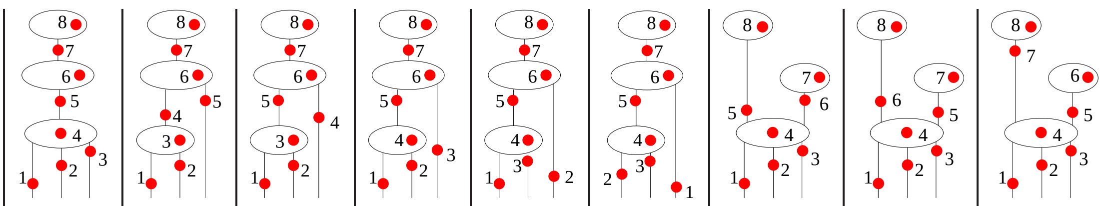
Theorem 3.9. *Let Δ be a h-transverse polygon such that Welschinger invariants are defined for the corresponding toric surface $Tor(\Delta)$ equipped with its tautological real structure. Then for any integer r such that $s - 2r \geq 0$, one has*

$$W(\Delta, r) = \sum \mu_r^{\mathbb{R}}(\mathcal{D}, m)$$

where the sum is taken over all marked floor diagrams of genus 0 and Newton polygon Δ.

Theorem 3.9 is a corollary of Proposition 5.9 proved in section 5.

Example 3.10. All marked floor diagrams of genus 0 and Newton polygon Δ_3 are depicted in Table 1 together with their real multiplicities. The first floor diagram has an edge of weight 2, but we didn't mention it in the picture to avoid confusion. According to Theorem 3.9 we find $W(\Delta_3, r) = 8 - 2r$.



$\mu^{\mathbb{C}}$	4	1	1	1	1	1	1	1	1
$\mu_0^{\mathbb{R}}$	0	1	1	1	1	1	1	1	1
$\mu_1^{\mathbb{R}}$	0	1	1	1	1	1	0	0	1
$\mu_2^{\mathbb{R}}$	0	1	1	1	1	1	-1	-1	1
$\mu_3^{\mathbb{R}}$	0	1	0	0	1	1	-1	-1	1
$\mu_4^{\mathbb{R}}$	0	1	0	0	0	0	-1	-1	1

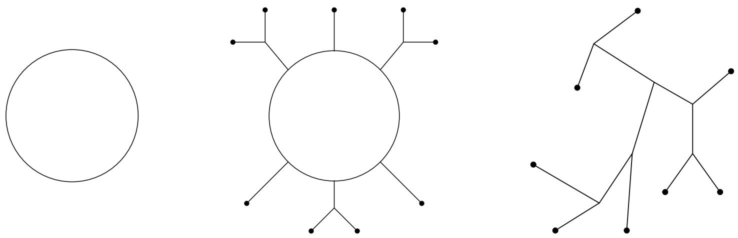
TABLE 1. Computation of $W(\Delta_3, r)$

4. ENUMERATIVE TROPICAL GEOMETRY

4.1. **Tropical curves.**

Definition 4.1. *An irreducible tropical curve C is a connected compact metric graph whose leaves are exactly the edges of infinite length. This means that $C \setminus End(C)$ is a complete metric space with inner metric. In other words the 1-valent vertices are at the infinite distance from all the other points of C. The genus of C is defined as its first Betti number $b_1(C)$.*

Example 4.2. Examples of tropical curves are depicted in Figure 5. 1-valent vertices are represented with bullets.



a) $g = 1$ b) $g = 1$ c) $g = 0$

FIGURE 5. Examples of tropical curves

Given e an edge of a tropical curve C, we choose a point p in the interior of e and a unit vector u_e of the tangent line to C at p. Of course, the vector u_e depends on the choice of p and is not well defined, but this will not matter in the following. We will sometimes need u_e to have a prescribed direction, and we will then precise this direction. The standard inclusion of $\mathbb{Z}^2$ in $\mathbb{R}^2$ induces a standard inclusion of $\mathbb{Z}^2$ in the tangent space of $\mathbb{R}^2$ at any point of $\mathbb{R}^2$.

Definition 4.3. *A map $f : C \backslash End(C) \to \mathbb{R}^2$ is called a tropical morphism if the following conditions are satisfied*

- *for any edge e of C, the restriction $f_{|e}$ is a smooth map with $df(u_e) = w_{f,e}u_{f,e}$ where $w_{f,e}$ is a non-negative integer and $u_{f,e} \in \mathbb{Z}^2$ is a primitive vector,*
- *for any vertex v of C whose adjacent edges are $e_1, \ldots, e_k$, one has the balancing condition*

$$\sum_{i=1}^{k} w_{f,e_i} u_{f,e_i} = 0$$

where u_{e_i} is chosen so that it points away from v.

Let $f : C \setminus \mathrm{End}(C) \to \mathbb{R}^2$ be a tropical morphism, and define $L(C, f)$ as the unordered list composed by elements $u_{f,e}$ repeated $w_{f,e}$ times where e goes through leaves of C and u_e is chosen so that it points to the 1-valent vertex. Then, there exists a unique, up to translation by a vector in $\mathbb{Z}^2$, lattice polygon $\Delta(C, f)$ such that the unordered list composed by the primitive vector normal to e and outward to $\Delta(C, f)$ repeated $l(e)$ times where e goes through edges of $\Delta(C, f)$ equals the list $L(C, f)$.

Definition 4.4. *The polygon $\Delta(C, f)$ is called the Newton polygon of the pair (C, f).*

Not any tropical curve admits a non-constant tropical morphism to $\mathbb{R}^2$. The tropical curve depicted in Figure 5a does not admit any tropical morphism since a circle cannot be mapped to a segment in $\mathbb{R}^2$ by a dilatation. However, up to *modification*, every tropical curve can be tropically immersed to $\mathbb{R}^2$ (see [Mik]).

The pair (C, f) where $f : C \setminus \mathrm{End}(C) \to \mathbb{R}^2$ is a tropical morphism with Newton polygon Δ is called a *parameterized tropical curve with Newton polygon* Δ. The integer $w_{f,e}$ is called the *weight* of the edge e. The genus of (C, f) is naturally defined as the genus of C.

Example 4.5. If C is the tropical curve depicted in Figure 5b (resp. c) then an example of the image $f(C)$ for some parameterization with Newton polygon Δ_3 is depicted in Figure 6a (resp. b). The second tropical morphism has an edge of weight 2.

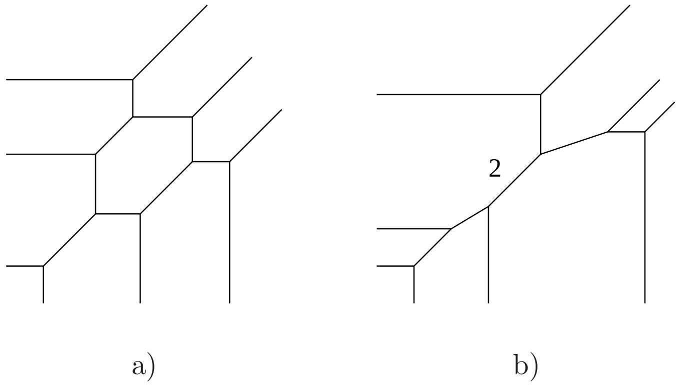


FIGURE 6. Images of tropical morphisms with Newton polygon Δ_3

Definition 4.6. *A tropical curve with n marked points is a $(n+1)$-tuple $(C, x_1, \ldots, x_n)$ where C is a tropical curve and the x_i's are n points on C.*

A parameterized tropical curve with n marked points is a $(n+2)$-tuple $(C, x_1, \ldots, x_n, f)$ where $(C, x_1, \ldots, x_n)$ is a tropical curve with n marked points, and (C, f) is a parameterized tropical curve.

Note that in this paper we do not require the marked points on a marked tropical curve to be distinct. In the following, we consider tropical curves (with n marked points) up to homeomorphism of metric graphs (which send the i^{th} point to the i^{th} point). The notions of vertices, edges, Newton polygon, … also make sense for a parameterized marked tropical curve as the corresponding notions for the underlying (parameterized) tropical curve.

4.2. **Complex multiplicity of a tropical curve.** Let us now turn to tropical enumerative geometry, and let's relate it first to complex enumerative geometry. More details about this section can be found in [Mik05] or [GM07].

Fix a lattice polygon Δ, a non-negative integer number g, and define $s = Card(\partial\Delta \cap \mathbb{Z}^2) - 1 + g$. Choose a collection $\omega = \{p_1, \ldots, p_s\}$ of s points in $\mathbb{R}^2$, and denote by $\mathcal{C}(\omega)$ the set of parameterized tropical curves with s-marked points $(C, x_1, \ldots, x_s, f)$ satisfying the following conditions

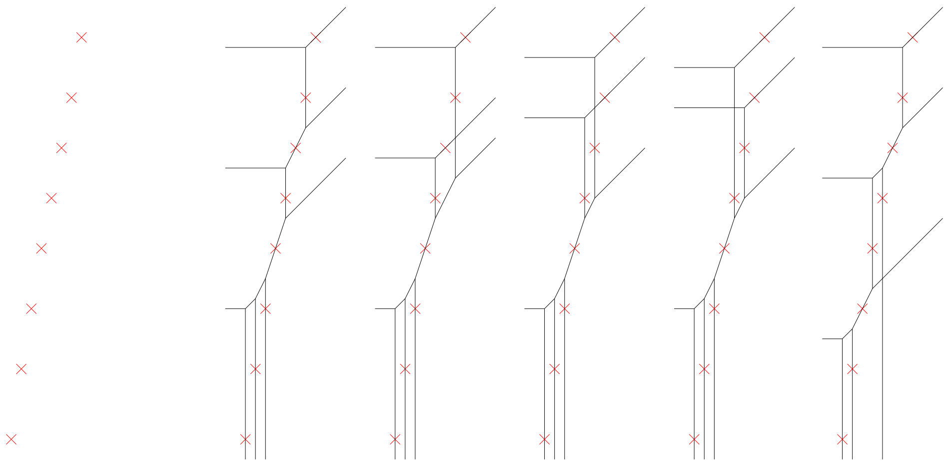
- the tropical curve C is irreducible and of genus g,
- $\Delta(C, f) = \Delta$,
- for any $1 \le i \le s$, $f(x_i) = p_i$.

Proposition 4.7 (Mikhalkin, [Mik05])**.** *For a generic configuration of points ω, the set $\mathcal{C}(\omega)$ is finite. Moreover, for any parameterized tropical curve $(C, x_1, \ldots, x_s, f)$ in $\mathcal{C}(\omega)$, the curve C has only 1 or 3-valent vertices, the set $\{x_1, \ldots, x_s\}$ is disjoint from Vert(C), any leaf of C is of weight 1, and f is a topological immersion. In particular, any neighborhood of any 3-valent vertex of C is never mapped to a segment by f.*

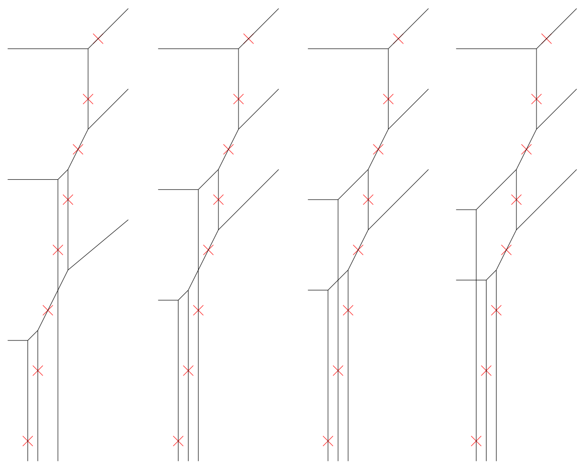
Given a generic configuration ω, we associate a complex multiplicity $\mu^{\mathbb{C}}(\widetilde{C})$ to any element $\widetilde{C} = (C, x_1, \ldots, x_s, f)$ in $\mathcal{C}(\omega)$. Let v be a vertex of $C \setminus \mathrm{End(C)}$ and e_1 and e_2 two of its adjacent edges. As v is trivalent, the balancing condition implies that the number $\mu^{\mathbb{C}}(v, f) = w_{f,e_1} w_{f,e_2} |\det(u_{f,e_1}, u_{f,e_2})|$ does not depend on the choice of e_1 and e_2.

Definition 4.8. *The complex multiplicity of an element $\widetilde{C}$ of $\mathcal{C}(\omega)$, denoted by $\mu^{\mathbb{C}}(\widetilde{C})$, is defined as*

$$\mu^{\mathbb{C}}(\widetilde{C}) = \prod_{v \in Vert(\widetilde{C})} \mu^{\mathbb{C}}(v, f)$$



a) b) $\mu^{\mathbb{C}} = 4$ c) $\mu^{\mathbb{C}} = 1$ d) $\mu^{\mathbb{C}} = 1$ e) $\mu^{\mathbb{C}} = 1$ f) $\mu^{\mathbb{C}} = 1$



g) $\mu^{\mathbb{C}} = 1$ h) $\mu^{\mathbb{C}} = 1$ i) $\mu^{\mathbb{C}} = 1$ j) $\mu^{\mathbb{C}} = 1$

FIGURE 7. $N(\Delta_3, 0) = 12$

Theorem 4.9 (Mikhalkin, [Mik05]). *For any lattice polygon Δ, any genus g, and any generic configuration ω of $Card(\partial\Delta \cap \mathbb{Z}^2) - 1 + g$ points in $\mathbb{R}^2$, one has*

$$N(\Delta, g) = \sum_{\widetilde{C} \in \mathcal{C}(\omega)} \mu^{\mathbb{C}}(\widetilde{C})$$

Example 4.10. Images $f(C)$ of all irreducible tropical curves of genus 0 and Newton polygon Δ_3 in $\mathcal{C}(\omega)$ for the configuration ω of 8 points depicted in Figure 7a are depicted in Figure 7b, ..., j. We verify that $N(\Delta_3, 0) = 12$ (compare with Table 1).

4.3. **Real multiplicities of a tropical curve.** We explain now how to adapt Theorem 4.9 to real enumerative geometry. Naturally, we need to consider tropical curves endowed with a real structure.

Definition 4.11. *A real parameterized tropical curve with n marked points is a $(n+3)$-uplet $(C, x_1, \dots, x_n, f, \phi)$ where $(C, x_1, \dots, x_n, f)$ is a parameterized marked tropical curve and $\phi : C \to C$ is an isometric involution such that*

- *there exists a permutation σ such that for any $1 \le i \le n$, $\phi(x_i) = x_{\sigma(i)}$,*
- *$f = f \circ \phi$.*

The *real and imaginary parts* of a real parameterized tropical curve with n marked points $\widetilde{C} = (C, x_1, \dots, x_n, f, \phi)$ are naturally defined as

$$\Re(\widetilde{C}) = \mathrm{Fix}(\phi) \quad \text{and} \quad \Im(\widetilde{C}) = C \setminus \Re(\widetilde{C})$$

Example 4.12. Two examples of real parameterized tropical curves with 4 marked points are depicted in Figure 8, the abstract curve is depicted on the left and its image by f in $\mathbb{R}^2$ is depicted on the right. Very close edges in the image represent edges which are mapped to the same edge by f. The parameterized tropical curve in Figure 8a has 2 equal marked points, and ϕ is the symmetry with respect to the non-leaf edge. In Figure 8b, ϕ exchanges the edges containing x_1 and x_2.

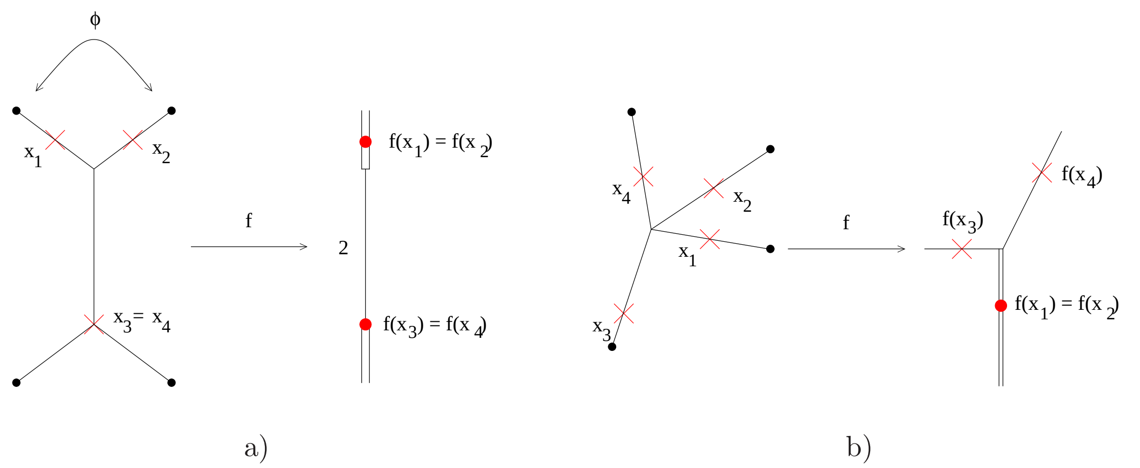


Figure 8. Real tropical curves

As usual, we fix a lattice polygon Δ and define $s = Card(\partial\Delta \cap \mathbb{Z}^2) - 1$. Let r be an non-negative integer such that $s - 2r \ge 0$, and choose a collection $\omega_r = \{p_1, \dots, p_{s-r}\}$ of $s - r$ points in $\mathbb{R}^2$. We should think of ω_r as the image under the map $(z, w) \mapsto (\log|z|, \log|w|)$ of a configuration $\{q_1, \dots, q_{s-2r}, q_{s-2r+1}, \overline{q_{s-2r+1}}, \dots, q_{s-r}, \overline{q_{s-r}}\}$ of s points in $(\mathbb{C}^*)^2$, where $\overline{z}$ is the complex conjugated of z. Hence, points p_i with $s - 2r + 1 \le i \le s - r$ represent pairs of complex conjugated points. Denote by $\mathbb{R}\mathcal{C}(\omega_r)$ the set of irreducible real parameterized tropical curves with s marked points $\widetilde{C} = (C, x_1, \dots, x_s, f, \phi)$ of genus 0 and Newton polygon Δ satisfying the following conditions

- for any $1 \le i \le s - 2r$, $f(x_i) = p_i$,
- for any $1 \le i \le r$, $f(x_{s-2r+2i-1}) = f(x_{s-2r+2i}) = p_{s-2r+i}$,
- if $1 \le i \le r$ and if $x_{s-2r+2i-1} = x_{s-2r+2i}$, then $x_{s-2r+2i}$ is a vertex of C,
- any edge in $\Re(\widetilde{C})$ has an odd weight.

Proposition 4.13. *For a generic configuration of points ω_r, the set $\mathbb{R}\mathcal{C}(\omega_r)$ is finite. Moreover, for any real parameterized curve $\widetilde{C} = (C, x_1, \dots, x_s, f, \phi)$ in $\mathbb{R}\mathcal{C}(\omega_r)$, the curve C has only 1, 3 or 4 valent vertices, any neighborhood of any 3 or 4-valent vertex of C is never mapped to a segment by f, any 4-valent vertex of C is adjacent to 2 edges in $\Im(\widetilde{C})$ and 2 edges in $\Re(\widetilde{C})$, and any leaf of C is of weight 1.*

Proof. Let $\widetilde{C}$ be an element of $\mathbb{R}\mathcal{C}(\omega_r)$. Passing through $s-r$ points in $\mathbb{R}^2$ in general position imposes $2(s-r)$ independent conditions on a tropical curve. Since all tropical maps are piecewise-linear, to prove the proposition it suffices to show that the dimension of the space of all real parameterized tropical curves with the same combinatorial type as $\widetilde{C}$ has dimension $2(s-r)$, and that any curve with this combinatorial type satisfies the proposition.

Recall that the space of all parameterized irreducible tropical curves (C, f) of genus 0 with x leaves and of a given combinatorial type is a polyhedral complex of dimension

$$x - 1 - \sum_{v \in \mathrm{Vert}(C) \setminus \mathrm{End}(C)} (\mathrm{val}(v) - 3) - n_c$$

where $\mathrm{val}(v)$ is the valence of a vertex v, and n_c is the number of edges of C contracted by f (see [Mik05]). Let $\widetilde{C} = (C, x_1, \ldots, x_s, f, \phi)$ be an element of $\mathbb{R}\mathcal{C}(\omega_r)$. We may prepare two auxiliary tropical curves $f^r : C^r \to \mathbb{R}^2$ and $f^i : C^i \to \mathbb{R}^2$ from $f : C \to \mathbb{R}^2$. We say that $v \in C$ is a *junction* vertex if any small neighborhood of v intersects both $\Re(\widetilde{C})$ and $\Im(\widetilde{C})$. We denote by J the number of junction vertices of C. Since any edge of $\Re(\widetilde{C})$ has odd weight, a junction vertex is at least 4-valent.

We define C^r to be the result of adding to $\Re(\widetilde{C})$ an infinite ray at each junction vertex of C. We define f^r so that it coincides with f on $\Re(\widetilde{C})$. The values of f^r at the new rays are determined by the balancing condition.

Connected components of $\Im(\widetilde{C})$ are naturally coupled in pairs exchanged by the map ϕ. To define C^i, we take $\Im(\widetilde{C})/\phi$ and replace all edges adjacent to a junction vertex with an infinite ray. We let $f^i : C^i \to \mathbb{R}^2$ to be the tropical map that agrees with f on $\Im(\widetilde{C})/\phi$. We denote by n^i the number of connected components of C^i. Note that $n^i \geq J$, and that equality holds if and only if each junction vertex is 4-valent.

We denote by x^r (resp. x^i) the number of leaves of C which are also leaves of C^r (resp. C^i). Since the curve C has genus 0, the curve C^r is connected and each component of C^i is adjacent to exactly one junction vertex. Hence, the space of parameterized tropical curves with the same combinatorial type as (C^r, f^r) has dimension

$$x^r + J - 1 - \sum_{v \in \mathrm{Vert}(C^r) \setminus \mathrm{End}(C^r)} (\mathrm{val}(v) - 3) - n_{c^r}$$

and the space of parameterized tropical curves with the same combinatorial type as (C^i, f^i) has dimension

$$x^i - \sum_{v \in \mathrm{Vert}(C^i) \setminus \mathrm{End}(C^i)} (\mathrm{val}(v) - 3) - n_{c^i}$$

To get f from f^r and f^i these maps must agree at each junction. Thus each connected component of C^i imposes one condition, and the space of real parameterized tropical curves with the same combinatorial type as (C, f) has dimension

$$x^r + x^i + J - n^i - 1 - \sum_{v \in \mathrm{Vert}(C^r) \setminus \mathrm{End}(C^r)} (\mathrm{val}(v) - 3) - \sum_{v \in \mathrm{Vert}(C^i) \setminus \mathrm{End}(C^i)} (\mathrm{val}(v) - 3) - n_{c^r} - n_{c^i}$$

If we consider, in addition, a configuration of s points on C then our dimension increases by s. Recall though that our points are constrained by the condition that the last $2r$ points are split into pairs invariant with respect to the involution ϕ. Furthermore, recall that if such a pair consists of the same point taken twice then it must be a vertex of C.

Denote with p the number of pairs of distinct points in C invariant with respect to ϕ and with q the number of pairs made from the vertices of C. Clearly we have $p + q = r$, and the dimension of allowed configurations is $s - p - 2q$. Since $f^i(C^i)$ passes through p generic points in $\mathbb{R}^2$, we have

$x_i \geq p$, and since $x^r + 2x^i \leq s+1$ we have $x^r + x^i \leq s+1-p$. Hence, the space of real parameterized tropical curves with s marked points with the same combinatorial type as $\widetilde{C}$ has dimension at most

$$2(s-r) + J - n^i - \sum_{v \in \mathrm{Vert}(C^r)\setminus \mathrm{End}(C^r)} (\mathrm{val}(v)-3) - \sum_{v \in \mathrm{Vert}(C^i)\setminus \mathrm{End}(C^i)} (\mathrm{val}(v)-3) - n_{c^r} - n_{c^i}$$

Since $\widetilde{C}$ is in $\mathbb{R}\mathcal{C}(\omega_r)$, its space of deformation must have dimension at least $2(s-r)$. Hence the curve C has exactly $s+1$ leaves, $n^i = J$, and any vertex of C which is not an end or a junction vertex is trivalent. $\square$

For a generic configuration ω_r and $(C, x_1, \ldots, x_s, f, \phi)$ in $\mathbb{R}\mathcal{C}(\omega_r)$, Proposition 4.13 implies that the real structure ϕ on C is uniquely determined by the marked parameterized tropical curve $(C, x_1, \ldots, x_s, f)$. Hence we will often omit to precise the map ϕ for elements of $\mathbb{R}\mathcal{C}(\omega_r)$. Moreover, $\Im(\widetilde{C})/\phi$ is a (possibly disconnected) non-compact graph, and a vertex v (resp. edge) inside $\Im(\widetilde{C})/\phi$ has a natural complex multiplicity $\mu^{\mathbb{C}}(v,f)$ (resp. weight) induced by the corresponding multiplicity of vertices (resp. edges) of C. If v is a 4-valent vertex of C, then by Proposition 4.13, there exist an edge $e_1 \in \Re(\widetilde{C})$ and an edge $e_2 \in \Im(\widetilde{C})$ adjacent to v. Define $\mu_r^{\mathbb{R}}(v,f) = w_{f,e_1} w_{f,e_2} |det(v_{f,e_1}, v_{f,e_2})|$. Define the integer $o_r^{\mathbb{R}}$ to be the number of vertices v in $\Re(\widetilde{C})$ satisfying one of the following conditions

- v is 3-valent and $\mu^{\mathbb{C}}(v,f) = 3 \ mod\ 4$,
- v is 4-valent adjacent to an edge $e \in \Im(\widetilde{C})$, and $\mu^{\mathbb{R}}(v,f) = w_{f,e} + 1 \ mod\ 2$.

Finally, define the integer $o_r^{\mathbb{C}}$ to be the number of vertices v of $\Im(\widetilde{C})/\phi$ with odd $\mu^{\mathbb{C}}(v,f)$.

Definition 4.14. *The r-real multiplicity of an element $\widetilde{C}$ of $\mathbb{R}\mathcal{C}(\omega_r)$, denoted by $\mu_r^{\mathbb{R}}(\widetilde{C})$, is defined as*

$$\mu_r^{\mathbb{R}}(\widetilde{C}) = (-1)^{o_r^{\mathbb{R}} + o_r^{\mathbb{C}}} \prod_{v \in Vert(\Im(\widetilde{C})/\phi)} \mu^{\mathbb{C}}(v,f) \prod_{v \in Vert(\widetilde{C}),\ f(v) \in \omega_r} \mu^{\mathbb{C}}(v,f) \prod_{v \in Vert(\widetilde{C}),\ v \text{ is 4-valent}} \mu^{\mathbb{R}}(v,f)$$

The tropical curves and their multiplicity we are considering here differ slightly from the one in [Shu06]. This difference comes from the fact that we are dealing with parameterization of tropical curves, and that Shustin deals with the cycles resulting as the images of the curves rather than parameterized curves.

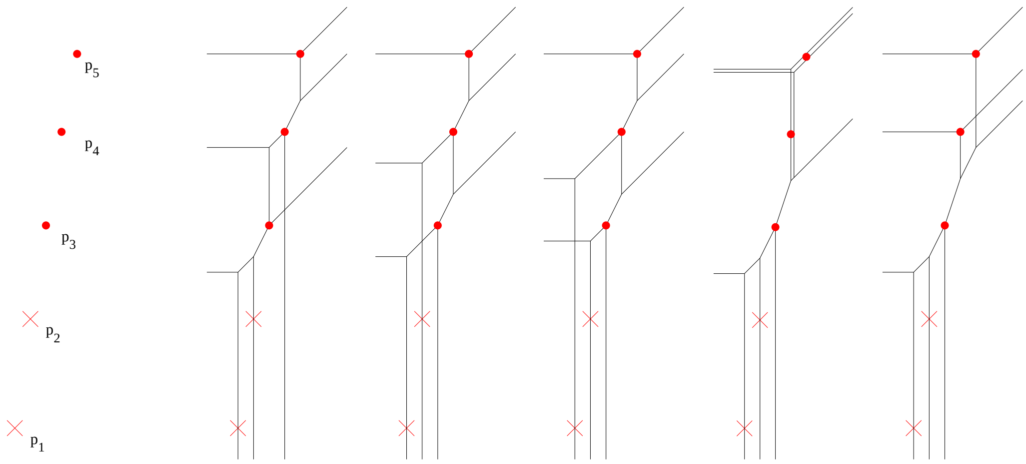
Remark 4.15. If $r = 0$, then for any real parameterized curve $(C, x_1, \ldots, x_s, f, \phi)$ in $\mathbb{R}\mathcal{C}(\omega_r)$, we have $\phi = Id$, and the map $(C, x_1, \ldots, x_s, f, \phi) \mapsto (C, x_1, \ldots, x_s, f)$ is a bijection from the set $\mathbb{R}\mathcal{C}(\omega_r)$ to the set of elements of $\mathcal{C}(\omega_r)$ with odd complex multiplicity.

Theorem 4.16 (Mikhalkin, [Mik05], Shustin, [Shu06])**.** *Let Δ be a lattice polygon such that Welschinger invariants are defined for the corresponding toric surface $Tor(\Delta)$ equipped with its tautological real structure. Then for any integer r such that $s - 2r \geq 0$, and any generic configuration ω_r of $s - r$ points in $\mathbb{R}^2$, one has*

$$W(\Delta, r) = \sum_{\widetilde{C} \in \mathbb{R}\mathcal{C}(\omega_r)} \mu_r^{\mathbb{R}}(\widetilde{C})$$

Remark 4.17. Theorem 4.16 implies that the right hand side of last equality does not depend on ω_r for smooth Del Pezzo toric surfaces $Tor(\Delta)$. However, this is not true in general and one can easily check that the sum of r-real multiplicities over all tropical curves in $\mathbb{R}\mathcal{C}(\omega_r)$ in the case of $r > 0$ does not have to be invariant if $Tor(\Delta)$ is singular (see also [ABLdM, Section 7.2]).

Example 4.18. If $\omega_3 = \{p_1, p_2, p_3, p_4, p_5\}$ is the configuration depicted in Figure 9a, then images of all parameterized tropical curves of genus 0 and Newton polygon Δ_3 in $\mathbb{R}\mathcal{C}(\omega_3)$ are depicted in Figures 9b, c, d, e, and f (compare with Table 1). Figure 9e) represents the image of 2 distinct marked parameterized tropical curves in $\mathbb{R}\mathcal{C}(\omega_3)$, depending on the position of marked points on the connected components of $\Im(\widetilde{C})$. Hence we verify that $W(\Delta_3,3)=2$.



a) b) $\mu_3^{\mathbb{R}}=1$ c) $\mu_3^{\mathbb{R}}=1$ d) $\mu_3^{\mathbb{R}}=1$ e) $\mu_3^{\mathbb{R}}=-1$ 2 choices f) $\mu_3^{\mathbb{R}}=1$

FIGURE 9. $W(\Delta_3,3)=2$

5. Proof of Theorems 3.6 and 3.9

Theorems 3.6 and 3.9 are obtained by applying Theorems 4.9 and 4.16 to configurations ω which are stretched in the direction $(0,1)$.

5.1. **Floors of a parameterized tropical curve.** As we fixed a preferred direction in $\mathbb{R}^2$, it is natural to distinguish between edges of parameterized tropical curves which are mapped parallely to this direction from the others.

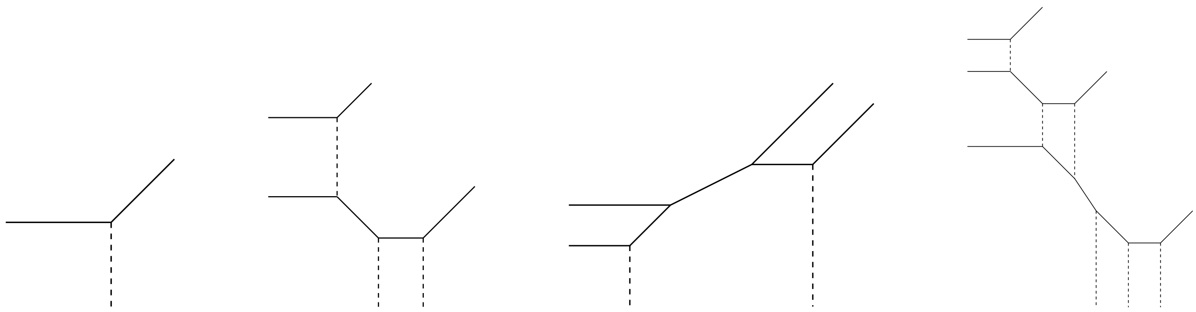
Definition 5.1. *An elevator of a parameterized tropical curve (C,f) is an edge e of C with $u_{f,e}=\pm(0,1)$. The set of elevators of (C,f) is denoted by $\mathcal{E}(f)$. If an elevator e is not a leaf of C, then e is said to be bounded. A floor of a parameterized tropical curve (C,f) is a connected component of the topological closure of $C\setminus(\mathcal{E}(f)\cup End(C))$.*

Naturally, a floor of a parameterized marked tropical curve is a floor of the underlying parameterized tropical curve.

Example 5.2. In Figure 10 are depicted some images of parameterized tropical curves. Elevators are depicted in dotted lines.

Let us fix a h-transverse polygon Δ, and a non-negative integer number g. Define $s=Card(\partial\Delta\cap\mathbb{Z}^2)-1+g$, and choose a generic configuration ω of s points in $\mathbb{R}^2$. If moreover $g=0$, choose r an non-negative integer such that $s-2r\geq 0$, and choose a collection ω_r of $s-r$ points in $\mathbb{R}^2$.

Proposition 5.3. *Let $I=[a;b]$ be a bounded interval of $\mathbb{R}$. Then, if ω (resp. ω_r) is a subset of $I\times\mathbb{R}$, then any vertex of any curve in $\mathcal{C}(\omega)$ (resp. $\mathbb{R}\mathcal{C}(\omega_r)$) is mapped to $I\times\mathbb{R}$.*



a) One floor b) Two floors c) One floor d) Three floors

FIGURE 10. Floors of tropical curves

Proof. Suppose that there exists an element $(C, x_1, \dots, x_s, f)$ in $\mathcal{C}(\omega)$ or $\mathbb{R}\mathcal{C}(\omega_r)$, and a vertex v of C such that $f(v) = (x_v, y_v)$ with $x_v < a$. Choose v such that no vertex of C is mapped by f to the half-plane $\{(x, y) \mid x < x_v\}$. Suppose that v is a trivalent vertex of C, and denote by e_1, e_2 and e_3 the three edges of C adjacent to v. For $1 \leq i \leq 3$, choose the vector u_{e_i} pointing away from v (see section 4.1). By assumption on v, this vertex is adjacent to a leaf of C, for example e_1, and since Δ is h-transverse we have $u_{f,e_1} = (-1, \alpha)$. Moreover, according to Propositions 4.7 and 4.13, we have $w_{f,e_1} = 1$. By the balancing condition, up to exchanging e_2 and e_3, we have $u_{f,e_2} = (-\beta, \gamma)$ with $\beta \geq 0$, and $u_{f,e_3} = (\delta, \varepsilon)$ with $\delta > 0$. Moreover, as no vertex of C is mapped to the half-plane $\{(x, y) \mid x < x_v\}$, the edge $f(e_2)$ is a leaf of C if $\beta > 0$. Then, by translating the vertex $f(v)$ (resp. and possibly $\phi(v)$) in the direction u_{f,e_3}, we construct a 1-parameter family of parameterized tropical curves in $\mathcal{C}(\omega)$ (resp. $\mathbb{R}\mathcal{C}(\omega_r)$), as depicted in two examples in Figure 11. This contradicts Propositions 4.7 and 4.13. If v is a 4-valent vertex of C, then we construct analogously a 1-parameter family of parameterized tropical curves in $\mathbb{R}\mathcal{C}(\omega_r)$. Alternatively, the contradiction may be derived from [Mik05, Lemma 4.17]. Hence, no vertex of C is mapped by f in the half-plane $\{(x, y) \mid x < a\}$.

The case where there exists an element $(C, x_1, \dots, x_s, f)$ in $\mathcal{C}(\omega)$ or $\mathbb{R}\mathcal{C}(\omega_r)$, and a vertex v of C such that $f(v) = (x_v, y_v)$ with $x_v > b$ works analogously. $\square$

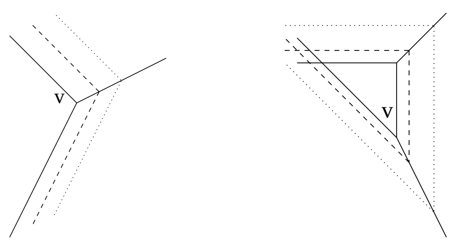


FIGURE 11. 1-parameter family of tropical curves

Corollary 5.4. *Let I be a bounded interval of $\mathbb{R}$. If ω (resp. ω_r) is a subset of $I \times \mathbb{R}$ and if the points of ω (resp. ω_r) are far enough one from the others, then any floor of any curve in $\mathcal{C}(\omega)$ (resp. $\mathbb{R}\mathcal{C}(\omega_r)$) can not contain more than one (resp. two) distinct marked point. If a floor of an element $\widetilde{C}$ in $\mathbb{R}\mathcal{C}(\omega_r)$ contains two distinct marked points, then they are contained in $\Im(\widetilde{C})$.*

Proof. Let $(C, x_1, \dots, x_s, f)$ be an element of $\mathcal{C}(\omega)$ or $\mathbb{R}\mathcal{C}(\omega_r)$ and choose a path γ in $C \setminus \mathcal{E}(C)$. The number of edges of C is bounded from above by a number which depends only on Δ and g, and according to the tropical Bézout Theorem, absolute value of the coordinates of the vector $w_{f,e}u_{f,e}$ for any edge e of C is bounded from above by a number which depends only on Δ. According to Proposition 5.3, all vertices of C are mapped by f to the strip $I\times\mathbb{R}$, so the length (for the Euclidean metric in $\mathbb{R}^2$) of $f(\gamma)$ is bounded from above by a number $l_{max}(\Delta, g)$ which depends only on Δ and g. Hence, if the distance between the points p_i is greater than $l_{max}(\Delta, g)$, two distinct marked points x_i which are not mapped to the same p_j cannot be on the same floor of C. □

For the remaining of this section, let us fix a bounded interval I of $\mathbb{R}$, a configuration $\omega = \{p_1, \ldots, p_s\}$, or possibly a configuration $\omega_r = \{p_1, \ldots, p_{s-r}\}$, such that the point p_i is very much higher than the points p_j if $j < i$. Here, *very much higher* means that we can apply Corollary 5.4. Actually, we prove in next corollary that any floor of any curve in $\mathcal{C}(\omega)$ or $\mathbb{R}\mathcal{C}(\omega_r)$ contains *exactly* one marked point. More precisely, we have the following statement.

Corollary 5.5. *Let $\widetilde{C}$ be an element of $\mathcal{C}(\omega)$ or $\mathbb{R}\mathcal{C}(\omega_r)$. Then, any floor of $\widetilde{C}$ contains exactly one marked point. Moreover, the curve $\widetilde{C}$ has exactly $Card(d_l(\Delta))$ floors and $Card(d_l(\Delta)) + g + d_-(\Delta) + d_+(\Delta) - 1$ elevators.*

Proof. Let us denote by f_i (resp b_i, $\widetilde{d_i}$) the number of floors (resp. bounded elevators, elevators) of $\widetilde{C}$ containing i marked points. According to Corollary 5.4, $f_i = 0$ as soon as $i \geq 3$, and since the points p_i are in general position, we have $b_i = \widetilde{d_i} = 0$ as soon as $i \geq 2$. What we have to prove is that $f_0 = f_2 = b_0 = \widetilde{d_0} = 0$. We have two expressions for the number s which gives us the equation

$$f_1 + 2f_2 + \widetilde{d_1} = d_+(\Delta) + d_-(\Delta) + 2Card(d_l(\Delta)) - 1 + g \tag{3}$$

According to tropical Bézout Theorem and Corollary 5.4, if a floor of C contains two marked points, then the intersection number of this floor with a generic tropical line is at least 2. Hence we have

$$f_0 + f_1 + 2f_2 \leq Card(d_l(\Delta)) \tag{4}$$

According to Propositions 4.7 and 4.13, we have $d_+ + d_-$ leaves of C which are elevators, thus

$$b_0 + b_1 = \widetilde{d_0} + \widetilde{d_1} - (d_+(\Delta) + d_-(\Delta)) \tag{5}$$

An Euler characteristic computation shows us that

$$f_0 + f_1 + f_2 - b_0 - b_1 \geq 1 - g \tag{6}$$

Combining Equations (3) with (4), then with Equation (5), and finally with Equation (6), we obtain

$$f_1 + f_2 \geq Card(d_l(\Delta))$$

which is compatible with Equation (4) if an only if $f_0 = f_2 = 0$. Moreover, in this case inequalities (6) and (4) are actually equalities, which implies $b_0 = \widetilde{d_0} = 0$. □

5.2. **From tropical curves to floor diagrams.** To a parameterized tropical curve (C, f), we associate the following oriented weighted graph, denoted by $\mathcal{F}(C, f)$: vertices of $\mathcal{F}(C, f)$ correspond to floors of (C, f), and edges of $\mathcal{F}(C, f)$ correspond to elevators of (C, f). Edges of $\mathcal{F}(C, f)$ inherit a natural weight from weight of (C, f). Moreover, $\mathbb{R}$ is naturally oriented, and edges of $\mathcal{F}(C, f)$ inherit this orientation, since they are all parallel to the coordinate axis $\{0\} \times \mathbb{R}$. Note that we do *not* consider the graph $\mathcal{F}(C, f)$ as a metric graph and that some leaves are non-compact.

Example 5.6. The graphs corresponding to parameterized tropical curves depicted in Figure 10 are depicted in Figure 12. Floors are depicted by ellipses, and elevators by segments. As all elevators have weight 1, we do not precise them on the picture. Orientation is implicitly from down to up.

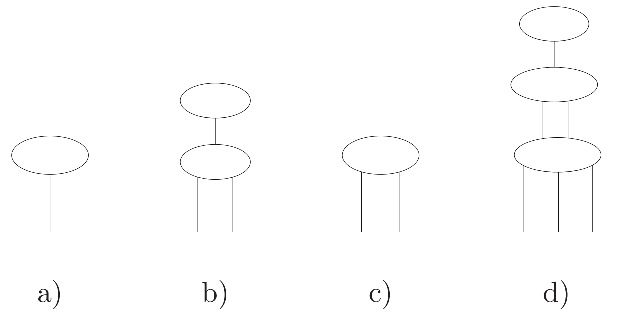


FIGURE 12. Graphs associated to tropical curves

Let $\widetilde{C}$ be a parameterized tropical curve in $\mathcal{C}(\omega)$ or $\mathbb{R}\mathcal{C}(\omega_r)$. Since $\widetilde{C}$ has exactly $Card(d_l(\Delta))$ floors, any floor ε of $\widetilde{C}$ has a unique leaf e with $u_{f,e} = (-1, -\alpha)$ where $u_{f,e}$ points to infinity. Hence the following map is well defined

$$\begin{array}{cccc} \theta : & \mathrm{Vert}(\mathcal{F}(\widetilde{C})) & \longrightarrow & \mathbb{Z} \\ & \varepsilon & \longmapsto & \alpha \end{array}$$

The following lemma follows directly from Corollary 5.5 and Definition 3.1 of a floor diagram.

Lemma 5.7. *The graph $\mathcal{F}(\widetilde{C})$ equipped with the map θ is a floor diagram of genus g and Newton polygon Δ.*

Let us denote by $\mathcal{D}(\widetilde{C})$ this floor diagram. Finally we associate to a parameterized tropical curve with n marked points $\widetilde{C} = (C, x_1, \ldots, x_n, f)$ in $\mathcal{C}(\omega)$ or $\mathbb{R}\mathcal{C}(\omega_r)$ a marking m of the floor diagram $\mathcal{D}(\widetilde{C})$. The natural idea is to map the points i to the floor or elevator of C containing x_i. However, it can happen if $\widetilde{C}$ is in $\mathbb{R}\mathcal{C}(\omega_r)$ that $x_i = x_{i+1}$ is a vertex v of C. In this case, according to Proposition 4.13 and Corollary 5.5, v is on a floor ε and is adjacent to an elevator e of $\mathcal{D}(\widetilde{C})$. If $u_{f,e} = (0, 1)$ points away from v (resp. to v), then we define $m(i) = \varepsilon$ and $m(i + 1) \in e$ (resp. $m(i + 1) = \varepsilon$ and $m(i) \in e$). If x_i is not a vertex of C, then we define $m(i)$ as the floor or a point on the edge of C which contains x_i.

The map $m : \{1, \ldots, Card(\partial\Delta \cap \mathbb{Z}) - 1 + g\} \to \mathcal{D}(\widetilde{C})$ is clearly an increasing map, hence it is a marking of the floor diagram $\mathcal{D}(\widetilde{C})$. In other words, we have a map $\Phi : \widetilde{C} \mapsto (\mathcal{D}(\widetilde{C}), m)$ from the set $\mathcal{C}(\omega)$ (resp. $\mathbb{R}\mathcal{C}(\omega_r)$) to the set of marked floor diagrams (resp. r-real marked floor diagrams with non-null r-real multiplicity) of genus g and Newton polygon Δ.

Example 5.8. All marked floor diagrams with a non-null complex multiplicity (resp. 3-real multiplicity) in Table 1 correspond exactly to parameterized tropical curves whose image in $\mathbb{R}^2$ are depicted in Figure 7 (resp. 9).

Theorems 3.6 and 3.9 are now a corollary of the next proposition.

Proposition 5.9. *The map Φ is a bijection. Moreover, for any element $\widetilde{C}$ in $\mathcal{C}(\omega)$ (resp. $\mathbb{R}\mathcal{C}(\omega_r)$), one has $\mu^{\mathbb{C}}(\widetilde{C}) = \mu^{\mathbb{C}}(\Phi(\widetilde{C}))$ (resp. $\mu_r^{\mathbb{R}}(\widetilde{C}) = \mu_r^{\mathbb{R}}(\Phi(\widetilde{C}))$).*

Proof. The fact that Φ is a bijection is clear when $Card(d_l(\Delta)) = 1$. Hence the map Φ is always a bijection since an element of $\mathcal{C}(\omega)$ (resp. $\mathbb{R}\mathcal{C}(\omega_r)$) is obtained by gluing, along elevators, tropical curves with a single floor which are uniquely determined by the points p_i they pass through.

Let $\widetilde{C} = (C, x_1, \ldots, x_s, f)$ be an element of $\mathcal{C}(\omega)$, and v a vertex of C. According to Corollary 5.4 and Corollary 5.5, v is adjacent to an elevator of weight w and to an edge e on a floor with $u_{f,e} = (\pm 1, \alpha)$ and $w_{f,e} = 1$. Hence, $\mu^{\mathbb{C}}(v, f) = w$. Since any leaf of C is of weight 1, it follows that $\mu^{\mathbb{C}}(\widetilde{C})$ is the product of the square of the multiplicity of all elevators of C, that is equal to $\mu^{\mathbb{C}}(\Phi(\widetilde{C}))$.

Let $\widetilde{C} = (C, x_1, \ldots, x_s, f, \phi)$ be an element of $\mathbb{R}\mathcal{C}(\omega_r)$. The same argument as before shows that $\mu_r^{\mathbb{R}}(\widetilde{C})$ and $\mu_r^{\mathbb{R}}(\Phi(\widetilde{C}))$ have equal absolute values. It remains us to prove that both signs coincide, and the only thing to check is that the number $o_r^{\mathbb{R}}$ is even. If v is a 4-valent vertex of C adjacent to an edge e in $\Im(\widetilde{C})$, then $\mu^{\mathbb{R}}(v) = w_{f,e}$. If v is a 3-valent vertex in $\Re(\widetilde{C})$ adjacent to an elevator e, then $\mu^{\mathbb{C}}(v) = w_{f,e}$. So if $\mu^{\mathbb{C}}(v) = 3 \ mod\ 4$, then e is bounded and the other vertex v' adjacent to e satisfy also $\mu^{\mathbb{C}}(v) = 3 \ mod\ 4$. Hence the number $o_r^{\mathbb{R}}$ is even as announced. □

6. Some applications

Here we use floor diagrams to confirm some results in classical enumerative geometry.

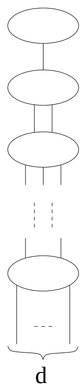


FIGURE 13. Unique floor diagram of maximal genus and Newton polygon Δ_d

6.1. **Degree of the discriminant hypersurface of the space of plane curves.**

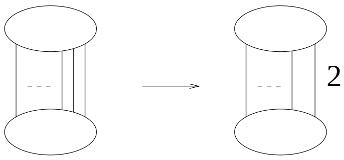
Proposition 6.1. *For any $d \geq 3$, one has*

$$N(\Delta_d, \frac{(d-1)(d-2)}{2} - 1) = 3(d-1)^2$$

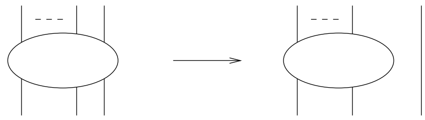
Proof. We see easily that the unique floor diagram $\mathcal{D}_{max}$ of genus $\frac{(d-1)(d-2)}{2}$ and Newton polygon Δ_d is the one depicted in Figure 13. Moreover, all floor diagrams of genus $\frac{(d-1)(d-2)}{2} - 1$ and Newton polygon Δ_d are obtained by decreasing the genus of $\mathcal{D}_{max}$ via one of the 2 moves depicted in Figure 14. There are $i-1$ different markings of the floor diagram obtained via the move of Figure 14a, and $2i+1$ different markings of the floor diagram obtained via the move of Figure 14b. Then we get

$$\begin{aligned} N(d, \tfrac{(d-1)(d-2)}{2} - 1) &= \textstyle\sum_{i=2}^{d-1} 4(i-1) + \sum_{i=2}^{d} (2i-1) \\ &= 3(d-1)^2 \end{aligned}$$

□



a) i edges $\rightarrow i-1$ edges
$\mu^{\mathbb{C}} = 4$



b) i outgoing edges $\rightarrow i-1$ outgoing edges
$\mu^{\mathbb{C}} = 1$

FIGURE 14. Decrease by 1 the genus of the floor diagram of maximal genus

6.2. **Asymptotic of Welschinger invariants.** In [Mik05], a combinatorial algorithm in terms of *lattice paths* has been given to enumerate complex and real curves in toric surfaces. The idea is that when we consider (the right number of) points which are sufficiently far one from the other but on the same line L with irrational slope, then all tropical curves passing through these points can be recovered inductively. Hence, if L is the line with equation $x + \varepsilon y$ with y a very small irrational number, then lattice paths and floor diagrams are two ways to encode the same tropical curves. However, in our opinion, floor diagrams are much easier to deal with. In particular, one does not have to consider reducible curves using floor diagrams.

As an example, we give a floor diagram proof of the following theorem that was initially proved with the help of the lattice paths.

Theorem 6.2 (Itenberg, Kharlamov, Shustin [IKS03] [IKS04]). *The sequence $(W(\Delta_d, 0))_{d\geq 1}$ satisfies the following properties :*

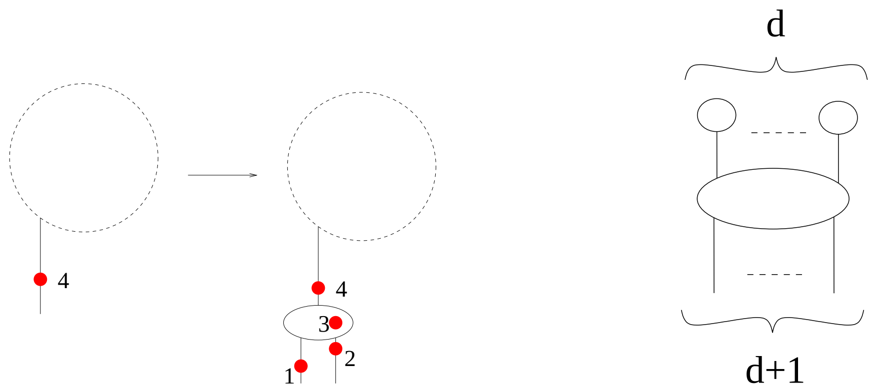
- *it is a sequence of positive numbers,*
- *it is an increasing sequence, and strictly increasing starting from $d = 2$,*
- *one has $\ln W(\Delta_d, 0) \sim \ln N(\Delta_d, 0) \sim 3d \ln d$ when d goes to infinity.*

Proof. As we have $\mu_0^{\mathbb{R}} = 1$ for any floor diagram, the numbers $W(\Delta_d, 0)$ are all non-negative. Moreover, we have $W(\Delta_1, 0) = 1$ so the positivity of these numbers will follow from the increasingness of the sequence $(W(\Delta_d, 0))_{d\geq 1}$.

Let $(\mathcal{D}_0, m_0)$ be a marked floor diagram of genus 0 and Newton polygon Δ_d. For convenience we use marking $m_0 : \{4, \ldots, 3d+2\} \rightarrow D_0$ (instead of the "usual" marking $\{1, \ldots, 3d-1\} \rightarrow \mathcal{D}_0$). Note that the point 4 has to be mapped to an edge in $\mathrm{Edge}^{-\infty}(\mathcal{D}_0)$. Out of $\mathcal{D}_0$, we can construct a new marked floor diagram $\mathcal{D}$ of genus 0 and Newton polygon Δ_{d+1} as indicated in Figure 15a. Both real multiplicities $\mu_0^{\mathbb{R}}(\mathcal{D}_0)$ and $\mu_0^{\mathbb{R}}(\mathcal{D})$ are the same, and two distinct marked floor diagrams $\mathcal{D}_0$ and $\mathcal{D}_0'$ give rise to two distinct marked floor diagrams $\mathcal{D}$ and $\mathcal{D}'$. Hence, we have $W(\Delta_{d+1}, 0) \geq W(\Delta_d, 0)$ for all $d \geq 1$. Moreover, if $d \geq 2$ then there exist marked floor diagrams with Newton polygon Δ_{d+1} which are not obtained out of a marked floor diagrams with Newton polygon Δ_d as described above. An example is given in Figure 15b, hence $W(\Delta_{d+1}, 0) > W(\Delta_d, 0)$ if $d \geq 2$.

We study now the logarithmic asymptotic of the sequence $(W(\Delta_d, 0))_{d\geq 1}$. Let $(\mathcal{D}_d)_{d\geq 1}$ be the sequence of floor diagrams constructed inductively in the following way : $\mathcal{D}_1$ is the floor diagram with Newton polygon Δ_1, and $\mathcal{D}_d$ is obtained out of $\mathcal{D}_{d-1}$ by gluing to each edge in $\mathrm{Edge}^{-\infty}(\mathcal{D}_{d-1})$ the piece depicted in Figure 16a. Floor diagrams $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$ are depicted in Figures 16b, c, d et e. The floor diagram $\mathcal{D}_d$ is of degree 2^{d-1} and we have $\mu_0^{\mathbb{R}}(\mathcal{D}_d) = 1$. If $\nu(\mathcal{D}_d)$ denotes the number of distinct markings of $\mathcal{D}_d$, then we have

$$\begin{aligned}
\forall d \geq 2 \qquad \nu(\mathcal{D}_d) &= \tfrac{\nu(\mathcal{D}_{d-1})^2}{2} C_{3\times 2^{d-1}-4}^{3\times 2^{d-2}-2} \\
&= \tfrac{(3\times 2^{d-1}-4)!}{2^{2^{d-1}-1}} \prod_{i=2}^{d} \frac{1}{\left((3\times 2^{d-i}-2)(3\times 2^{d-i}-3)\right)^{2^{i-1}}}
\end{aligned}$$



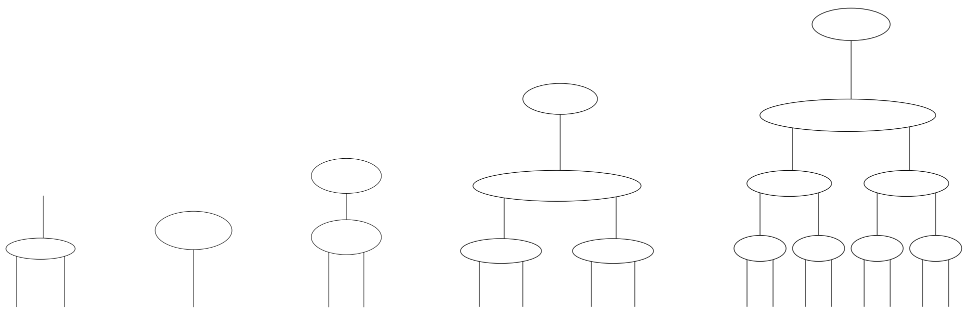
a) From Δ_d to Δ_{d+1} b) Not obtained from Δ_d

FIGURE 15. The numbers $W(\Delta_d, 0)$ are increasing

Hence we get

$$\frac{(3 \times 2^{d-1} - 4)!}{2^{2^d} \prod_{i=1}^{d} (3 \times 2^{d-i})^{2^i}} \le \nu(\mathcal{D}_d) \le (3 \times 2^{d-1} - 4)!$$

The Stirling Formula implies that $\ln d! \sim d \ln d$, and we see easily that both right and left hand side of the inequality have the same logarithmic asymptotic, namely $3 \times 2^{d-1} \ln(2^{d-1})$. As we have $\ln \nu(\mathcal{D}_d) \le \ln W(\Delta_{2^{d-1}}, 0) \le \ln N(\Delta_{2^{d-1}}, 0)$, the result follows from the increasingness of the sequence $(W(\Delta_d, 0))_{d \ge 1}$ and from the equivalence $\ln(N(\Delta_d, 0)) \sim 3d \ln d$ proved in [DFI95]. $\square$



a) b) $\mathcal{D}_1$ c) $\mathcal{D}_2$ d) $\mathcal{D}_3$ e) $\mathcal{D}_4$

FIGURE 16. Asymptotic of the numbers $W(\Delta_d, 0)$

6.3. **Recursive formulas.** Floor diagrams allow one to write down easily recursive formulas in a Caporaso-Harris style (see [CH98]) for both complex and real enumerative invariants. The recipe to extract such formulas is explained in [ABLdM] in the particular case of the numbers $W(\Delta_d, r)$.

As an example we briefly outline here how to reconstruct Vakil's formula [Vak00], which relates some enumerative invariants of Hirzebruch surfaces.

The Hirzebruch surface $\mathbb{F}_n$ of degree n, with $n \ge 0$, is the compactification of the line bundle over $B = \mathbb{C}P^1$ with first Chern class n. If $\mathbb{F}_n \supset F \approx \mathbb{C}P^1$ denotes the compactification of a fiber, then the second homology group of $\mathbb{F}_n$ is the free abelian group generated by B and F. In a suitable coordinate system, a generic algebraic curve in $\mathbb{F}_n$ of class $aB+bF$, with $a,b\geq 0$, has the h-transverse Newton polygon $\Delta_{n,a,b}$ with vertices $(0,0)$, $(na+b,0)$, $(0,a)$, and (b,a) (see [Bea83] for more details about Hirzebruch surfaces).

Before stating the theorem, we need to introduce some notations. In the following, $\alpha=(\alpha_1,\alpha_2,\dots)$ denotes a sequence of non-negative integers, and we set

$$|\alpha|=\sum_{i=1}^{\infty}\alpha_i,\qquad I\alpha=\sum_{i=1}^{\infty}i\alpha_i,\qquad I^{\alpha}=\prod_{i=1}^{\infty}i^{\alpha_i}$$

If a and b are two integer numbers, $\binom{a}{b}$ denotes the binomial coefficient. If a and $b_1,b_2,\dots,b_k$ are integer numbers then $\binom{a}{b_1,\dots,b_k}$ denotes the multinomial coefficient, i.e.

$$\binom{a}{b_1,\dots,b_k}=\prod_{i=1}^{k}\binom{a-\sum_{j=1}^{i-1}b_j}{b_i}$$

Theorem 6.3 (Vakil, [Vak00])**.** *For any $n\geq 0$, any $g\geq 0$, and any $b\geq 1$, one has*

$$N(\Delta_{n,2,b},g)=N(\Delta_{n+1,2,b-1},g)+\sum_{\substack{I\beta\leq n\\ |\beta|=g+1}}\binom{2n+2b+g+2}{n-I\beta}\binom{\beta_1+b}{b}\binom{|\beta|+b}{\beta_1+b,\beta_2,\beta_3,\dots}I^{2\beta}$$

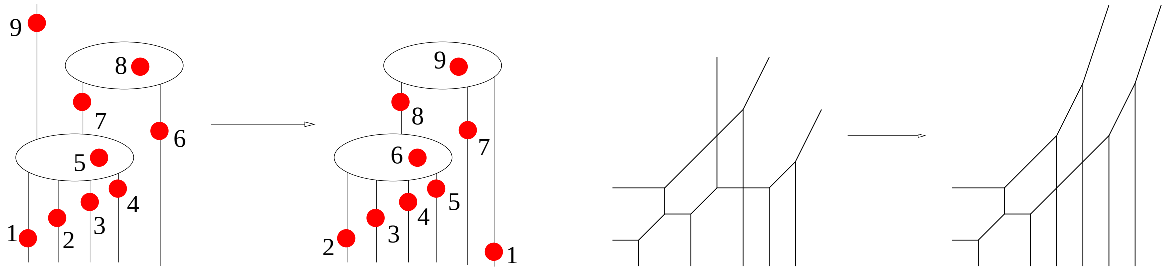
Proof. We want to enumerate marked floor diagrams of genus g and Newton polygon $\Delta_{n,2,b}$. As these floor diagrams have only two floors, our task is easy. Let $\mathcal{D}$ be such a marked floor diagrams of genus g and Newton polygon $\Delta_{n,2,b}$. Then, the marking m is defined on the set $\{1,\dots,s\}$ where $s=2(n+2)+2b-1+g$.

Suppose that $m(s)$ is a floor of $\mathcal{D}$. These marked floor diagrams are easy to enumerate, their contribution to the number $N(\Delta_{n,2,b},g)$ is the second term on the right hand side of the equality.

Suppose that $m(s)$ is on an edge e in $\mathrm{Edge}^{+\infty}(\mathcal{D})$. Define a new floor diagram $\mathcal{D}'$ as follows : $\mathrm{Vert}(\mathcal{D}')=\mathrm{Vert}(\mathcal{D}')$, $\mathrm{Edge}(\mathcal{D}')=(\mathrm{Edge}(\mathcal{D})\setminus\{e\})\cup\{e'\}$, where e' is in $\mathrm{Edge}^{-\infty}(\mathcal{D})$ and is adjacent to the other floor than e. Define a marking m' on $\mathcal{D}'$ as follows : $m'(i)=m(i-1)$ if $i\geq 2$ and $m(1)\in e'$. Now, the marked floor diagram $(\mathcal{D}',m')$ is of genus g and Newton polygon $\Delta_{n+1,2,b-1}$ (see Figure 17a). Moreover, we obtain in this way a bijection between the set of marked floor diagrams of genus g and Newton polygon $\Delta_{n,2,b}$ such that $m(s)\in\mathrm{Edge}^{+\infty}(\mathcal{D})$, and marked floor diagrams of genus g and Newton polygon $\Delta_{n+1,2,b-1}$. Hence, we get the first term of the right hand side of the equality, and the theorem is proved. □

Remark 6.4. Our proof of Theorem 6.3 is a combinatorial game on marked floor diagrams that can be obtained as the translation to the floor diagram language of Vakil's original proof : take the highest point p of the configuration, and specialize it to the exceptional section E. Then, either a curve C we are counting breaks into 2 irreducible components, which give the second term, or C has now a prescribed point on E. Blowing up this point and blowing down the strict transform of the fiber, the curve is transformed to a curve in $\mathbb{F}_{n+1}$ with a prescribed point on B (which is the image under the blow down of the second intersection point of C with the fiber).

The effect of such a blow up and down in tropical geometry can be easily seen, since intersection points with E correspond to leaves going up, and intersection with B correspond to leaves going



a) b)

FIGURE 17. From $\mathbb{F}_n$ to $\mathbb{F}_{n+1}$

down. An example is given in Figure 17b which correspond to the operation on marked floor diagram depicted in Figure 17a.

7. Further computations

One can adapt the technics of this paper to compute other real and complex enumerative invariants of algebraic varieties. In addition to genus 0 Gromov-Witten invariants and Welschinger invariants of higher dimensional spaces, as announced in [BM07], one can compute in this way characteristic numbers of the projective plane (at least in genus 0 and 1), as well as Gromov-Witten and Welschinger invariants of the blown up projective plane. Details will appear soon.

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Université Pierre et Marie Curie, Paris 6, 175 rue du Chevaleret, 75 013 Paris, France
E-mail address: brugalle@math.jussieu.fr

Université de Genève, 2-4 rue du Lièvre, Case postale 64, 1211 Genève, Suisse
E-mail address: grigory.mikhalkin@unige.ch