FLOOR DIAGRAMS OF PLANE TROPICAL CURVES RELATIVE TO A CONIC

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Floor diagrams in [BM08] were obtained by stretching a configuration of points in $\mathbb{C}P^2$ relatively to a line. Another way to think about it is that we specialize one by one the points to the normal bundle of the line in question. If we play the same game with respect to a conic, everything works the same. In particular, we end up with tropical curves having very simple floor decomposition. Since enumerative invariants of $\mathbb{C}P^2$ relative to a conic underly these new floor diagrams, these latter contain floors of degree 1 and 2 (respectively of divergence 2 and 4 in what follows), floors of degree 1 being special.

Note that if we stretch our configurations of points relatively to a smooth curve of degree 3 or more, then some superabundant situations appear and the problem becomes much more complicated.

Given $d \geq 1$ and $g \geq 0$, we denote by $N(d,g)$ the number of complex irreducible algebraic curves of degree d and genus g in $\mathbb{C}P^2$ passing through a generic configuration of $3d-1+g$ points. We also denote by $W(d,r)$ the Welschinger invariant of $\mathbb{R}P^2$ of degree d for configuration of points with exactly r pairs of complex conjugated points.

1. Floor diagrams

Let Γ be a finite oriented graph. We say that Γ is *acyclic* if it does not contain any non-trivial *oriented* cycle. We denote the set of its vertices with $\overline{Vert}(\Gamma)$ and the set of its (open) edges with $Edge(\Gamma)$. We denote by $Vert^\infty(\Gamma)$ the set of sources (i.e. the vertices such that all their adjacent edges are outgoing), and with $Edge^\infty(\Gamma)$ the set of edges adjacent to a source. Finally we put $Vert(\Gamma) = \overline{Vert(\Gamma)} \setminus Vert^\infty(\Gamma)$.

We say that Γ is a *weighted graph* if each edge of Γ is prescribed a natural weight, i.e. we are given a function $w : Edge(\Gamma) \to \mathbb{N}$. The weight allows one to define the *divergence* at the vertices. Namely, for a vertex $v \in Vert(\Gamma)$ we define the divergence $div(v)$ to be the sum of the weights of all incoming edges minus the sum of the weights of all outgoing edges.

Definition 1.1. *A connected weighted oriented graph $\mathcal{D}$ is called a* floor diagram of genus g, degree d, and relative degree 2 *if the following conditions hold*

- *The oriented graph $\mathcal{D}$ is acyclic.*
- *We have $div(v) = 2$ or 4 for any $v \in Vert(\mathcal{D})$ and $div(v) = -1$ for every $v \in Vert^\infty(\mathcal{D})$.*
- *If $div(v) = 2$ for $v \in Vert(\mathcal{D})$, then v is a sink (i.e. all its adjacent edges are oriented toward this vertex).*
- *The first Betti number $b_1(\mathcal{D})$ equals g.*
- *The set $Vert^\infty(\mathcal{D})$ consists of $2d$ elements.*

We call a vertex $v \in Vert(\mathcal{D})$ *a floor of divergence $div(v)$*. Note that there are slight differences with the definition of floor diagrams in [BM07] and [BM08].

Here are the convention we use to depict floor diagrams : vertices of $\mathcal{D}$ are represented by ellipses. Edges of $\mathcal{D}$ are represented by vertical lines, and the orientation is implicitly from down to up. We write the weight of an edge close to it only if this weight is at least 2.

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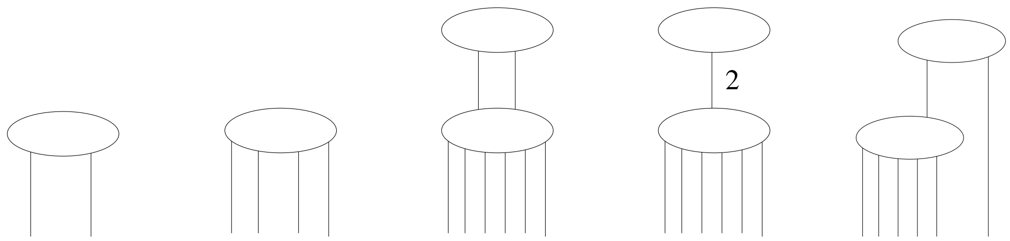
Example 1.2. Figure 1 depicts all floor diagrams of relative degree 2 and of degree 1, 2 and 3.



a) $d = 1$, $g = 0$ b) $d = 2$, $g = 0$ c) $d = 3$, $g = 1$ d) $d = 3$, $g = 0$ e) $d = 3$, $g = 0$

FIGURE 1. Examples of floor diagrams of relative degree 2

A map m between two partially ordered sets is said *increasing* if

$$m(i) > m(j) \Longrightarrow i > j$$

A floor diagram inherit a partial ordering from the orientation of its underlying graph.

Definition 1.3. *A marking of a floor diagram $\mathcal{D}$ of genus g, degree d and relative degree 2 is an increasing map $m : \{1, \ldots, 3d - 1 + g\} \to \mathcal{D}$ such that*

(1) *for any edge or vertex x of $\mathcal{D}$, the set $m^{-1}(x)$ consists at most of one element.*
(2) *a vertex of $\mathcal{D}$ is in the image of m if and only if it is of divergence 4.*

A floor diagram of relative degree 2 enhanced with a marking is called a *marked floor diagram* and is said to be marked by m.

Definition 1.4. *Two marked floor diagrams $(\mathcal{D}, m)$ and $(\mathcal{D}', m')$ are called equivalent if there exists a homeomorphism of weighted oriented graphs $\phi : \mathcal{D} \to \mathcal{D}'$ such that $m = m' \circ \phi$.*

Hence, if $m(i)$ is an edge e of $\mathcal{D}$, only the knowledge of e is important to determine the equivalence class of $(\mathcal{D}, m)$, not the position of $m(i)$ on e. From now on, we consider marked floor diagrams up to equivalence.

2. ENUMERATION OF COMPLEX CURVES

The complex multiplicity of a marked floor diagram is defined as in [BM08].

Definition 2.1. *The complex multiplicity of a marked floor diagram $\mathcal{D}$, denoted by $\mu^{\mathbb{C}}(\mathcal{D})$, is defined as*

$$\mu^{\mathbb{C}}(\mathcal{D}) = \prod_{e \in Edge(\mathcal{D})} w(e)^2$$

Note that the complex multiplicity of a marked floor diagram depends only on the underlying floor diagram.

Theorem 2.2. *For any degree $d \geq 1$ and any genus $g \geq 0$, one has*

$$N(d, g) = \sum \mu^{\mathbb{C}}(\mathcal{D})$$

where the sum is taken over all marked floor diagrams of genus g, degree d and relative degree 2.

Example 2.3. Counting how many markings admit the floor diagrams depicted in Figure 1 we verify that $N(1, 0) = N(2, 0) = N(3, 1) = 1$ (Figures 1a, b and c), and $N(3, 0) = 4 + 8 = 12$ (Figures 1d and e).

As in the case of Gromov-Witten invariants of $\mathbb{C}P^2$ relative to a line, one can modify slightly Theorem 2.2 to compute all Gromov-Witten invariants of $\mathbb{C}P^2$ relative to a conic. One can also write recursive formulas following the method exposed in [ABLdM].

3. Enumeration of real curves

Now suppose that our relative conic is real. There exist two topological types of smooth conics in $\mathbb{R}P^2$, and we can either choose this conic with a non-empty or an empty real part. This will give rise to two possible real floor diagrams. Here we restrict ourselves to the two simplest situations: either all points are real and the conic has a non-empty real part, or all, except maybe 1, points are complex. It is interesting to remark that in the latter case, the final floor diagrams do not depend wether the conic has an empty real part or not.

It is in principle possible to mix these two possibilities, and I guess this should lead to interesting results.

3.1. **Real floor diagrams with respect to a totally real configuration.**

Here we study floor diagrams when all points are real. Since all points are lye in the normal bundle of the conic, this latter necessarily has a non-empty real part. In this case, the real multiplicity of a marked floor diagram is as in [BM08].

Definition 3.1. *The totally real multiplicity of a marked floor diagram, denoted by $\mu^{\mathbb{R}}_{tot}(\mathcal{D}, m)$, is defined as*

$$\mu^{\mathbb{R}}_{tot}(\mathcal{D}, m) = 1$$

if all edges of $\mathcal{D}$ are of odd weight, and as

$$\mu^{\mathbb{R}}_{tot}(\mathcal{D}, m) = 0$$

otherwise.

As in the complex case, the totally real multiplicity of a marked floor diagram only depends on its underlying graph.

Theorem 3.2. *For any degree $d \geq 1$ one has*

$$W(d, 0) = \sum \mu^{\mathbb{R}}_{tot}(\mathcal{D}, m)$$

where the sum is taken over all marked floor diagrams of genus 0, of degree d and relative degree 2.

Example 3.3. Out of our computations in Example 2.3 we verify that $W(1, 0) = W(2, 0) = 1$ (Figures 1a, and b), and $W(3, 0) = 0 + 8 = 8$ (Figures 1d and e).

One can modify Theorem 3.2 to compute Welschinger invariants $W(d, r)$ for any couple (d, r), exactly as in [BM08]. We don't do it here since it doesn't seem to enlighten the situation and would have made the notation much more heavy.

3.2. **Real floor diagrams with respect to a real configuration containing at most one real point.**

This case is much more interesting, and is the tropical/floor diagram version of [Wel]. In the opposite of the precedding case, we restrict ourselves here to the case when all points, except maybe 1, are complex. Next definition can be extended to the general case as in [BM08], but so far we state it only in the case of real configurations with at most one real point.

Definition 3.4. *Let $\mathcal{D}$ be a floor diagram of genus 0 and relative degree 2, and $\rho : \mathcal{D} \to \mathcal{D}$ a continuous involution of weighted oriented graphs. The pair $(\mathcal{D}, \rho)$ is called a real floor diagram of relative degree 2 if ρ has only one fixed point.*

Note that if $(\mathcal{D},\rho)$ is a real floor diagram of relative degree 2 then the fixed point of ρ is necessarily a floor v of $\mathcal{D}$, with $div(v)=4$ if $\mathcal{D}$ has even degree and with $div(v)=2$ otherwise.

Let us denote by $\mathcal{D}_1$ the quotient of $\mathcal{D}\setminus\{v\}$ by ρ.

Definition 3.5. *An imaginary-real marking of a real floor diagram $(\mathcal{D},\rho)$ of genus 0, degree d and relative degree 2 is an increasing map $m:\{1,\dots,\left[\frac{3d-1}{2}\right]\}\to\mathcal{D}_1$ such that*

(1) *for any edge or vertex x of $\mathcal{D}_1$, the set $m^{-1}(x)$ consists at most of one element.*
(2) *a vertex of $\mathcal{D}_1$ is in the image of m if and only if it is of divergence 4.*

The imaginary-real multiplicity of a real marked floor diagram $(\mathcal{D},\rho,m)$, denoted by $\mu^{\mathbb{R}}_{im}(\mathcal{D},\rho,m)$, is defined as

$$\mu^{\mathbb{R}}_{im}(\mathcal{D},\rho,m)=2^{2-b_0(\mathcal{D}_1)}\prod_{e\in Edge(\mathcal{D}_1)}w^2(e)$$

We have an obvious notion of equivalent imaginary-real marked floor diagrams. Note that the imaginary-real multiplicity of a real floor diagram of relative degree 2 still only depends on the underlying graph.

Theorem 3.6. *For any degree $d\geq 1$, one has*

$$W\left(d,\left[\frac{3d-1}{2}\right]\right)=2^{\left[\frac{3d-5}{2}\right]}(-1)^{\frac{(d-1)(d-2)}{2}}\sum\mu^{\mathbb{R}}_{im}(\mathcal{D},\rho,m)$$

where the sum is taken over all imaginary-real marked floor diagrams of genus 0, of degree d and relative degree 2.

Example 3.7. The floor diagrams on Figures 1a and b are both real, and we verify that $W(1,1)=W(2,2)=1$.

On the opposite, none of the floor diagrams on Figures 1d and e are real, so we get $W(3,4)=0$.

It is also easy to see that there does not exist any real marked floor diagram of genus 0, degree 4 and relative degree 2, i.e. $W(4,5)=0$.

In degree 5, there exists only one real marked floor diagram of genus 0 and relative degree 2 (See Figure 2a). We verify that $W(5,7)=2^6$.

In degree 6, there exists only two real marked floor diagrams of genus 0 and relative degree 2 (See Figures 2b and c). We verify that $W(6,8)=2^6(8+2\times 4)=2^{10}$.

In degree 7, there exist only two real marked floor diagram of genus 0 and relative degree 2 (See Figures 2d and e). We verify that $W(7,10)=-2^8(2\times 20+2\times 8)=-14336$.

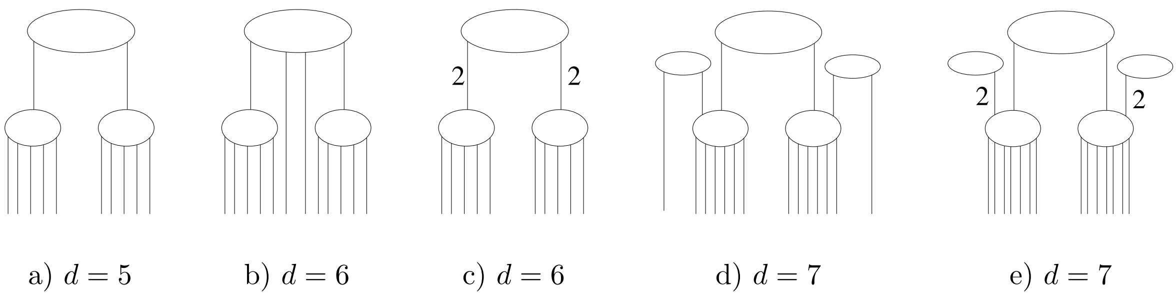


FIGURE 2. Real floor diagrams of genus 0, of relative degree 2, and of degree 5, 6, and 7

3.3. **The mixed case.** Work in progress!

References

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