# Enumeration of curves via floor diagrams

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**Abstract**

In this note we compute some enumerative invariants of real and complex projective spaces by means of some enriched graphs called floor diagrams.

## 1. Introduction: enumerative invariants of real and complex projective spaces

The subject of this note is the number of curves passing through configuration of linear spaces in $\mathbb{RP}^n$ and $\mathbb{CP}^n$. To set up an enumerative problem we fix integer numbers $d \geq 1$, $g \geq 0$ and $n \geq 2$. Then we look at the algebraic curves of degree $d$ and genus $g$ in $\mathbb{RP}^n$ or $\mathbb{CP}^n$. Namely, by the curves of genus $g$ in $\mathbb{CP}^n$ we mean the images of Riemann surfaces of genus $g$ under holomorphic maps to projective spaces; by the curves in $\mathbb{RP}^n$ we mean the real points of those curves in $\mathbb{CP}^n$ that are invariant under the involution of complex conjugation.

By the Riemann-Roch formula, the space of all such curves has dimension greater or equal than $(n+1)d + (n-3)(1-g)$. Furthermore, we always have equality in the case when $n = 2$ or in the case when $g = 0$ for $n > 2$. These are the two cases that we are concerned with in this note.

Let us fix a generic configuration $\mathcal{P}$ of projective-linear subspaces (of different dimensions) in $\mathbb{RP}^n$ or $\mathbb{CP}^n$, so that we have $l_j$ subspaces of dimension $j = 0, \ldots, n-2$. These subspaces are known as *constraints of dimension $j$*. It can be shown that if

$$\sum_{j=0}^{n-2} l_j(n-1-j) = (n+1)d + (n-3)(1-g) \tag{1}$$

then the number of curves of degree $d$ and genus $g$ passing via $\mathcal{P}$ is finite (recall that our assumption was that $g = 0$ whenever $n > 2$). Furthermore, if we work in $\mathbb{CP}^n$ then this number does not depend on the choice of $\mathcal{P}$ and is known as the Gromov-Witten number, which we denote by $N_{d,g}^{(n)}(l_0, \ldots, l_{n-2})$. These

invariants are well-known; they were computed by Kontsevich (see [4]) in the case of $g = 0$ and arbitrary $n$ and by Caporaso and Harris (see [2]) in the case of $n = 2$ and arbitrary $g$.

Much less is known if we work in $\mathbb{RP}^n$. Certainly in this case the number of curves of degree $d$ and genus $g$ depends not only on the numbers $l_j$ but also on the choice of the configuration $\mathcal{P}$. Nevertheless, Welschinger proved in [7], [8] that in the case when $n = 2, 3$, $g = 0$ and $l_j = 0$ for any $j > 0$, there is a consistent choice of signs $\pm 1$ such that the number of the corresponding real curves counted with the sign depends only on $d$ (note that the assumption $l_j = 0$ for $j > 0$ allows us to determine $l_0$ once we fix $d$ by (1) ). These numbers are known as the Welschinger numbers, we denote them with $W_d^{(n)}$.

The technique for computation of the numbers $W_d^{(2)}$ was given in [5] as an application of tropical geometry. It was used in [3] to produce several estimates for these numbers. In particular, we have $W_d^{(2)} > 0$ for any $d$. By symmetry reason we have $W_d^{(3)} = 0$ for any even $d$, but the values of $W_d^{(3)}$ remained unknown for odd $d$ even for $d = 5$.

In this note we announce a technique which allows a simultaneous computation of $N_{d,g}^{(n)}(l_0, \ldots, l_{n-2})$ and $W_d^{(n)}$ (for $n = 2, 3$). The details will be given in [1] along with a somewhat more general computation of the number of real curves in the case when the configuration $\mathcal{P}$ contains pairs of complex conjugate subspaces.


## 2. Floor diagrams

Let $\Gamma$ be a finite oriented graph. We say that $\Gamma$ is *acyclic* if it does not contain any non-trivial *oriented cycle* (in the same time the first Betti number of $\Gamma$ may be high). We denote the set of its vertices with $\overline{\mathrm{Vert}}(\Gamma)$ and the set of its (open) edges with $\mathrm{Edge}(\Gamma)$. We denote by $\mathrm{Vert}^\infty(\Gamma)$ the set of sinks (i.e. the vertices such that all their adjacent edges are incoming), and with $\mathrm{Edge}^\infty(\Gamma)$ the set of edges adjacent to a sink. Finally we put $\mathrm{Vert}(\Gamma) = \overline{\mathrm{Vert}}(\Gamma) \smallsetminus \mathrm{Vert}^\infty(\Gamma)$.

We say that $\Gamma$ is a *weighted graph* if each edge of $\Gamma$ is prescribed a natural weight, i.e. we are given a function $w : \mathrm{Edge}(\Gamma) \to \mathbb{N}$. The weight allows one to define the *divergence* at the vertices. Namely, for a vertex $v \in \mathrm{Vert}(\Gamma)$ we define the divergence $\mathrm{div}(v)$ to be the sum of the weights of all outgoing edges minus the sum of the weights of all incoming edges.

**Definition 2.1** *A connected weighted oriented graph $\mathcal{D}$ is called a* floor diagram *of genus $g$ and degree $d$ if the following conditions hold.*
– *The oriented graph $\mathcal{D}$ is acyclic.*
– *We have $\mathrm{div}(v) > 0$ for any $v \in \mathrm{Vert}(\mathcal{D})$ and $\mathrm{div}(v) = -1$ for every $v \in \mathrm{Vert}^\infty(\mathcal{D})$.*
– *The first Betti number $b_1(\mathcal{D})$ equals $g$.*
– *The set $\mathrm{Vert}^\infty(\mathcal{D})$ consists of $d$ elements.*
We call a vertex $v \in \mathrm{Vert}(\mathcal{D})$ *a floor of degree $\mathrm{div}(v)$.*

Let $l_0, \ldots, l_{n-2} \geq 0$ be integer numbers subject to (1) and $\mathcal{P} = \{x_1^{(0)}, \ldots, x_{l_0}^{(0)}, \ldots x_1^{(n-2)}, \ldots, x_{l_{n-2}}^{(n-2)}\}$ be the ordered set of $\sum_{j=0}^{n-2} l_j$ elements. We define $\dim(x_k^{(j)}) = j$. Let $m : \mathcal{P} \to \mathcal{D} \smallsetminus \mathrm{Vert}^\infty(\mathcal{D})$ be a map. Let $\mathcal{D}_{>e}$ be the component of $\mathcal{D} \smallsetminus e$ that contains the arrowhead of $e$. We define the *height* of $e$ by $h(e) = 0$ if $g > 0$ and otherwise by

$$h(e) = \sum_{q \in \mathcal{P}, \; m(q) \in \mathcal{D}_{>e}} \big(n - 1 - \dim(q)\big) + 1 - w(e) - (n+1) \sum_{v \in \mathrm{Vert}(\mathcal{D}) \cap \mathcal{D}_{>e}} \mathrm{div}(v).$$

(The meaning of the height is the dimension of the constraint we have to put on $e$ to make $\mathcal{D}_{>e}$ rigid.)

**Definition 2.2** *The map $m$ is called the* marking *of $\mathcal{D}$ if it satisfies to the following properties.*

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– *If $m(q) = m(q')$ for $q < q' \in \mathcal{P}$ then $m(q)$ is a vertex and $\dim(q) > 0$.*
– *For any $v \in \mathrm{Vert}(\mathcal{D})$, there exists $q \in \mathcal{P}$ such that $m(q) = v$.*
– *If $q' > q$ and $m(q') < m(q)$, then there exists $q''$ such that $m(q) = m(q'')$ and $q'' > q'$.*

A floor diagram $\mathcal{D}$ enhanced with a marking is called the *marked floor diagram*, and is said to be *marked by $\mathcal{P}$*. Two marked floor diagrams are called *equivalent* if they can be identified by a homeomorphism of oriented graphs. They are called to be of the *same combinatorial type* if there exists a bijection $\sigma : \mathcal{P} \to \mathcal{P}$ that preserves the dimension of the constraints and such that $(\mathcal{D}, m)$ is equivalent to $(\mathcal{D}, m \circ \sigma)$.

To each marked floor diagram $\mathcal{D}$, we may associate its complex and real *multiplicities* $\mu^{\mathbb{C}}(\mathcal{D})$ and $\mu^{\mathbb{R}}(\mathcal{D})$. These multiplicities record the number of complex and real curves (respectively) encoded by the diagram.

The definition is inductive by $n$. Take a floor $v \in \mathrm{Vert}(\mathcal{D})$ of degree $\mathrm{div}(v)$. For the maximum element $x_k^{(j)}$ of $m^{-1}(v)$ (recall that $\mathcal{P}$ is ordered), take a linear space of dimension $j$. For each other element $x_{k'}^{(j')}$ in $m^{-1}(v)$, take a linear space of dimension $j' - 1$. Take a linear space of dimension $h(e)$ for each edge $e$ incoming to $v$. For each edge $e$ outgoing from $v$ take a linear space of dimension

$$n - 1 - h(e) - \sum_{q \in \mathcal{P}, \ m(q) \in e} \big( n - 1 - \dim(q) \big).$$

If any of these numbers is outside of the range between $0$ and $(n-2)$ then we set both the real and complex multiplicity of $\mathcal{D}$ equal to $0$. Otherwise denote the number of resulting $j$-dimensional linear spaces with $l_j^{(v)}$ and define $\mu^{\mathbb{C}}(v) = \mathrm{div}(v)^{l_{n-2}^{(v)}} N_{\mathrm{div}(v),0}^{(n-1)}(l_0^{(v)}, \ldots, l_{(n-3)}^{(v)})$ and

$$\mu^{\mathbb{C}}(\mathcal{D}, m) = \prod_{v \in \mathrm{Vert}(\mathcal{D})} \mu^{\mathbb{C}}(v) \prod_{e \in \mathrm{Edge}(\mathcal{D})} (w(e))^{1 + \#(m^{-1}(e))}.$$

Here $\#(m^{-1}(e))$ is the number of elements in the inverse image of an *open* edge $e$. To fix the base of induction for $n = 1$ we define $N_{1,0}^{(1)} = 1$ and $N_{a,b}^{(1)} = 0$ for $(a, b) \neq (1, 0)$. If $g = 0$ and $l_j = 0$ for $j \geq 1$ then we also define the real multiplicity

$$\mu^{\mathbb{R}}(\mathcal{D}) = \prod_{v \in \mathrm{Vert}(\mathcal{D})} W^{(n-1)}(\mathrm{div}(v)) \ \ if \ \ w(\mathrm{Edge}(\mathcal{D})) \cap 2\mathbb{N} = \emptyset \ \ and \ \ \mu^{\mathbb{R}}(\mathcal{D}) = 0 \ otherwise.$$

We put $W_1^{(1)} = 1$ and $W_d^{(1)} = 0$ for $d > 1$.

Note that for $n = 2$, both real and complex multiplicities have a very simple form. For any diagram with the non-zero multiplicity we have $h(e) = 0$ for any $e \in \mathrm{Edge}(\mathcal{D})$ and $\mathrm{div}(v) = 1$ for any $v \in \mathrm{Vert}(\mathcal{D})$, so $\mu^{\mathbb{C}}(\mathcal{D})$ is the square of the product of the weight of all edges of $\mathcal{D}$, and $\mu^{\mathbb{R}}(\mathcal{D}) = \mu^{\mathbb{C}}(\mathcal{D}) \ mod \ 2$. Note also that all marked floor diagrams of the same combinatorial type have the same multiplicities.

In the examples of combinatorial types of marked floor diagrams we give below, the convention we use are the following : the graph $\mathcal{D}$ is oriented from up to down, vertices are represented by ellipses with its degree written inside, edges are represented by solid lines close of which is written its weight (except if it is 1), and the images of $m$ are represented by a set of points on $\mathcal{D}$. Below any combinatorial type, we write the number of marked floor diagram of this combinatorial type, and the complex and real (if any) multiplicities of such a floor diagram.

*Example 1 In Figure 1, we depict all 2-dimensional marked floor diagrams of degree 3 with non null multiplicity. In Figure 2 (resp. 3), we depict all 3-dimensional marked floor diagrams of degree 5 and marked by 10 points (resp. of degree 2 and marked by 8 lines) with non null multiplicity.*
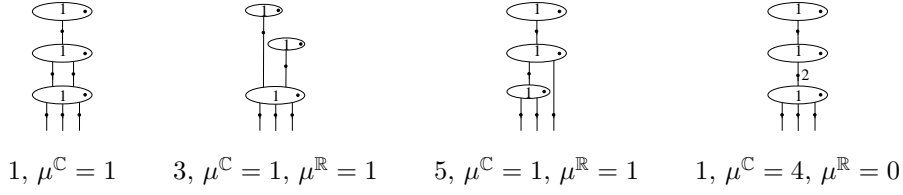
$$1, \mu^{\mathbb{C}} = 1 \qquad 3, \mu^{\mathbb{C}} = 1, \mu^{\mathbb{R}} = 1 \qquad 5, \mu^{\mathbb{C}} = 1, \mu^{\mathbb{R}} = 1 \qquad 1, \mu^{\mathbb{C}} = 4, \mu^{\mathbb{R}} = 0$$

Figure 1. 2-dimensional marked floor diagrams of degree 3



$$3, \mu^{\mathbb{C}} = 12, \mu^{\mathbb{R}} = 8 \qquad 21, \mu^{\mathbb{C}} = 1, \mu^{\mathbb{R}} = 1 \qquad 6, \mu^{\mathbb{C}} = 8, \mu^{\mathbb{R}} = 0$$
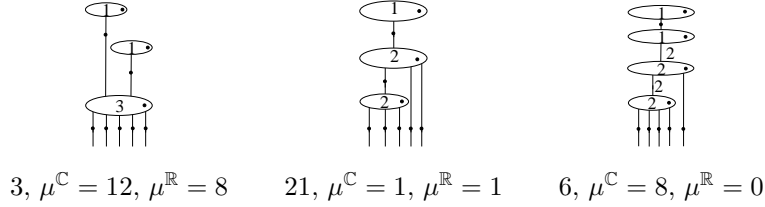
Figure 2. 3-dimensional marked floor diagrams of degree 5, genus 0, and marked by 10 points



$$5, \mu^{\mathbb{C}} = 1 \quad 3, \mu^{\mathbb{C}} = 1 \quad 10, \mu^{\mathbb{C}} = 1 \quad 12, \mu^{\mathbb{C}} = 1 \quad 3, \mu^{\mathbb{C}} = 1 \quad 5, \mu^{\mathbb{C}} = 1 \quad 3, \mu^{\mathbb{C}} = 1$$

$$10, \mu^{\mathbb{C}} = 1 \quad 12, \mu^{\mathbb{C}} = 1 \quad 3, \mu^{\mathbb{C}} = 1 \quad 1, \mu^{\mathbb{C}} = 8 \quad 3, \mu^{\mathbb{C}} = 4 \quad 3, \mu^{\mathbb{C}} = 2$$
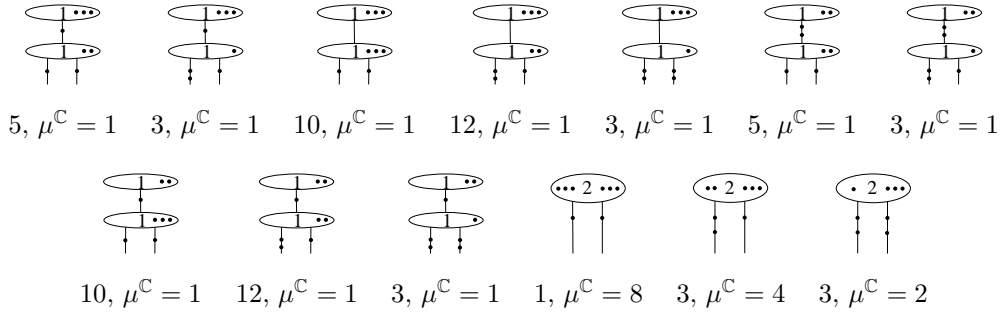
Figure 3. 3-dimensional marked floor diagrams of degree 2, genus 0, and marked by 8 lines

## 3. Main Formulas

**Theorem 1** *For $n = 2$ or $g = 0$, the number $N_{d,g}^{(n)}(l_0, \ldots, l_{n-2})$ is equal to the sum of the complex multiplicity of all floor diagrams of degree $d$, genus $g$ marked by $\mathcal{P}$.*

*Example 2 Using marked floor diagrams depicted in Figures 1, 2, and 3, we verify that $N_{3,1}^{(2)}(9) = 1$, $N_{3,0}^{(2)}(8) = 12$, $N_{5,0}^{(3)}(10,0) = 105$, and $N_{2,0}^{(3)}(0,8) = 92$.*

*Example 3 Using Theorem 1, one can compute the numbers $N_{d,\frac{(d-1)(d-2)}{2}-1}^{(2)}\left(\frac{d(d+3)}{2} - 1\right) = 3(d-1)^2$ (the degree of the discriminant) and $N_{d,\frac{(d-1)(d-2)}{2}-2}^{(2)}\left(\frac{d(d+3)}{2} - 2\right) = \frac{3}{2}(d-1)(d-2)(3d^2 - 3d - 11)$, cf. [6].*

**Theorem 2** *For $n = 2$ or $n = 3$, the Welschinger invariant $W_d^{(n)}$ is equal to $(-1)^{\frac{n(d-1)(d-2)}{2}}$ times the sum of the real multiplicity of all $n$-dimensional floor diagrams of degree $d$, genus 0 marked by $\mathcal{P}$ with $l_j = 0$ for $j > 0$.*

The two-dimensional Welschinger invariants were studied by Itenberg, Kharlamov and Shustin [3] who obtained a number of estimates on these invariants. Theorem 1 and 2 allow one not only to recover these results but also to show that similar results hold for the 3-dimensional invariants $W^{(3)}$.

*Example 4 We list the values of $N_{d,0}^{(3)}(2d,0)$ and $W_d^{(3)}$ up to degree 7.*

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $N_{d,0}^{(3)}(2d,0)$ | 1 | 0 | 1 | 4 | 105 | 2576 | 122129 |
| $W_d^{(3)}$ | 1 | 0 | $-1$ | 0 | 45 | 0 | $-14589$ |

**Proposition 3** *The 3-dimensional Welschinger invariants have the following properties :*

– *For any $k > 1$, $|W_{2k+1}^{(3)}| > |W_{2k-1}^{(3)}|$.*

– *The sequences $(W_{(2k+1)}^{(3)})$ and $(N_{2k+1,0}^{(3)}(4k+2,0))$ are logarithmically equivalent, i.e.*

$$\log(|W^{(3)}(2k+1)|) \sim 4k \log k \sim \log(N_{2k+1,0}^{(3)}(4k+2,0)).$$

– *For any $d \geq 1$, $|W_d^{(3)}| = N_{d,0}^{(3)}(2d,0) \mod 4$.*

To deduce Theorems 1 and 2 we use tropical geometry. The main technical ingredient is the following lemma (stated in the notation of Section 1, and where $\mathcal{P}$ is here a configuration of tropical linear subspaces in $\mathbb{R}^n$).

**Lemma 4** *Let $HC$ be a hypercube containing all vertices of all elements of $\mathcal{P}$. Then $HC$ contains all the vertices of any tropical curve of degree $d$ and genus $g$ in $\mathbb{R}^n$ passing through elements of $\mathcal{P}$.*

## 4. Further remarks

– Results of this note can be extended from projective spaces to some other (but not all) toric varieties.
– As in [5], one can count how many real curves pass through some special real configurations $\mathcal{P}$. For example, one can choose 8 real lines in $\mathbb{RP}^3$ such that the 92 conics passing through these lines are real.

## References

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