FLOOR DECOMPOSITIONS OF TROPICAL CURVES IN ANY DIMENSION

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ABSTRACT. We reduce the computation of Gromov-Witten invariants of $\mathbb{C}P^n$ in genus 0 to the combinatorial enumeration of floor diagrams. This formula was announced in [BM07].

This is a preliminary draft of the extended version of [BM07]. The final version of the paper will also contain the computation of Welschinger invariants of $\mathbb{R}P^3$, and will be available soon.

1. ENUMERATIVE INVARIANTS OF $\mathbb{C}P^n$

Given $d \geq 1$ and $n \geq 2$ two integer numbers, we look at irreducible rational algebraic curves of degree d in $\mathbb{C}P^n$. Such a curve is given by an algebraic map $f : \mathbb{C}P^1 \to \mathbb{C}P^n$, i.e. by $n+1$ polynomials $P_0, \ldots, P_n$ of degree d with no common factor. Moreover, the P_i's are determined up to a common multiplicative constant, and up to a reparametrization of $\mathbb{C}P^1$. In particular, the space of all rational curves of degree d in $\mathbb{C}P^n$ has dimension $(n+1)d + n - 3$.

Let $l_0, \ldots, l_{n-2} \geq 0$ be integer numbers such that

$$\sum_{j=0}^{n-2} l_j(n-1-j) = (n+1)d + n - 3 \qquad (1)$$

and let us fix a configuration ω of linear subspaces of $\mathbb{C}P^n$ which contains exactly l_i spaces of dimension j for all $j = 0, \ldots, n-2$. If the configuration ω is generic, then the number of rational curves of degree d in $\mathbb{C}P^n$ intersecting all elements of ω is finite and does not depend on ω. This number is known as a Gromov-Witten number of $\mathbb{C}P^n$, and we denote it by $N_d^n(l_0, \ldots, l_{n-2})$. For practical purposes which will appear later, we define $N_1^1 = 1$ and $N_d^1 = 0$ for $d \geq 2$. We refer to the excellent book [KV06] for more details about Gromov-Witten numbers. These invariants are well-known, and were computed by Kontsevich for the first time (see [KM94]). Later, Vakil proposed in [Vak00b] another way of computing them, based on the Caporaso-Harris method (see [CH98]) in enumeration of planar curves.

In this text we give a purely combinatorial way of computing Gromov-Witten numbers of $\mathbb{C}P^n$ in genus 0 via enumeration of floor diagrams (Theorem 2.10). This formula was announced in [BM07] and is based on tropical geometry. Note that floor diagrams also allow one to enumerate planar curves of any genus, see [BM08]. The main interest of the floor diagrams technique is that it allows a simultaneous enumeration of complex and real curves. For example this method provides a very practical way to compute Welschinger invariants (the case of $\mathbb{R}P^3$ will be added later in this text, see [BM08] for the planar case), as well as an efficient tool in the study of maximality of real enumerative problems (see [BP] for the case of conics). Note that this technique has strong connections with the Caporaso and Harris method (see [CH98]), extended later by Vakil (see [Vak00a] and [Vak00b]), and with the neck-stretching method in symplectic field theory (see [EGH00]). In particular it allows one to compute Gromov-Witten and Welschinger invariants by induction on n and d, and to write recursive formulas computing these invariants (see for example [ABLdM11] or [BGM]).

We define floor diagrams and their markings in section 2, and we relate them with Gromov-Witten invariants in Theorem 2.10. The proof of this latter is based on tropical geometry, whose objects are introduced in section 3.1. In particular this section contains the Correspondence Theorem relating tropical and complex enumerative geometries of rational curves (Theorem 3.10). Another major ingredient in the proof of Theorem 2.10 is the enumeration of *reducible* rational curves, and we prove in section 4 a Correspondence Theorem for

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those curves (Theorem 4.5). This theorem could be proved along the same lines as Theorem 3.10, however we decided to illustrate tropical techniques by derivating Theorem 4.5 from Theorem 3.10 and basic facts about tropicalisations. Section 5 is devoted to the end of the proof of Theorem 2.10 via floors, elevators and walls of tropical varieties.

Notation: Given a graph Γ, we denote by $\text{Vert}(\Gamma)$ the set of its vertices, by $\text{Vert}^\infty(\Gamma)$ the set of its 1-valent vertices, by $\text{Edge}(\Gamma)$ the set of its edges, and by $\text{Edge}^\infty(\Gamma)$ the set of its edges adjacent to a 1-valent vertex. Elements of $\text{Edge}^\infty(\Gamma)$ are also called *ends* of Γ. The complement of $\text{Vert}^\infty(\Gamma)$ in $\text{Vert}(\Gamma)$ is denoted by $\text{Vert}^0(\Gamma)$, and the complement of $\text{Edge}^\infty(\Gamma)$ in $\text{Edge}(\Gamma)$ is denoted by $\text{Edge}^0(\Gamma)$. If in addition Γ is oriented, we denote by $\text{Source}(\Gamma)$ the set of its *source*, i.e. vertices v of Γ such that all their adjacent edges are oriented outward v. The complement of $\text{Source}(\Gamma)$ in $\text{Vert}(\Gamma)$ is denoted by $\text{Vert}^\circ(\Gamma)$.

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2. Floor diagrams

Definition 2.1. *A floor diagram (of genus 0) $\mathcal{D}$ is an oriented tree equipped with a map $w : Edge(\mathcal{D}) \to \mathbb{Z}_{>0}$, called the weight function, such that*

- *the set $Source(\mathcal{D})$ is contained in $Vert^\infty(\mathcal{D})$;*
- *any edge e adjacent to a source has weight 1;*
- *all vertices in $Vert^\circ(\mathcal{D})$ have positive divergence.*

The divergence of a vertex v of $\mathcal{D}$, denoted by $\text{div}(v)$, is equal to the sum of the weights of all incoming edges minus the sum of the weights of all outgoing edges. The *degree* of a floor diagram $\mathcal{D}$ is the sum of the divergence of its vertices in $\text{Vert}^\circ(\mathcal{D})$.

Remark 2.2. The orientation convention on floor diagrams we use in this paper is the same than in [BM08], which is the opposite than the one used in [BM07].

Here are the convention we use to depict floor diagrams : vertices in $\text{Vert}^\circ(\mathcal{D})$ are represented by ellipses, and edges of $\mathcal{D}$ are represented by vertical lines. The orientation is implicitly from down to up, and we precise the weight of an edge only if this weight is at least 2. We do not draw vertices in $\text{Source}(\mathcal{D})$.

Example 2.3. Figure 1 depicts some examples of floor diagram of genus 0 and degree d.

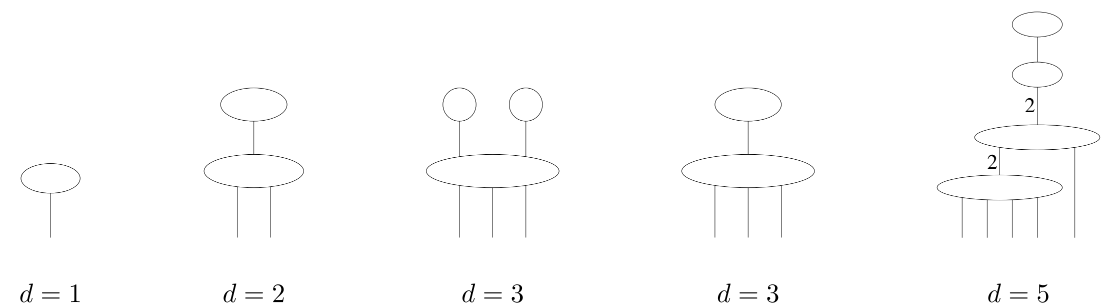


Figure 1. Examples of floor diagrams

Let $n \geq 2$ and $l_0, \ldots, l_{n-2} \geq 0$ be integer numbers subject to (1), and let

$$\mathcal{P} = \{x_1^{(0)}, \ldots, x_{l_0}^{(0)}, \ldots, x_1^{(n-2)}, \ldots, x_{l_{n-2}}^{(n-2)}\}$$

be a set of $\sum_{j=0}^{n-2} l_j$ distinct elements equipped with some total ordering $<$. We define $\dim(x_k^{(j)}) = j$. The orientation of a floor diagram $\mathcal{D}$ induces a partial order on $\mathcal{D}$: we define $a < b$ if there exists an oriented path in $\mathcal{D}$ from a to b.

Definition 2.4. *A map $m : \mathcal{P} \to \mathcal{D}\setminus Source(\mathcal{D})$ is called a* marking *of $\mathcal{D}$ if it satisfies the following conditions:*

- *$Vert^\circ(\mathcal{D}) \subset m(\mathcal{P})$;*
- *if $q < q'$ and $m(q) > m(q')$, then $m(q') \in Vert^\circ(\mathcal{D})$ and there exists $q'' < q$ such that $m(q'') = m(q')$.*

A floor diagram $\mathcal{D}$ enhanced with a marking is called a *marked floor diagram*, and is said to be *marked by* $\mathcal{P}$. We also say that $\mathcal{D}$ is an n-dimensional floor diagram marked by l_0 points, l_1 lines, …, l_{n-2} codimension 2 planes.

Definition 2.5. *Two marked floor diagrams $(\mathcal{D}, m)$ and $(\mathcal{D}', m')$ are called equivalent if there exists a isomorphism of oriented graphs $\phi : \mathcal{D} \to \mathcal{D}'$ such that $w = w' \circ \phi$, and $m = m' \circ \phi$.*

They are called to be of the same combinatorial type *if there exists a bijection $\sigma : \mathcal{P} \to \mathcal{P}$ that preserves the dimension of the constraints and such that $(\mathcal{D}, m)$ is equivalent to $(\mathcal{D}, m \circ \sigma)$.*

From now on, we consider marked floor diagrams up to equivalence. To any (equivalence class of) marked floor diagram $(\mathcal{D}, m)$, we assign a integer called its *complex multiplicity*, denoted by $\mu_{\mathbb{C}}(\mathcal{D}, m)$. This multiplicity records the number of complex curves encoded by the diagram.

We first need to associate a integer number to any edge e of $\mathcal{D}$ as follows. Let $\mathcal{D}_{<e}$ be the component of $\mathcal{D} \setminus e$ that contains elements of $\mathcal{D}$ lower than e. Then we define

$$h(e) = \sum_{q\in\mathcal{P},\ m(q)\in\mathcal{D}_{<e}} \big(n - 1 - \dim(q)\big) + 1 - w(e) - (n + 1) \sum_{v\in\mathrm{Vert}^\circ(\mathcal{D})\cap\mathcal{D}_{<e}} \mathrm{div}(v).$$

Given a vertex v in $\mathrm{Vert}^\circ(\mathcal{D})$, let us do the following: for the minimum element $x_k^{(j)}$ of $m^{-1}(v)$, we take a linear space of dimension j; for each other element $x_{k'}^{(j')}$ in $m^{-1}(v)$, we take a linear space of dimension $j' - 1$; we take a linear space of dimension $h(e)$ for each edge e outgoing from v; for each edge e incoming to v we take a linear space of dimension

$$n - 1 - h(e) - \sum_{q\in\mathcal{P},\ m(q)\in e} \big(n - 1 - \dim(q)\big).$$

If any of these numbers is outside of the range between 0 and $(n - 2)$ then we set

$$\mu_{\mathbb{C}}(v) = 0.$$

Otherwise denote the number of resulting j-dimensional linear spaces with $l_j^{(v)}$ and define

$$\mu_{\mathbb{C}}(v) = \mathrm{div}(v)^{l_{n-2}^{(v)}} N^{n-1}_{\mathrm{div}(v)}(l_0^{(v)}, \dots, l_{(n-3)}^{(v)}).$$

Note that we express $\mu_{\mathbb{C}}(v)$ in terms of $N^{n-1}_{\mathrm{div}(v)}(l_0^{(v)}, \dots, l_{(n-3)}^{(v)})$, i.e. the corresponding Gromov-Witten number for dimension $n - 1$. In next definition, if A is a set then $|A|$ denotes its cardinal.

Definition 2.6. *The complex multiplicity of a marked floor diagram $(\mathcal{D}, m)$ is defined as*

$$\mu_{\mathbb{C}}(\mathcal{D}, m) = \prod_{v\in Vert^\circ(\mathcal{D})} \mu_{\mathbb{C}}(v) \prod_{e\in Edge(\mathcal{D})} w(e)^{1+|(m^{-1}(e)|}.$$

Note that two marked floor diagrams of the same combinatorial type have the same complex multiplicity. In the special case $n = 2$, both real and complex multiplicities have a very simple form: for any diagram with a non-zero multiplicity we have $h(e) = 0$ for any $e \in \mathrm{Edge}(\mathcal{D})$, and $\mathrm{div}(v) = 1$ for any $v \in \mathrm{Vert}^\circ(\mathcal{D})$. Furthermore, each edge is marked by a single element (of dimension 0). This implies that

$$\mu_{\mathbb{C}}(\mathcal{D}, m) = \prod_{e\in\mathrm{Edge}(\mathcal{D})} w(e)^2.$$

We refer to [BM08] for more details about floor decomposition of planar curves.

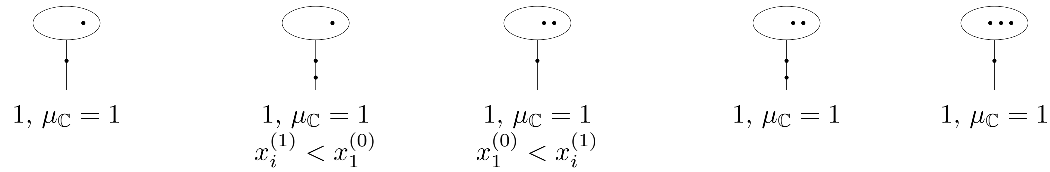


FIGURE 2. 3-dimensional marked floor diagrams of degree 1 marked by two points, one point and two lines, and four lines

Example 2.7. In Figure 2, we depict all combinatorial types of 3-dimensional floor diagrams of degree 1 with non-null multiplicity marked by either two points, a point and two lines, or four lines. The chosen order on $\mathcal{P}$ plays a role only in the case of one point and two lines, for which we depict the combinatorial types corresponding to two different orders.

For each combinatorial type, we depict the floor diagram together with the image of the marking m. In addition we write below the number of marked floor diagram of this combinatorial type, and the complex multiplicity of such a floor diagram.

Example 2.8. In Figure 3, we depict all combinatorial types of 3-dimensional floor diagrams of degree 3, 4, and 5, marked by respectively 6, 8, and 10 points and with non-null multiplicity. Note that there does not exist a 3-dimensional marked floor diagram of degree 2 marked by 4 points with non-null multiplicity.

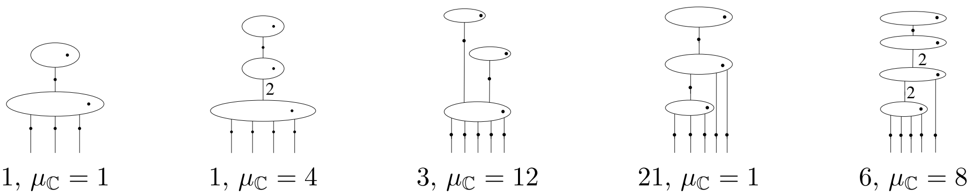


FIGURE 3. 3-dimensional marked floor diagrams of degree 3, 4, and 5 respectively marked by 6, 8, and 10 points

Example 2.9. In Figure 4, we depict all combinatorial types with non-null multiplicity of 3-dimensional floor diagrams of degree 2 marked by 8 lines.

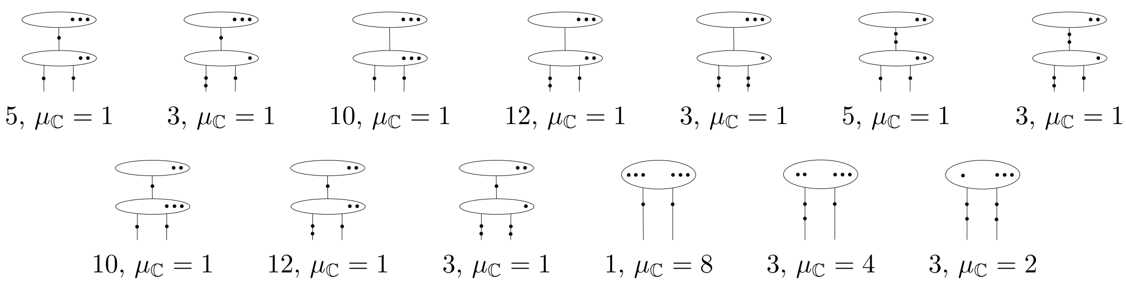


FIGURE 4. 3-dimensional marked floor diagrams of degree 2 marked by 8 lines

Enumeration of marked floor diagrams and complex rational curves in $\mathbb{C}P^n$ are related by the following theorem which will be proved in section 5.

Theorem 2.10. *For any $n \geq 2$ and $l_0, \ldots, l_{n-2} \geq 0$ integer numbers subject to equality (1), the number $N_d^n(l_0, \ldots, l_{n-2})$ is equal to the sum of the complex multiplicity of all n-dimensional floor diagrams of degree d marked by l_0 points, l_1 lines, …, l_{n-2} codimension 2 planes.*

Example 2.11. Using marked floor diagrams depicted in Figures 2, 3, and 4, we verify that

$$N_1^3(2,0) = N_1^3(1,2) = 1, \quad N_1^3(0,4) = 2, \quad N_2^3(4,0) = 0, \quad N_2^3(0,8) = 92,$$
$$N_3^3(6,0) = 1, \quad N_4^3(8,0) = 4, \quad N_5^3(10,0) = 105.$$

3. Enumerative tropical geometry

3.1. **Tropical curves.** To fix notation and convention, we remind the basic definitions of tropical curves and morphisms.

Definition 3.1. *A rational tropical curve C is a finite compact tree equipped with a complete inner metric on $C \setminus Vert^\infty(C)$, and without any 2-valent vertex.*

By definition, the leaves of C are at the infinite distance from all the other points of C.

Given e an edge of a tropical curve C, we choose a point p in the interior of e and a unit vector u_e of the tangent line to C at p. Of course, the vector u_e depends on the choice of p and is not well defined, but this will not matter in the following. We will sometimes need u_e to have a prescribed direction at any point p from the interior of e, and we will then precise this direction. The standard inclusion of $\mathbb{Z}^n$ in $\mathbb{R}^n$ induces a standard inclusion of $\mathbb{Z}^n$ in the tangent space of $\mathbb{R}^n$ at any point of $\mathbb{R}^n$.

Definition 3.2. *A map $f : C \setminus Edge^\infty(C) \to \mathbb{R}^n$ is called a tropical morphism if the following conditions are satisfied*

- *for any edge e of C, the restriction $f_{|e}$ is a smooth map with $df(u_e) = w_{f,e}u_{f,e}$ where $w_{f,e}$ is a non-negative integer and $u_{f,e} \in \mathbb{Z}^n$ is a primitive vector,*
- *for any vertex v of C whose adjacent edges are $e_1, \dots, e_k$, one has the balancing condition*

$$\sum_{i=1}^{k} w_{f,e_i} u_{f,e_i} = 0$$

where u_{e_i} is chosen so that it points away from v. As remarkred...

The integer $w_{f,e}$ is called the *weight of the edge e with respect to f*. When no confusion is possible, we will speak about the weight of an edge, without referring to the morphism f. By abuse, we will denote $f : C \to \mathbb{R}^n$ instead of $f : C \setminus \mathrm{Vert}^\infty(C) \to \mathbb{R}^n$. If $w_{f,e} = 0$, we say that the morphism f *contracts* the edge e. The morphism f is called *minimal* if it does not contract any edge in $\mathrm{Edge}^\infty(C)$.

Given $u = (u_1, \dots, u_n)$ a vector in $\mathbb{R}^n$, we set

$$d_u = \max\{0, u_1, \dots, u_n\}.$$

Definition 3.3. *The degree of a tropical morphism $f : C \to \mathbb{R}^n$ is defined by*

$$\sum_{e \in Edge^\infty(C)} w_{f,e} d_{u_{f,e}}$$

where $u_{f,e}$ is chosen so that it points to its adjacent 1-valent vertex.

A tropical morphism $f : C \to \mathbb{R}^n$ of degree d is said to be transverse at infinity if C has exactly $(n+1)d$ non-contracted ends.

Note that if $f : C \to \mathbb{R}^n$ is a minimal tropical morphism of degree d transverse at infinity, then for any $i = 1, \dots, n+1$ the curve C has exactly d edges $e \in \mathrm{Edge}^\infty(C)$ with $u_{f,e} = U_i$, where $u_{f,e}$ is chosen so that it points to its adjacent 1-valent vertex and $U_1 = (-1, 0, \dots, 0)$, $U_2 = (0, -1, 0, \dots, 0)$, $\dots$, $U_n = (0, \dots, 0, -1)$, and $U_{n+1} = (1, \dots, 1)$.

Two tropical morphisms $f_1 : C_1 \to \mathbb{R}^n$ and $f_2 : C_2 \to \mathbb{R}^n$ are said to be *isomorphic* if there exists an isomorphism of metric graphs $\phi : C_1 \to C_2$ such that $f_1 = f_2 \circ \phi$. In this text, we consider tropical curves and tropical morphisms up to isomorphism.

Two tropical morphisms $f_1 : C_1 \to \mathbb{R}^n$ and $f_2 : C_2 \to \mathbb{R}^n$ are said to have the same *combinatorial type* if there exists a homeomorphism of graphs $\phi : C_1 \to C_2$ such that for all edges e of C_1 one has

$$w_{f_1,e} = w_{f_2,\phi(e)} \quad \textit{and} \quad u_{f_1,e} = u_{f_2,\phi(e)}$$

Given a combinatorial type α of tropical morphisms, we denote by $\mathcal{M}_\alpha$ the space of all such tropical morphisms having this combinatorial type. Given $f \in \mathcal{M}_\alpha$, we say that $\mathcal{M}_\alpha$ is the *rigid deformation space* of f.

Lemma 3.4 (Mikhalkin, [Mik05]). *Let α be a combinatorial type of tropical morphisms $f : C \to \mathbb{R}^n$ where C is a rational tropical curve. Then the space $\mathcal{M}_\alpha$ is an open convex polyhedral cone in the vector space $\mathbb{R}^{n+|Edge^0(\alpha)|}$, and*

$$\dim \mathcal{M}_\alpha = |Edge^\infty(\alpha)| + n - 3 - \sum_{v \in Vert^0(\alpha)} (val(v) - 3).$$

Proof. We remind the proof in order to fix notations we will need later. If $\mathrm{Vert}^0(\alpha) \neq \emptyset$, choose a root vertex v_1 of α, and an ordering $e_1, \ldots, e_{|\mathrm{Edge}^0(\alpha)|}$ of the edges in $\mathrm{Edge}^0(\alpha)$. Given $f : C \to \mathbb{R}^n$ in $\mathcal{M}_\alpha$, we write $f(v_1) = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and we denote by $l_i \in \mathbb{R}^*$ the length of the edge $e_i \in \mathrm{Edge}^0(C)$. Then

$$\mathcal{M}_\alpha = \{(x_1, \ldots, x_n, l_1, \ldots, l_{|\mathrm{Edge}^0(\alpha)|} \quad | \quad l_1, \ldots, \ldots, l_{|\mathrm{Edge}^0(\alpha)|} > 0\} = \mathbb{R}^n \times \mathbb{R}_{>0}^{|\mathrm{Edge}^0(\alpha)|}.$$

If $\mathrm{Vert}^0(C) = \emptyset$, then $\mathcal{M}_\alpha = \mathbb{R}^n / \mathbb{R}u_{f,e}$, where e is the only edge of α. □

Other choices of v_1 and of the ordering of elements of $\mathrm{Edge}^0(\alpha)$ provide other coordinates on $\mathcal{M}_\alpha$, and the change of coordinates is given by an element of $GL_{n+|\mathrm{Edge}^0(\alpha)|}(\mathbb{Z})$.

3.2. **Approximable tropical linear spaces.** We consider the logarithm with base t coordinate-wise on $(\mathbb{C}^*)^n$

$$\begin{array}{cccc} \mathrm{Log}_t : & (\mathbb{C}^*)^n & \longrightarrow & \mathbb{R}^n \\ & (z_i) & \longmapsto & (\log_t(|z_i|)) \end{array}.$$

For the sake of simplicity, we say that $\mathcal{L} \subset (\mathbb{C}^*)^N$ is a linear space if it is given by a system of equations of degree 1, i.e. if it compactifies as a projective linear space in the compactification of $(\mathbb{C}^*)^n$ given by the embedding $(z_1, \ldots, z_n) \mapsto [z_1 : \ldots : z_n : 1]$.

Definition 3.5. *Let $(\mathcal{L}_t)_{t \in \mathbb{R}_0}$ be a family of linear spaces of dimension k in $(\mathbb{C}^*)^n$. If the limit*

$$L = \lim_{t \to +\infty} Log_t(\mathcal{L}_t)$$

exists in the sense of Hausdorff metric on compact sets of $\mathbb{R}^n$, we call L an approximable tropical linear space of dimension k in $\mathbb{R}^n$.

The term approximable emphasizes that we only consider in this text tropical linear spaces which arise as limit of amoebas of complex linear spaces. There exists tropical linear spaces in $\mathbb{R}^n$ of dimension and codimension at least 2 which are not approximable (see [Spe08]).

A tropical linear space of dimension k is a finite polyhedral complex of pure dimension k (see [MS] or [Spe]).

Example 3.6. The tropical polynomial "$0 + x_1 + \ldots + x_n$" defines a tropical hyperplane H_0^n in $\mathbb{R}^n$. Recall that we defined the vectors $U_1 = (-1, 0, \ldots, 0)$, $U_2 = (0, -1, 0, \ldots, 0)$, …, $U_n = (0, \ldots, 0, -1)$, and $U_{n+1} = (1, \ldots, 1)$ in $\mathbb{R}^n$. The tropical hyperplane H_0^n can be described as follows: given $1 \leq i < j \leq n+1$, we denote by $E_{i,j}$ the convex polyhedron of $\mathbb{R}^n$ obtained by taking all non-negative real linear combinations of all the vectors U_k but U_i and U_j; then we have $H_0^n = \cup_{1 \leq i < j \leq n+1} E_{i,j}$.

A *generic* approximable tropical hyperplane H of $\mathbb{R}^n$ is the translation of H_0^n along some vector $\vec{v}$ of $\mathbb{R}^n$. If $\vec{v} = (v_1, \ldots, v_n)$ and $\alpha_0, \ldots, \alpha_n$ are any $n+1$ non-null complex numbers, then we have $H = \lim_{t \to +\infty} \mathrm{Log}_t(\mathcal{H}_t)$ where $\mathcal{H}_t$ is given by the equation

$$\alpha_0 + \alpha_1 t^{v_1} z_1 + \ldots + \alpha_n t^{v_n} z_n = 0.$$

A tropical plane in $\mathbb{R}^3$ is depicted in Figure 5a.

Example 3.7. Let $H_1, \ldots, H_{n-k}$ be $n-k$ tropical hyperplanes as described in Example 3.6. Then the tropical (or stable) intersections of these tropical hyperplanes in $\mathbb{R}^n$ (see [Mik06] or [RGST05]) is an approximable tropical linear space L of dimension k. We called such a approximable tropical linear space a *complete tropical linear space*. If $H_i = \lim_{t\to+\infty} \mathrm{Log}_t(\mathcal{H}_{i,t})$ where $\mathcal{H}_{i,t}$ is given by the equation

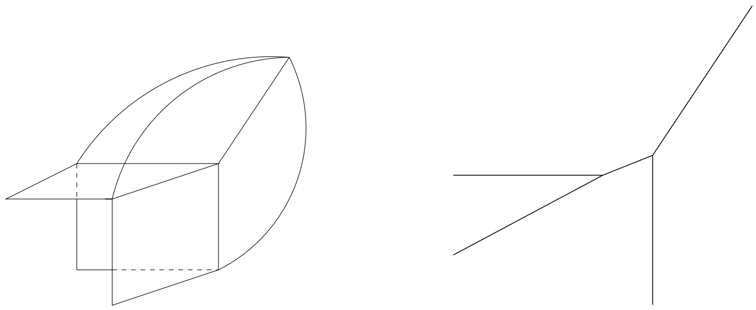
$$\alpha_{i,0} + \alpha_{i,1}t^{v_{i,1}}z_1 + \ldots + \alpha_{i,n}t^{v_{i,n}}z_n = 0,$$

then we have

$$L = \lim_{t\to+\infty} \mathrm{Log}_t(\cap_{i=1}^{n-k}\mathcal{H}_{i,t}) \qquad (2)$$

for a generic choice of the non-null complex numbers $\alpha_{i,j}$.

Note that if the configuration of tropical hyperplanes $H_1, \ldots, H_{n-k}$ is chosen generically, then the tropical intersection of the H_i's reduces to their set-theoretic intersection and equality (2) holds for *any* choice of the complex numbers $\alpha_{i,j}$. An example of a complete tropical line in $\mathbb{R}^3$ is depicted in Figure 5b. Its four unbounded rays have direction U_1, U_2, U_3, and U_4.



a) b)

FIGURE 5. A tropical plane and a tropical line in $\mathbb{R}^3$

3.3. **System of linear equations and tropical structure on classical affine spaces.** Let $A \in \mathcal{M}_{p,n}(\mathbb{Z})$ be a matrix of rank p and $B \in \mathbb{R}^n$, and consider the classical affine subspace $E = Ker(A) + B$ of $\mathbb{R}^n$. Since the direction of E is defined over $\mathbb{Z}$, the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ restrict to a lattice on $Ker(A)$. A $\mathbb{Z}$-*basis* of E is a basis of this lattice.

In addition to defining the direction of E as a classical vector space in $\mathbb{R}^n$, the matrix A also defines a tropical structure on E, i.e. a weight $w(E)$ on E (see [Mik06]). If $p = 1$, then $w(E)$ is simply the gcd of the entries of A. If $p > 1$, then E is, as a tropical variety, the tropical intersection of all the tropical hypersurfaces defined by the rows of A. The tropical structure defined by A on E is denoted by $w(E)E$. If $w(E) = 1$, then we say that A defines the *primitive* tropical structure on E.

3.4. **Correspondence Theorem.** Let $l_0, \ldots, l_{n-2} \geq 0$ be integer numbers subject to equality (1), and let us fix a configuration $\omega^{\mathbb{T}}$ of approximable tropical linear subspaces of $\mathbb{R}^n$ which contains exactly l_j spaces of dimension j for all $j = 0, \ldots, n-2$. We define by $\mathcal{S}(d, \omega^{\mathbb{T}})$ the set of all minimal tropical morphisms $f : C \to \mathbb{R}^n$, with C a rational tropical curve, such that $f(C)$ intersects all tropical linear spaces in $\omega^{\mathbb{T}}$.

Definition 3.8. *The configuration $\omega^{\mathbb{T}}$ is said to be generic if the set $\mathcal{S}(d, \omega^{\mathbb{T}})$ is finite and if any tropical morphism $f : C \to \mathbb{R}^n$ in $\mathcal{S}(d, \omega^{\mathbb{T}})$ satisfies the following conditions:*

- *f is transverse to infinity*
- *C is trivalent (i.e. all its vertices in $Vert^0(C)$ are adjacent to 3 edges);*
- *for any tropical linear space L in $\omega^{\mathbb{T}}$, the set $f(C) \cap L$ consists of a single point which is in the relative interior of a face of maximal dimension of L;*
- *$f(Vert^0(C)) \cap \left(\cup_{L\in\omega^{\mathbb{T}}} L\right) = \emptyset$.*

Proposition 3.9. *Let $\omega^{\mathbb{T}} = \{L_1, \ldots, L_k\}$. Then there exists a dense open subset $\mathcal{U}$ of $(\mathbb{R}^n)^k$ such that for any $(\vec{v}_1, \ldots, \vec{v}_k) \in \mathcal{U}$ the configuration $\omega'^{\mathbb{T}} = \{L_1 + \vec{v}_1, \ldots, L_k + \vec{v}_k\}$ is generic.*

Proof. The case when all L_i's are also classical affine subspaces of $\mathbb{R}^n$ is done in [NS06]. The general case then follows as in [BBM, Section 4]. □

From now on we suppose that the configuration $\omega^{\mathbb{T}}$ is generic. Let $f : C \to \mathbb{R}^n$ be an element of $\mathcal{S}(d, \omega^{\mathbb{T}})$, and let us denote by α its combinatorial type. Given $L \in \omega^{\mathbb{T}}$, we denote by λ_L the set of elements of $\mathcal{M}_\alpha$ in a small neighborhood U_f of f which pass through L, and we denote by w_L the weight of the edge of C which intersect L. If U_f is small enough, then λ_L spans a classical affine space Λ_L defined over $\mathbb{Z}$ in $\mathcal{M}_\alpha$. Since $\omega^{\mathbb{T}}$ is generic we have

$$\bigcap_{L \in \omega^{\mathbb{T}}} \Lambda_q = \{f\}.$$

The $\omega^{\mathbb{T}}$*-multiplicity* of $f : C \to \mathbb{R}^n$, denoted by $\mu_{\omega^{\mathbb{T}}}(f)$, is defined as the tropical intersection number in $\mathcal{M}_\alpha$ of all the tropical hypersurfaces $\mu_L \Lambda_L$:

$$\mu_{\omega^{\mathbb{T}}}(f) = \prod_{L \in \omega^{\mathbb{T}}} w_L \Lambda_L.$$

Theorem 3.10 (Correspondence Theorem, see [Mik05], [NS06], [Mik]). *If $\omega^{\mathbb{T}}$ is a generic configuration of complete tropical linear spaces, we have*

$$N_d^n(l_0, \ldots, l_{n-2}) = \sum_{f \in \mathcal{S}(d, \omega^{\mathbb{T}})} \mu_{\omega^{\mathbb{T}}}(f).$$

Proof. As explained in [BBM, Section 6] the multiplicity of f reduces to local computations, which are done in [NS06]. □

The multiplicity of a tropical morphism in $\mathcal{S}(d, \omega^{\mathbb{T}})$ has a geometric signification: Theorem 3.10 is obtained by degenerating the standard complex structure on $(\mathbb{C}^*)^n$ via the following self-diffeomorphism of $(\mathbb{C}^*)^n$

$$\begin{array}{cccc} H_t: & (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^n \\ & (z_i) & \longmapsto & \left(|z_i|^{\frac{1}{\log t}} \frac{z_i}{|z_i|}\right) \end{array}.$$

Namely, for any tropical complete linear space L in $\omega^{\mathbb{T}}$, there exists a family $(\mathcal{L}_{t,L})_{t>0}$ of complex linear spaces in $(\mathbb{C}^*)^n$ of dimension $\dim(L)$ such that the sets $\text{Log} \circ H_t(\mathcal{L}_{t,L})$ converges to L when $t \to \infty$, in the Hausdorff metric on compact subsets of $(\mathbb{R}^*)^n$. The map Log is defined by $\text{Log}(z_i) = (\log |z_i|)$. Hence for each t, we associate a configuration $\omega_t = \{\mathcal{L}_{t,L} : L \in \omega^{\mathbb{T}}\}$ of linear subspaces of $(\mathbb{C}^*)^n$. For t big enough, the configuration ω_t is generic if ω is generic, so the complex rational curves of degree d passing through all the linear spaces in ω_t form a finite set $\mathcal{C}(d, \omega_t)$. It turns out, and this is the core of Theorem 3.10, that the set $\text{Log} \circ H_t(\mathcal{C}(d, \omega_t))$ converges to the set $\mathcal{S}(d, \omega^{\mathbb{T}})$, and that for any tropical morphism $f \in \mathcal{S}(d, \omega^{\mathbb{T}})$ there exist exactly $\mu_{\omega^{\mathbb{T}}}(f)$ complex curves in $\mathcal{C}(d, \omega_t)$ whose image under $\text{Log} \circ H_t$ converges to $f(C)$.

In order compute the $\omega^{\mathbb{T}}$-multiplicity of a tropical morphism, we need explicit equations for the tropical spaces $w_L \Lambda_L$. Let $f : C \to \mathbb{R}^n$ be an element of $\mathcal{S}(d, omega^{\mathbb{T}})$, and denote by α its combinatorial type. Let us choose a vertex $v_1 \in \text{Vert}^0(C)$, and some ordering of edges in $\text{Edge}^0(C)$. This induces coordinates in $\mathcal{M}_\alpha$ as explained in the proof of Lemma 3.4. If $v \in \text{Vert}^0(C)$, we denote by $(v_1 v)$ the path joining v_1 to v in C. In particular we have

$$f(v) = f(v_1) + \sum_{e \in (v_1 v)} l_e w_{f,e} u_{f,e} \tag{3}$$

where the vectors $u_{f,e}$ are oriented toward v_1.

Proposition 3.11. *Let L be a tropical linear space in $\omega^{\mathbb{T}}$, and let $(e_0, \ldots, e_{\dim(L)})$ be a $\mathbb{Z}$-basis of the face of L which contains the point $f(C) \cap L$. If e is the edge of C which intersect L and v is a vertex adjacent to e, then the tropical structure on $w_L \Lambda_L$ is defined by the equation*

$$e_1 \wedge \ldots \wedge e_{\dim(L)} \wedge w_{f,e} u_{f,e} \wedge f(v) = 0.$$

Proof. By hypothesis, the space Λ_L is characterized by the condition that there exists some real numbers $a_1, \dots, a_j, a_e$ such that

$$p + \sum_{i=1}^{\dim(L)} a_i e_i = f(v) + a_e u_{f,e}$$

where p is some fixed point on the corresponding face of L. Now the result follows from elementary operations on the rows of the system

$$e_1 \wedge \dots \wedge e_{\dim(L)} \wedge u_{f,e} \wedge f(v) = 0$$

using the fact that $(e_0, \dots, e_{\dim(L)})$ is a $\mathbb{Z}$-basis of the face of L. □

Note that since $\omega^{\mathbb{T}}$ is generic, the vectors $e_0, \dots, e_{\dim(L)}$, and $u_{f,e}$ are linearly independent, and we verify that Λ_L has dimension $n - 1 - \dim(L)$.

Example 3.12. Let $\omega^{\mathbb{T}}$ be the generic configuration composed of the following three approximable tropical linear spaces in $\mathbb{R}^3$:

- L_1 is the affine line passing through the point $(0, 0, 0)$ and with direction $(-1, 0, 0)$;
- L_2 is the affine line passing through the point $(0, 0, 10)$ and with direction $(0, -1, 0)$;
- L_3 is the point $(-2, -1, 20)$.

The set $\mathcal{S}(1, \omega^{\mathbb{T}})$ has a unique element $f : C \to \mathbb{R}^3$ whose image is depicted in Figure 6: the edge of C passing through L_1 and L_2 has direction $(0, 0, -1)$ and the edge passing through L_3 has direction $(-1, 0, 0)$. The

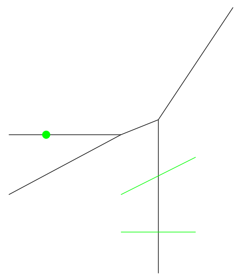


FIGURE 6. A tropical line in $\mathbb{R}^3$ passing through two lines and a point

space $\mathcal{M}_\alpha$ has dimension 4 with coordinates (x, y, z, l) where (x, y, z) are the coordinates in $\mathbb{R}^3$ of the vertex of C adjacent to the edge with direction $(-1, 0, 0)$, and l is the length of the bounded edge of C. Then the $\omega^{\mathbb{T}}$-multiplicity of f is given by the determinant of the following linear system

$$\begin{cases} y + l &= 0 \\ x + l &= 0 \\ y &= 0 \\ z &= 0 \end{cases}$$

so we get

$$\mu_{\omega^{\mathbb{T}}}(f) = \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 1.$$

Hence we find again that $N_1^3(1, 2) = 1$. Note that in this example the two tropical lines L_1 and L_2 are not complete, however one can think of them as edges of two complete tropical lines (see section 5).

Remark 3.13. The moduli spaces $\mathcal{M}_\alpha$ we are considering are moduli spaces of tropical morphisms with no marked points. One can of course reformulate the tropical enumerative problems considered here using moduli space of *stable*, or *marked*, tropical morphisms (see for example [GKM]), as one does in classical algebraic geometry. Let us check that the multiplicities of a marked tropical morphism f' and of the corresponding minimal tropical morphism f coincide.

For this, we need to define the multiplicity of such a marked tropical morphism. To each constraint L in $\omega^{\mathbb{T}}$, we associate a tropical space Λ'_L in the corresponding moduli space as follows: if $(e_1,\ldots,e_m)$ is a $\mathbb{Z}$-basis of $L^\perp$, the orthogonal of the direction of L, then we consider the tropical structure on Λ'_L defined by the system of equations

$$(4) \qquad \left\{\begin{array}{c} \langle f'(x_L), e_1\rangle = 0 \\ \vdots \\ \langle f'(x_L), e_m\rangle = 0 \end{array}\right.$$

where x_L is the marked point corresponding to L and $\langle\ ,\ \rangle$ denotes the standard Euclidean product on $\mathbb{R}^n$. Then the multiplicity $\mu_{\omega^{\mathbb{T}}}(f')$ of f' is given by the tropical intersection product of all the Λ'_L, i.e. $\mu_{\omega^{\mathbb{T}}}(f')$ is equal to the absolute value of the determinant of the matrix defined by considering all equations (4). By an easy elimination procedure, we see that $\mu_{\omega^{\mathbb{T}}}(f') = \mu_{\omega^{\mathbb{T}}}(f)$.

4. Enumeration of reducible curves

4.1. **Complex reducible curves.** To any connected nodal complex algebraic curve $\mathcal{C}$, we can associate its dual graph $\Gamma_{\mathcal{C}}$ whose vertices correspond to irreducible components of $\mathcal{C}$ and whose edges between two vertices correspond to nodes involving the two corresponding components. We say that $\mathcal{C}$ is rational if its arithmetic genus is 0 (i.e. any of its irreducible component is rational and $\Gamma_{\mathcal{C}}$ is a tree).

Let us fix a tree Γ and a function $deg : \mathrm{Vert}(\Gamma) \to \mathbb{N}$, and let us define $d = \sum_{v\in \mathrm{Vert}(\Gamma)} deg(v)$. We denote by $\mathcal{R}(\Gamma, deg)$ the space of connected rational curves $\mathcal{C}$ in $\mathbb{C}P^n$ of degree d equipped with an isomorphism of graphs $\phi : \Gamma_{\mathcal{C}} \to \Gamma$ such the map $deg \circ \phi$ is the degree of the corresponding irreducible components of $\mathcal{C}$. Then the space $\mathcal{R}(\Gamma, deg)$ has dimension $(n+1)d + n - 3 - |\mathrm{Edge}(\Gamma)|$. Let us fix $n-1$ functions $\gamma_j : \mathrm{Vert}(\Gamma) \to \mathbb{N}$ for $j = 0,\ldots,n-2$, and a function $\delta : \mathrm{Edge}(\Gamma) \to \{0,\ldots,n-1\}$ such that

$$(5) \qquad \sum_{v\in\mathrm{Vert}(\Gamma)} \sum_{j=0}^{n-2} \gamma_j(v)\,(n-1-j) + \sum_{e\in\Gamma} (n-\delta(e)) = (n+1)d + n - 3 - |\mathrm{Edge}(\Gamma)|.$$

For any v in $\mathrm{Vert}(\Gamma)$, choose a configuration ω_v of linear subspaces of $\mathbb{C}P^n$ containing exactly $\gamma_j(v)$ linear spaces of dimension j for $j = 0,\ldots,n-2$. For any edge e of Γ, choose a linear subspace $\mathcal{L}_e$ of $\mathbb{C}P^n$ of dimension $\delta(e)$. Then if the configuration of linear spaces

$$\bigcup_{v\in\mathrm{Vert}(\Gamma)} \omega_v \bigcup_{e\in\mathrm{Edge}(\Gamma)} \{\mathcal{L}_e\}$$

is generic, there exist a finite number of curves $\mathcal{C}$ in $\mathcal{R}(\Gamma, deg)$ such that the component v passes through all linear spaces in ω_v for $v \in \mathrm{Vert}(\Gamma)$, and such that the node e lies on $\mathcal{L}_e$ for $e \in \mathrm{Edge}(\Gamma)$. Moreover this number only depends on the graph Γ and the functions deg, $\gamma_0,\ldots,\gamma_{n-2}$, and δ. We denote this number $N^n_{deg}(\Gamma, \gamma_0,\ldots,\gamma_{n-2}, \delta)$.

Note that when $\mathrm{Edge}(\Gamma) = \emptyset$, we find again the Gromov-Witten invariants of $\mathbb{C}P^n$ defined in section 1. It turns out that all the numbers $N^n_{deg}(\Gamma, \gamma_0,\ldots,\gamma_{n-2}, \delta)$ can easily be expressed in terms of these Gromov-Witten invariants. For this purpose, we need to introduce some notation. Let $v \in \mathrm{Vert}(\Gamma)$ and $e \in \mathrm{Vert}(\Gamma)$, and denote by Γ' and Γ'' the two connected components of $\Gamma\setminus\{e\}$, in such a way that Γ' contains v. Similarly, the curve $\mathcal{C}$ minus the node corresponding to e has two connected components $\mathcal{C}'$ and $\mathcal{C}''$, and we may assume that the dual graph of $\mathcal{C}'$ is Γ' and the dual graph of $\mathcal{C}''$ is Γ''. The functions deg, $\gamma_0,\ldots,\gamma_{n-2}$, and δ induce respectively the functions deg', $\gamma'_0,\ldots,\gamma'_{n-2}$, and δ' on Γ' and deg'', $\gamma''_0,\ldots,\gamma''_{n-2}$, and δ'' on Γ''. Let us denote by $h_v(e)$ the expected dimension of the space of curves $\mathcal{R}(\Gamma', deg')$ such that the component v' passes through all linear spaces in $\omega_{v'}$ for $v' \in \mathrm{Vert}(\Gamma')$, and such that the node e' lies on $\mathcal{L}_{e'}$ for $e' \in \mathrm{Edge}(\Gamma')$. Defining $d' = \sum_{v'\in\Gamma'} deg'(v')$, we get

$$h_v(e) = (n+1)d' + n - 3 - |\mathrm{Edge}(\Gamma')| - \sum_{v'\in\mathrm{Vert}(\Gamma')} \sum_{j=0}^{n-2} \gamma'_j(v')\,(n-1-j) - \sum_{e'\in\Gamma'} (n-\delta'(e')).$$

Note that in particular, if $h_v(e)$ is not in the range $0,\ldots,n-1$ for some vertex v and edge e, then $N^n_{deg}(\Gamma, \gamma_0,\ldots,\gamma_{n-2}, \delta) = 0$.

For each vertex v of Γ, we take $\gamma_j(v)$ linear subspaces of dimension j, and a linear subspace of dimension $n-1-h_v(e)$ for each edge e adjacent to v. Let us denote by γ_j^v the number of resulting linear subspaces of dimension j. If any of the number $h_v(e)$ is negative, we set

$$\nu(v) = 0$$

and otherwise we set

$$\nu(v) = deg(v)^{\gamma_{n-1}^v} N^n_{deg(v)}(\gamma_0^v, \dots, \gamma_{n-2}^v).$$

Proposition 4.1. *With the above notation we have*

$$N^n_{deg}(\Gamma, \gamma_0, \dots, \gamma_{n-2}, \delta) = \prod_{v \in Vert(\Gamma)} \nu(v).$$

Proof. We prove the proposition by induction on the number of vertices of Γ. First, the proposition holds by definition if $|\text{Vert}(\Gamma)| = 1$, so let us suppose now that $|\text{Vert}(\Gamma)| \geq 2$. Let v be a vertex in $\text{Vert}^\infty(\Gamma)$, denote by $e \in \text{Edge}(\Gamma)$ its adjacent edge, and by v' the other vertex adjacent to e. Let V' be the algebraic variety in $\mathbb{C}P^n$ given by the union of all irreducible rational curves of degree $deg(v)$ passing through all linear spaces in ω_v. We consider the set S of all rational curves in $\mathcal{R}(\Gamma'', deg'')$ such that the component v'' passes through all linear spaces in $\omega_{v''}$ for $v'' \in \text{Vert}(\Gamma'')$, and such that the node e'' lies on $\mathcal{L}_{e''}$ for $e'' \in \text{Edge}(\Gamma'')$. We define V'' as the algebraic variety in $\mathbb{C}P^n$ given by the union of the component of $\mathcal{C}''$ corresponding to v' when $\mathcal{C}''$ runs over all elements of S. By definition, the number $N^n_{deg}(\Gamma, \gamma_0, \dots, \gamma_{n-2}, \delta)$ is equal to the number of intersection points in $V' \cap V'' \cap L_e$, i.e. is equal to the product of the degrees of V' and V''.

Since the configuration of linear spaces is generic, we have $\dim(V') = h_v(e) + 1$, so that the degree of V' is its number of intersection points with a generic linear space in $\mathbb{C}P^n$ of codimension $h_v(e)+1$. This number is by definition precisely $\nu(v)$.

For the same reason, we have $\dim(V'') = h_{v'}(e) + 1$. If $h_{v'}(e) = 0$, then the degree of V'' is precisely $deg(v') N^n_{deg''}(\Gamma'', \gamma_0'', \dots, \gamma_{n-2}'', \delta'')$. Otherwise, the degree of V'' is by definition $N^n_{deg''}(\Gamma'', \hat{\gamma}_0, \dots, \hat{\gamma}_{n-2}, \delta'')$ where $\hat{\gamma}_0, \dots, \hat{\gamma}_{n-2}$ are defined by $\hat{\gamma}_i = \gamma_i''$ with the following exception:

$$\hat{\gamma}_{n-1-h_{v'}(e)}(v') = \gamma_{n-1-h_{v'}(e)}(v') + 1.$$

The result follows by induction. □

4.2. **Tropical reducible curves.** We first extend Definition 3.1 to nodal tropical curves.

Definition 4.2. *A nodal rational tropical curve C is a finite compact connected 1-dimensional simplicial complex with Euler characteristic 1 equipped with a complete inner metric on $C \setminus (Vert^\infty(C) \cup Node(C))$ where $Node(C)$ is a subset of $Edge(C) \setminus Edge^\infty(C)$.*
An edge e in $Node(C)$ is called a node of C and is equipped with a metric which makes it isometric to $\mathbb{R}$.
A connected component of $C \setminus Node(C)$ is called an irreducible component of C.

By definitions, a node of C has infinite length. Definition 3.2 extends word by word to the case of a tropical morphism $f : C \to \mathbb{R}^n$ when C is a nodal tropical curve. Note that in this case, Definition 3.2 forces f to contract each node e of C.

As in section 3.1, given a combinatorial type α of tropical morphism $f : C \to \mathbb{R}^n$ from a nodal rational tropical curve C, the rigid deformation space $\mathcal{M}_\alpha$ of f has dimension $n + |\text{Edge}^0(C)| - |\text{Node}(C)|$, and coordinates on this space is given by a choice of a root vertex and an ordering of edges in $\text{Edge}^0(C) \setminus \text{Node}(C)$. In particular, if α is minimal and of degree d, then $\mathcal{M}_\alpha$ has dimension at most $(n+1)d + n - 3 - |\text{Node}(\alpha)|$.

Let us enumerate nodal rational tropical curves following section 4.1. As for a complex curve, we can associate its dual graph Γ_C to any connected nodal tropical curve C: vertices of Γ_C correspond to irreducible components of $C \setminus \text{Node}(C)$ and edges of Γ_C correspond to nodes of C.

Let us fix a tree Γ and a function $deg : \text{Vert}(\Gamma) \to \mathbb{N}$, and we define $d = \sum_{v \in \text{Vert}(\Gamma)} deg(v)$. We denote by $\mathcal{R}^{\mathbb{T}}(\Gamma, deg)$ the space of tropical morphisms $f : C \to \mathbb{R}^n$ of degree d where C is a connected rational tropical curve equipped with an isomorphism of graphs $\phi : \Gamma_C \to \Gamma$ such the map $deg \circ \phi$ is the degree of the corresponding irreducible components of C. We also fix $n-1$ functions $\gamma_j : \text{Vert}(\Gamma) \to \mathbb{N}$ for $j = 0, \dots, n-2$, and a function $\delta : \text{Edge}(\Gamma) \to \{0, \dots, n-1\}$ such that Equation (5) is satisfied. For any v in $\text{Vert}(\Gamma)$, choose a configuration $\omega_v^{\mathbb{T}}$ of approximable tropical linear subspaces of $\mathbb{R}^n$ containing exactly $\gamma_j(v)$ linear spaces of dimension j for $j = 0, \dots, n-2$. For any edge e of Γ, choose an approximable tropical linear subspace L_e of $\mathbb{R}^n$ of dimension $\delta(e)$. We define

$$\omega^{\mathbb{T}} = \bigcup_{v \in \mathrm{Vert}(\Gamma)} \omega_v^{\mathbb{T}} \bigcup_{e \in \mathrm{Edge}(\Gamma)} \{L_e\}.$$

Let us denote by $\mathcal{RS}^{\mathbb{T}}(\Gamma, \omega^{\mathbb{T}})$ the set of all tropical morphisms $f : C \to \mathbb{R}^n$ in $\mathcal{R}(\Gamma, deg)$ such that the component v passes through all linear spaces in $\omega_v^{\mathbb{T}}$ for $v \in \mathrm{Vert}(\Gamma)$, and such that the node e is mapped to L_e by f for $e \in \mathrm{Edge}(\Gamma)$.

Definition 4.3. *The configuration $\omega^{\mathbb{T}}$ is said to be generic if the set $\mathcal{RS}^{\mathbb{T}}(\Gamma, \omega^{\mathbb{T}})$ is finite and if any tropical morphism $f : C \to \mathbb{R}^n$ in $\mathcal{RS}^{\mathbb{T}}(d, \omega^{\mathbb{T}})$ satisfies the following conditions:*
- *f is transverse to infinity*
- *C is trivalent;*
- *for any vertex v of Γ and any tropical linear space L in $\omega_v^{\mathbb{T}}$, the set $f(C) \cap L$ consists of a single point which is in the relative interior of a face of maximal dimension of L;*
- *$f(Vert^0(C)) \cap \left(\cup_{v \in Vert(\Gamma)} \cup_{L \in \omega_v^{\mathbb{T}}} L \right) = \emptyset$.*

The following statement is proved using the same standard techniques in enumerative geometry than in the proof of Proposition 3.9.

Proposition 4.4. *Let $\omega^{\mathbb{T}} = \{L_1, \dots, L_k\}$. Then there exists a dense open subset $\mathcal{U}$ of $(\mathbb{R}^n)^k$ such that for any $(\vec{v}_1, \dots, \vec{v}_k) \in \mathcal{U}$ the configuration $\omega'^{\mathbb{T}} = \{L_1 + \vec{v}_1, \dots, L_k + \vec{v}_k\}$ is generic.*

From now on we suppose that the configuration $\omega^{\mathbb{T}}$ is generic. As in section 3.4, we assign a multiplicity to each element $f : C \to \mathbb{R}^n$ of $\mathcal{RS}^{\mathbb{T}}(\Gamma, \omega^{\mathbb{T}})$. Let us denote by α its combinatorial type. As in section 3.4, any constraint $L \in \omega^{\mathbb{T}}$ defines a classical affine space Λ_L defined over $\mathbb{Z}$ in $\mathcal{M}_\alpha$. In the case when $L \in \cup_{v \in \mathrm{Vert}(\Gamma)} \omega_v^{\mathbb{T}}$, we define the weight w_L of Λ_L as in section 3.4, i.e. w_L is equal to the weight of the edge e of C such that $f(e)$ intersects L. In the case when $L \in \bigcup_{e \in \mathrm{Edge}(\Gamma)} \{L_e\}$, we define the weight w_L of Λ_L to be equal to 1. Since $\omega^{\mathbb{T}}$ is generic we have

$$\bigcap_{L \in \omega^{\mathbb{T}}} \Lambda_q = \{f\},$$

and we define the $\omega^{\mathbb{T}}$*-multiplicity* of $f : C \to \mathbb{R}^n$ as the tropical intersection number in $\mathcal{M}_\alpha$ of all the hypersurfaces $w_L \Lambda_L$ for $L \in \omega^{\mathbb{T}}$. Note that if $\mathrm{Node}(C) = \emptyset$, we are again in the situation described in section 3.4.

Theorem 4.5 (Correspondence Theorem for reducible rational curves)**.** *If $\omega^{\mathbb{T}}$ is a generic configuration of complete tropical linear spaces, with the notation above we have*

$$N^n_{deg}(\Gamma, \gamma_0, \dots, \gamma_{n-2}, \delta) = \sum_{f \in \mathcal{RS}^{\mathbb{T}}(\Gamma, \omega^{\mathbb{T}})} \mu_{\omega^{\mathbb{T}}}(f).$$

Proof. The proof follows from a combination of Theorem 3.10 and Proposition 4.1. Recall the self-diffeomorphism H_t of $(\mathbb{C}^*)^n$ we defined in section 3.4. For each tropical complete linear space L in $\omega^{\mathbb{T}}$, we choose a family $(\mathcal{L}_{t,L})_{t>0}$ of complex linear spaces in $(\mathbb{C}^*)^n$ of dimension $\dim(L)$ such that the sets $\mathrm{Log} \circ H_t(\mathcal{L}_{t,L})$ converges to L when $t \to \infty$.

Let v be a vertex in $\mathrm{Vert}^\infty(\Gamma)$, denote by $e \in \mathrm{Edge}(\Gamma)$ its adjacent edge, and by v' the other vertex adjacent to e. We use the notation $\nu, h, \Gamma', \Gamma'', deg', deg'', \gamma'_j, \gamma''_j, \delta'$, and δ'' as in section 4.1. For each $t > 0$, we also construct the algebraic varieties V'_t and V''_t in $\mathbb{C}P^n$ as in the proof of Proposition 4.1. Since the configuration $\omega^{\mathbb{T}}$ is generic, for t big enough the configuration

$$\bigcup_{v \in \mathrm{Vert}(\Gamma)} \omega_{v,t} \bigcup_{e \in \mathrm{Edge}(\Gamma)} \{\mathcal{L}_{t,L_e}\}$$

is also generic, so we have

$$\dim(V'_t) = h_v(e) + 1, \quad \dim(V''_t) = h_{v'}(e) + 1, \quad \text{and} \quad N^n_{deg}(\Gamma, \gamma_0, \dots, \gamma_{n-2}, \delta) = |V'_t \cap V''_t \cap \mathcal{L}_{t,L_e}|.$$

Let us denote by V' the *tropicalisation* of the family (V'_t). As a set, V' is defined by

$$V' = \lim_{t\to+\infty} \mathrm{Log}(V_t).$$

The set V' can be equipped with the structure of a finite rational polyhedral complex of pure dimension $h_v(e)+1$ (see [BG84]), where rational means that each of the faces of V' has a direction defined over $\mathbb{Z}$. Given a face F of maximal dimension of V', we define its *weight* $w(F)$ as follows: pick a point $p = (p_1, \dots, p_n)$ in the relative interior of F, and choose a classical affine subspace G in $\mathbb{R}^n$ defined over $\mathbb{Z}$ of codimension $h_v(e)+1$ such that $F \cap G = \{p\}$; choose a $\mathbb{Z}$-basis $(e_1, \dots, e_{h_v(e)+1})$ of F, and a $\mathbb{Z}$-basis $(e_{h_v(e)+2}, \dots, e_n)$ of G; the $\mathbb{Z}$-basis $(e_{h_v(e)+2}, \dots, e_n)$ induces a linear map $\mathbb{Z}^{n-h_v(e)-1} \to \mathbb{Z}^n$ which in its turn induces a monomial map $\phi : (\mathbb{C}^*)^{n-h_v(e)-1} \to (\mathbb{C}^*)^n$; we define by G_t the image of $\phi((\mathbb{C}^*)^{n-h_v(e)-1})$ translated in $(\mathbb{C}^*)^n$ along the vector $(t^{p_1}, \dots, t^{p_n})$; the weight $w(F)$ of F is the integer such that there are exactly $w(F)|\det(e_1, \dots, e_n)|$ intersection points of V'_t and G_t (counted with multiplicity) whose image under Log_t converge to p when $t \to \infty$. Note that $w(F)$ is a positive integer and does not depend on the choice of the point $(p_1, \dots, p_n)$ (see for example [Spe]).

In the same way, we define the tropicalisation V'' of the family (V''_t), and the tropicalisation W of the variety $V'_t \cap V''_t \cap \mathcal{L}_{t,L_e}$. Note that W is a finite set of points, and that the weight $w(p)$ of $p \in W$ is equal to the number of points of $V'_t \cap V''_t \cap \mathcal{L}_{t,L_e}$ whose limit under $\mathrm{Log} \circ H_t$ is p.

Analogously, we define U' as the subset of $\mathbb{R}^n$ given by the union of all $f(C)$ where $f : C \to \mathbb{R}^n$ is a tropical morphism from an irreducible tropical rational curve of degree $deg(v)$ passing through all linear spaces in $\omega^{\mathbb{T}}_v$. We consider the set $S^{\mathbb{T}}$ of all tropical rational curves in $\mathcal{R}^{\mathbb{T}}(\Gamma'', deg'')$ such that the component v'' passes through all tropical linear spaces in $\omega^{\mathbb{T}}_{v''}$ for $v'' \in \mathrm{Vert}(\Gamma'')$, and such that the node e'' is mapped to $\mathcal{L}_{e''}$ for $e'' \in \mathrm{Edge}(\Gamma'')$. We define U'' as the subset of $\mathbb{R}^n$ given by the union of the component of C'' corresponding to v' when C'' runs over all elements of $S^{\mathbb{T}}$.

Note that U' and U'' are not necessarily tropical varieties, and might contain cells of dimension higher than the expected one. However, there exists a natural surjection $\psi : \mathcal{RS}^{\mathbb{T}}(\Gamma, \omega^{\mathbb{T}}) \to U' \cap U'' \cap L_e$, which is also injective if $\delta(e) > 0$. In particular the set $U' \cap U'' \cap L_e$ is finite. Since we have $V' \cap V'' \cap L_e \subset U' \cap U'' \cap L_e$, the set $V' \cap V'' \cap L_e$ is also finite and is the underlying set of W. Hence, for each point p in W the sets V' and U' on one hand and V'' and U'' on the other hand locally coincide at p, and the weight $w(p)$ is equal to the tropical intersection number of V', V'', and L_e at p. Recall that $N^n_{deg}(\Gamma, \gamma_0, \dots, \gamma_{n-2}, \delta) = d'd''$ where d' is the degree of V' and d'' the degree of V''. Hence to conclude the proof of the theorem, it remains us to prove that

$$\sum_{f \in \mathcal{RS}^{\mathbb{T}}(\Gamma, \omega^{\mathbb{T}})} \mu_{\omega^{\mathbb{T}}}(f) = d'd''. \tag{6}$$

Suppose that L_e is a point. In this case we define

$$\Omega^{\mathbb{T}\prime} = \omega^{\mathbb{T}}_v \bigcup \{L_e\} \quad \text{and} \quad \Omega^{\mathbb{T}\prime\prime} = \bigcup_{\substack{v'' \in \mathrm{Vert}(\Gamma) \\ v'' \neq v}} \omega^{\mathbb{T}}_{v''} \bigcup \{L_e\}.$$

In the notation of section 3.4, we clearly have that $\mathcal{RS}^{\mathbb{T}}$ is in bijection with $\mathcal{S}(deg(v), \Omega^{\mathbb{T}\prime}) \times \mathcal{RS}^{\mathbb{T}}(\Gamma'', \Omega^{\mathbb{T}\prime\prime})$, and thanks to Remark 3.13 we have

$$\sum_{f \in \mathcal{RS}^{\mathbb{T}}(\Gamma, \omega^{\mathbb{T}})} \mu_{\omega^{\mathbb{T}}}(f) = \sum_{f \in \mathcal{S}(deg(v), \Omega^{\mathbb{T}\prime})} \mu_{\Omega^{\mathbb{T}\prime}}(f) \sum_{f \in \mathcal{RS}^{\mathbb{T}}(\Gamma'', \Omega^{\mathbb{T}\prime\prime})} \mu_{\Omega^{\mathbb{T}\prime\prime}}(f).$$

Theorem 3.10 gives us that $\sum_{f \in \mathcal{S}(deg(v), \Omega^{\mathbb{T}\prime})} \mu_{\Omega^{\mathbb{T}\prime}}(f) = d'$ and induction gives us that $\sum_{f \in \mathcal{RS}^{\mathbb{T}}(\Gamma'', \Omega^{\mathbb{T}\prime\prime})} \mu_{\Omega^{\mathbb{T}\prime\prime}}(f) = d''$, so that Equality (6) is proved.

Let us suppose now that $\delta(e) > 0$. As we said above, this implies that the natural map $\psi : \mathcal{RS}^{\mathbb{T}}(\Gamma, \omega^{\mathbb{T}}) \to U' \cap U'' \cap L_e$ is a bijection. We are going to prove that if $f : C \to \mathbb{R}^n$ is an element of $\mathcal{RS}^{\mathbb{T}}(\Gamma, \omega^{\mathbb{T}})$, then $\mu_{\omega^{\mathbb{T}}}(f)$ is equal to the tropical intersection multiplicity of V', V'', and L_e at $\psi(f)$ which will prove Equality

(6). To compute $\mu_{\omega^{\mathbb{T}}}(f)$ we may suppose that L_e is an affine space of $\mathbb{R}$, whose primitive tropical structure is given by some matrix $A \in \mathcal{M}_{n-l,n}(\mathbb{Z})$.

Choose a point on e, and let C' be the topological closure of the component of $C \setminus \{p\}$ corresponding to v, and let C'' be the topological closure of the other. The tropical structure on C induces a structure of a tropical rational curve on C', as well as on C''. The tropical morphism f induces two tropical morphisms $f' : C' \to \mathbb{R}^n$ and $f'' : C'' \to \mathbb{R}^n$ with combinatorial type respectively α' and α''. Note that both f' and f'' contract the end corresponding to e. Let α be the combinatorial type of f, and let us fix coordinates on $\mathcal{M}_\alpha$ by choosing the root vertex v_1 to be a vertex adjacent to e, and by ordering edges of C first by the edges of C' and then the edges of C''. If we denote by $N = \dim(\mathcal{M}_\alpha)$, $N' = \dim(\mathcal{M}_{\alpha'})$, and $N'' = \dim(\mathcal{M}_{\alpha''})$ then we have

$$N = N' + N'' - n.$$

We also order the equations of the spaces Λ_L in $\mathcal{M}_\alpha$ as follows: first the condition that v_1 is mapped to L_e, then the conditions relative to edges contained in C', and then the conditions relative to edges contained C''. The condition that v_1 is mapped to L_e is just given by equations of L_e itself, hence $\mu_{\omega^{\mathbb{T}}}(f)$ is equal to $\prod_{L\in\omega^{\mathbb{T}}} w_L |\det(M)|$ where

$$M = \begin{pmatrix} A & 0 & 0 \\ B & C & 0 \\ D & 0 & E \end{pmatrix}$$

with

$$B \in \mathcal{M}_{N'-h_v(e)-1,n}(\mathbb{Z}),\ C \in \mathcal{M}_{N'-h_v(e)-1,N'-n}(\mathbb{Z}),\ D \in \mathcal{M}_{N''-h_{v'}(e)-1,n}(\mathbb{Z}),\ E \in \mathcal{M}_{N''-h_{v'}(e)-1,N''-n}(\mathbb{Z}).$$

Let us consider the set $Z' \subset \mathcal{M}_\alpha$ defined by the system of linear equations given by the matrix $M' = (B\ C\ 0)$, the set $Z'' \subset \mathcal{M}_\alpha$ defined by the system of linear equations given by the matrix $M'' = (D\ 0\ E)$, and the evaluation map

$$\begin{array}{cccc} ev: & \mathcal{M}_\alpha & \longrightarrow & \mathbb{R}^n \\ & f & \longmapsto & f(v_1) \end{array}.$$

The map ev is linear on $\mathcal{M}_\alpha$, and a neighborhood of f in Z' (resp. Z'') is mapped by ev to a neighborhood of $\psi(f)$ in V' (resp. V''). In particular $Ker(ev) \cap Ker(M') = 0$, i.e. $Ker(C) = 0$. Since $h_v(e) \le n-1$, we have $rk(C) = N' - n$, and so by means of elementary operations on the row of M' we obtain the matrix

$$\begin{pmatrix} B_0 & 0 & 0 \\ B_1 & C_0 & 0 \end{pmatrix}$$

where $B_0 \in \mathcal{M}_{n-1-h_v(e),n}(\mathbb{Q})$ is of rank $n-1-h_v(e)$. Moreover we have $ev(Z') = Ker(B_0) + \psi(f)$ as classical affine spaces. In the same way, by means of elementary operations on the row of M'' we obtain the matrix

$$\begin{pmatrix} D_0 & 0 & 0 \\ D_1 & E_0 & 0 \end{pmatrix}$$

where $D_0 \in \mathcal{M}_{n-1-h_{v'}(e),n}(\mathbb{Q})$ is of rank $n-1-h_{v'}(e)$, and $ev(Z'') = Ker(D_0) + \psi(f)$. Hence we get that

$$\det(M) = \det \begin{pmatrix} A & 0 & 0 \\ B_0 & 0 & 0 \\ B_1 & C_0 & 0 \\ D_0 & 0 & 0 \\ D_1 & 0 & E_0 \end{pmatrix} = \det \begin{pmatrix} A & 0 & 0 \\ B_0 & 0 & 0 \\ D_0 & 0 & 0 \\ 0 & C_0 & 0 \\ 0 & 0 & E_0 \end{pmatrix} = \det \begin{pmatrix} A \\ B_0 \\ D_0 \end{pmatrix} \det(C_0)\det(E_0).$$

The vector space $Ker(B_0)$ is the direction of the affine span of a neighborhood of V' at $\psi(f)$, however the two tropical structures might be different, i.e. as a tropical cycle the tropical variety defined by $Ker(B_0) + \psi(f)$ is equal to $w'V'$ with $w' \in \mathbb{Q}_{>0}$. In the same way, the tropical cycle defined by $Ker(E_0) + \psi(f)$ is equal to $w''V''$ with $w'' \in \mathbb{Q}$. With this notation, we have that

$$\mu_{\omega^{\mathbb{T}}}(f) = \left(\prod_{L\in\omega^{\mathbb{T}}} w_L\right) w'w''|\det(C_0)\det(E_0)|(V'.V''.L_e)_{\psi(f)}$$

where $(V'.V''.L_e)_{\psi(f)}$ denotes the tropical intersection multiplicity of V', V'', and L_e at $\psi(f)$. Hence it remains us to prove that

$$\left(\prod_{L\in\omega^{\mathbb{T}}} w_L\right) w'w''|\det(C_0)\det(E_0)| = 1.$$

In order to prove this, let L' be a tropical linear space of dimension $n-1-h_v(e)$, which is also a classical affine space, and which contains L_e. The primitive tropical structure on L' is defined by a matrix

$$(A\ A') \in \mathcal{M}_{h_v(e)+1,n}(\mathbb{Z}).$$

We define $\Omega^{\mathbb{T}\prime} = \omega_v^{\mathbb{T}} \bigcup \{L'\}$. Recall that f' denotes the tropical morphism with one marked point induced by f on C', thanks to Theorem 3.10 and Remark 3.13 we get that

$$(V'.L')_{\psi(f)} = \mu_{\Omega'^{\mathbb{T}}}(f').$$

On the other hand, we also have

$$\det\begin{pmatrix} A & 0\\ A' & 0\\ B_0 & 0\\ 0 & C_0\end{pmatrix} = \det\begin{pmatrix} A\\ A'\\ B_0\end{pmatrix}\det(C_0) = w'|\det(C_0)|(V'.L')_{\psi(f)}$$

and so

$$\mu_{\Omega'^{\mathbb{T}}}(f') = \left(\prod_{L\in\Omega'^{\mathbb{T}}} w_L\right) w'|\det(C_0)|(V'.L')_{\psi(f)}$$

from which we deduce that

$$\left(\prod_{L\in\Omega'^{\mathbb{T}}} w_L\right) w'|\det(C_0)| = 1.$$

Note that the weights w_L for f and f' coincide for $L\in\Omega^{\mathbb{T}\prime}$.

In the same way, let L'' be a tropical linear space of dimension $n-1-h_{v'}(e)$, which is also a classical affine space, and which contains L_e. We define

$$\Omega^{\mathbb{T}\prime\prime} = \bigcup_{\substack{v''\in \mathrm{Vert}(\Gamma)\\ v''\neq v}} \omega_{v''}^{\mathbb{T}} \bigcup\{L''\}.$$

Then we have

$$(V''.L'')_{\psi(f)} = \mu_{\Omega''^{\mathbb{T}}}(f'') = \left(\prod_{L\in\Omega''^{\mathbb{T}}} w_L\right) w''|\det(E_0)|(V''.L'')_{\psi(f)}$$

from which we deduce that

$$\left(\prod_{L\in\Omega''^{\mathbb{T}}} w_L\right) w''|\det(E_0)| = 1.$$

Now the result follows since we have $\left(\prod_{L\in\Omega'^{\mathbb{T}}} w_L\right)\left(\prod_{L\in\Omega''^{\mathbb{T}}} w_L\right) = \left(\prod_{L\in\omega^{\mathbb{T}}} w_L\right)$. □

5. Proof of Theorem 2.10

We denote by $\pi:\mathbb{R}^n\to\mathbb{R}^{n-1}$ the linear projection forgetting the last coordinate.

5.1. **Floors, elevators, and walls.** The starting idea of floor decomposition is to compute the numbers $N_d^n(l_1, \ldots, l_{n-2})$ by induction on the dimension n. As easy at it sounds, this approach does not work straightforwardly and one has to work carefully: it is easy to compute that through one point p and two tropical lines L_1 and L_2 in $\mathbb{R}^3$ passes exactly 1 tropical line L. However, there exists infinitely many tropical lines in $\mathbb{R}^2$ passing trough $\pi(p)$, $\pi(L_1)$, and $\pi(L_2)$, and without knowing L, it is not clear at all which one of these planar lines is $\pi(L)$.

To make the induction work, we first stretch the configuration ω is the direction $U_n = (0, \ldots, 0, -1)$. Then the tropical curves we are counting break in several *floors* for which we can apply induction.

Example 5.1. Let us explain how to use the floor decomposition technique in a simple case, see also Example 3.12. Let use choose a point $L_3 = (-2, -1, a_3)$ in $\mathbb{R}^3$ and two tropical lines L_1 and L_2 in $\mathbb{R}^3$ such that L_1 (resp. L_2) consists of only one edge with direction $(-1, 0, 0)$ (resp. $(0, -1, 0)$) contained in the horizontal plane with equation $z = a_1$ (resp. $z = a_2$). If a_3 is much more bigger than a_2 which in its turn is much more bigger than a_1, then the unique tropical line L in $\mathbb{R}^3$ passing through p, L_1, and L_2 is depicted in Figure 7, and $\pi(L)$ is the unique tropical line in $\mathbb{R}^2$ passing through $\pi(p)$ and $\pi(L_1) \cap \pi(L_2)$.

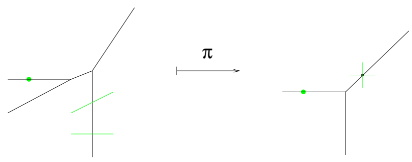
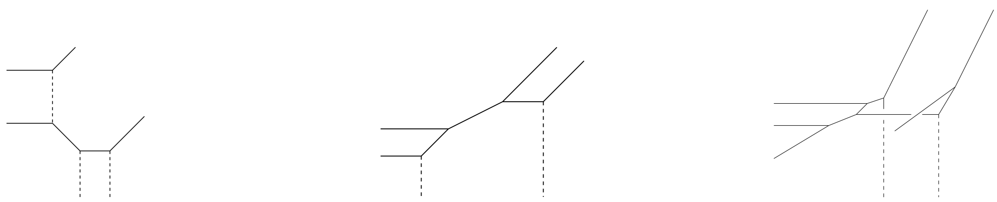


FIGURE 7. Floor decomposition technique to compute $N_1^3(1, 2) = 1$

Definition 5.2. *Let C be a rational tropical curve and $f : C \to \mathbb{R}^n$ a tropical morphism. An elevator of f is an edge e of C with $u_{f,e} = (0, \ldots, 0, \pm 1)$. A floor of f is a connected component of the topological closure of $C \setminus Vert^\infty(C)$ minus all its elevators.*

Note that if $f : C \to \mathbb{R}^n$ is a tropical morphism of degree d in $\mathbb{R}^n$ and $\mathcal{F}$ is a floor of f, then C induces a structure of tropical curve on $\mathcal{F}$ and $\pi \circ f_{|\mathcal{F}} : \mathcal{F} \to \mathbb{R}^{n-1}$ is a tropical morphism of degree $1 \leq d' \leq d$. The integer d' is called the *degree* of $\mathcal{F}$.

Example 5.3. Examples of planar and spatial conics decomposed into floors and elevators are depicted in Figure 8. Elevators are depicted in dotted lines.



a) A planar conic with two floors b) A planar conic with one floor c) A spatial conic with one floor

FIGURE 8

Definition 5.2 extends to any tropical varieties in $\mathbb{R}^n$, however we keep on restricting ourselves to complete tropical linear spaces. Note that if F is an unbounded face of dimension j of a complete tropical linear space of dimension j, then there exists a subset I of cardinal j of $\{1, \ldots, n+1\}$ such that the direction of the affine span of F is equal to $Vect(U_i,\ i \in I)$.

Definition 5.4. *Let L be a complete tropical linear space in $\mathbb{R}^n$. The wall (resp. floor) of L is the union of all faces of L which contain (resp. do not contain) the direction* $(0, \ldots, 0, -1)$.

Note that if L is a complete tropical linear space of dimension j in $\mathbb{R}^n$ with wall W and floor F, then $\pi(W)$ is a complete tropical linear space of dimension $j - 1$ in $\mathbb{R}^{n-1}$, and $\pi(F) = \pi(L)$ is a complete tropical linear space of dimension j in $\mathbb{R}^{n-1}$.

Example 5.5. A tropical plane L in $\mathbb{R}^3$ together with its wall W are depicted in Figure 9. Clearly, $\pi(L) = \pi(F) = \mathbb{R}^2$ and $\pi(W)$ is a tropical line in $\mathbb{R}^2$.

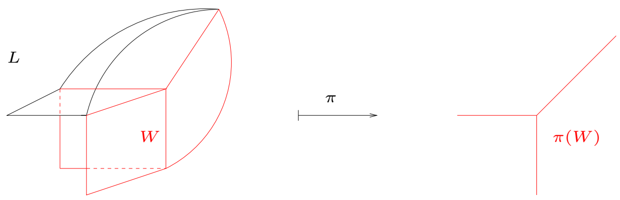


FIGURE 9. $\pi(L) = \mathbb{R}^2$ and $\pi(W)$ is a tropical line

Let us fix some integers $d \geq 1$, $n \geq 2$, $l_0 \geq 0$, $\ldots$, $l_{n-2} \geq 0$ subject to equality (1), and let us choose a generic configuration $\omega^{\mathbb{T}}$ of complete tropical linear spaces in $\mathbb{R}^n$ containing exactly l_j spaces of dimension j. As before, we denote by $\mathcal{S}(d, \omega^{\mathbb{T}})$ is the set of all rational tropical morphisms $f : C \to \mathbb{R}^n$ of degree d such that $f(C)$ intersects all tropical linear spaces in $\omega^{\mathbb{T}}$.

Definition 5.6. *An element L of $\omega^{\mathbb{T}}$ is called a vertical (resp. horizontal) constraint for $f \in \mathcal{S}(d, \omega^{\mathbb{T}})$ if $f(C) \cap L$ lies in the wall (resp. floor) of L.*

Let us denote by $\mathrm{Vert}(L)$ the set of vertices of the complete tropical linear space L, and let us fix a hypercube $\mathcal{H}_{n-1}$ in $\mathbb{R}^{n-1}$ such that the cylinder $\mathcal{H}_{n-1} \times \mathbb{R}$ contains the set $\cup_{L \in \omega^{\mathbb{T}}} \mathrm{Vert}(L)$. Given two points $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ in $\mathbb{R}^n$, we define $|p - q|_n = |p_n - q_n|$. Finally, we define $R_{\mathcal{H}}$ to be the length of the edges of $\mathcal{H}_{n-1}$, and

$$R(\omega) = \min_{L \neq L' \in \omega^{\mathbb{T}},\ p \in \mathrm{Vert}(L),\ q \in \mathrm{Vert}(L')} |p - q|_n.$$

The following lemma and corollary are the starting observation of the technique.

Lemma 5.7. *For any a tropical morphism $f : C \to \mathbb{R}^n$ in $\mathcal{S}(d, \omega^{\mathbb{T}})$, we have $f(\mathit{Vert}^0(C)) \subset \mathcal{H}_{n-1} \times \mathbb{R}$.*

Proof. Given a real number a, we define

$H_a = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = a\}$, $H_a^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > a\}$, $H_a^- = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 < a\}$.

Let us write $\mathcal{H}_{n-1} = [-A; A]^{n-1}$, and let us suppose that there exists $f : C \to \mathbb{R}^n$ in $\mathcal{S}(d, \omega^{\mathbb{T}})$, and $v \in \mathrm{Vert}^0(C)$ such that $f(v) \in H_{-A}^-$. Let a be the minimal real number such that H_a contains the image by f of a vertex of C. By assumption we have $a < -A$. If $v \in \mathrm{Vert}^0(C)$ is mapped to H_a by f and is adjacent to an end e of C mapped to H_a^-, we deduce from $w_{f,e} = 1$ that C has at most one edge adjacent to v mapped to H_a^+. This implies that for ε a real number small enough, there exists a tropical morphism $f_\varepsilon : C_\varepsilon \to \mathbb{R}^n$ of the same combinatorial type as f and such that (see Figure 10):

- if $f(v) \notin H_a$, then $f_\varepsilon(v_\varepsilon) = f(v)$;
- if $f(v) \in H_a$, then $f_\varepsilon(v_\varepsilon) \in H_{a+\varepsilon}$;

where v_ε is the vertex of C_ε corresponding to the vertex v of C. In particular, we have $f_0 = f$.

Now we prove that for ε small enough, the tropical morphism f_ε is in $\mathcal{S}(d, \omega^{\mathbb{T}})$. First, note that since a tropical linear space L in $\omega^{\mathbb{T}}$ is complete, any unbounded face of L intersecting the half-space H_{-A}^- contains the vector $(-1, 0, \ldots, 0)$. Moreover since $\cup_{L \in \omega^{\mathbb{T}}} \mathrm{Vert}(L) \subset \mathcal{H}_{n-1} \times \mathbb{R}$, any face of L intersecting H_{-A}^- is necessarily unbounded. Hence if e is an end of C with $u_{f,e} = (-1, 0, \ldots, 0)$ whose adjacent vertex in $\mathrm{Vert}^0(C)$ is mapped to H_α, then $f(e)$ does not intersect any tropical linear space in $\omega^{\mathbb{T}}$. Suppose that an edge e of C with $f(e) \subset H_\alpha$ intersects the constraint L at a point p. Then by denoting U a small neighborhood of p in $\mathbb{R}^n$ and by e_ε the edge of C_ε corresponding to e, we have that $f_\varepsilon(e_\varepsilon) \cap (U + (\varepsilon, 0, \ldots, 0)) = (f(e) \cap U) + (\varepsilon, 0, \ldots, 0)$ for ε small enough. In particular $f_\varepsilon(e_\varepsilon)$ intersects the constraint L. In conclusion the morphism f_ε meets all constraints in $\omega^{\mathbb{T}}$, i.e. $f_\varepsilon \in \mathcal{S}(d, \omega^{\mathbb{T}})$. Hence the set $\mathcal{S}(d, \omega^{\mathbb{T}})$ contains infinitely many tropical morphisms, which contradicts the assumption that $\omega^{\mathbb{T}}$ is generic. Hence we have $f(\mathrm{Vert}^0(C)) \subset H_{-A}^+$. The proof that $f(\mathrm{Vert}^0(C)) \subset H_A^-$ is analogous. □

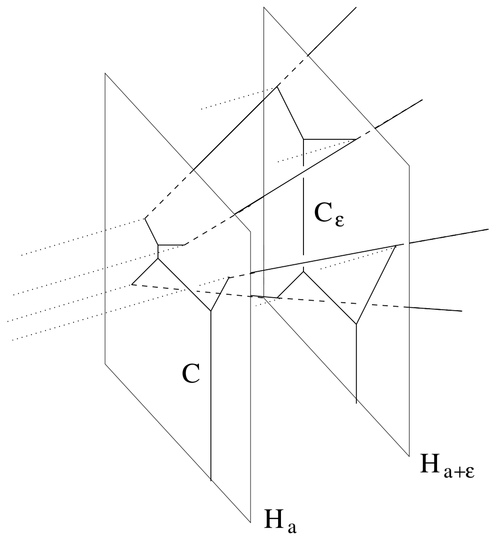


FIGURE 10. An infinite family of tropical morphisms

Corollary 5.8. *There exists a real number $D(n,d)$, depending only in n and d, such that if $R(\omega) \geq R_{\mathcal{H}} \times D(n,d)$ then for each morphism $f : C \to \mathbb{R}^n$ in $\mathcal{S}(d,\omega^{\mathbb{T}})$ and for each floor $\mathcal{F}$ of f, $f(\mathcal{F})$ meets one and exactly one horizontal constraint.*

Proof. If $R(\omega)$ is big enough compared to $R_{\mathcal{H}}$, then the floors of two distinct constraints in $\omega^{\mathbb{T}}$ do not intersect in $\mathbb{R}^n$. The number of edges of C and the absolute values of the coordinates of the vectors $w_{f,e}u_{f,e}$ for all edges of C are bounded from above by a number which depends only on d. So according to Lemma 5.7, if (xy) is a path in C which do not contain any elevator and such that $f(x)$ and $f(y)$ are in $\mathcal{H}_{n-1} \times \mathbb{R}$, then the last coordinate of $f(y)$ can be bounded in terms of the last coordinate of $f(x)$ and a quantity which depends only on d and $R_{\mathcal{H}}$. Hence if $R(\omega)$ is big enough compared to $R_{\mathcal{H}}$, a floor of f cannot meet two distinct horizontal constraints.

Suppose that f has a floor $\mathcal{F}$ which does not meet any horizontal constraints. Then, by translating slightly $\mathcal{F}$ in the direction $(0,\ldots,0,1)$, one construct infinitely many tropical morphisms in $\mathcal{S}(d,\omega^{\mathbb{T}})$. □

Definition 5.9. *If $R(\omega) \geq R_{\mathcal{H}}D(n,d)$, we say that ω is a (d,n)-decomposing configuration.*

Note that imposing to a configuration $\omega^{\mathbb{T}}$ to be (d,n)-decomposing is the same than imposing conditions on the relative position of the vertices of elements of $\omega^{\mathbb{T}}$. In particular, it makes sense to say that a configuration $\omega^{\mathbb{T}}$ is (d,n)-decomposing even if its elements do not satisfy equality (1).

The choice of the preferred direction U_n provides a natural partial order among the tropical linear spaces.

Definition 5.10. *Let L and L' in $\mathbb{R}^n$ be two complete tropical linear spaces. We say that L is higher than L', and denote $L \gg L'$, if any vertex of L has greater last coordinate than all vertices of L'.*

Note that a (d,n)-decomposing configuration is totally ordered for $\gg$.

5.2. **From tropical curves to floor diagrams.** Recall that we fixed some integers $d \geq 1$, $n \geq 2$, $l_0 \geq 0$, $\ldots$, $l_{n-2} \geq 0$ subject to equality (1), and a (n,d)-decomposing configuration $\omega^{\mathbb{T}}$ of complete tropical linear spaces in $\mathbb{R}^n$ containing exactly l_j spaces of dimension j. We define the following complete tropical linear spaces in $\mathbb{R}^{n-1}$

$$L'_L = \pi(W), \quad \text{and} \quad \widetilde{L}'_A = \bigcap_{L\in A} \pi(L)$$

where W is the wall of L, and $A \subset \omega^{\mathbb{T}}$. For simplicity we also denote by $\widetilde{L}'_L$ the complete tropical linear space $\widetilde{L}'_{\{L\}}$.

Definition 5.11. *The configuration $\omega^{\mathbb{T}}$ is said to be generic as an (n,d)-decomposing configuration if $\omega^{\mathbb{T}}$ is generic and if for any partition of $\omega^{\mathbb{T}}$ in subsets $A_0, A_1, \ldots, A_l$, the configuration made of the complete tropical linear spaces L'_L with $L \in A_0$, and $\widetilde{L}'_{A_1}, \ldots, \widetilde{L}'_{A_l}$ is generic.*

Once again, it follows from standard techniques in tropical enumerative geometry that the space of generic (n,d)-decomposing configurations is an open dense subset of the set of all (n,d)-decomposing configurations. From now on, we assume that $\omega^{\mathbb{T}}$ is a generic (n,d)-decomposing configuration.

To a tropical morphism $f : C \to \mathbb{R}^n$ in $\mathcal{S}(d,\omega^{\mathbb{T}})$, we associate the following floor diagram, denoted by $\mathcal{D}(f)$: edges of $\mathcal{D}(f)$ correspond to elevators of f, and vertices of $\mathcal{D}(f)$ correspond to floors of f and to vertices in $\text{Vert}^\infty(C)$ which are adjacent to an elevator of f; edges of $\mathcal{D}(f)$ inherit a natural weight from weights of $f : C \to \mathbb{R}^n$; moreover $\mathbb{R}$ is naturally oriented, and edges of $\mathcal{D}(f)$ inherit this orientation, since they are all parallel to the coordinate axis $\{(0,\ldots,0)\} \times \mathbb{R}$. It is immediate from Definitions 2.1 and 3.8 that $\mathcal{D}(f)$ is a floor diagram of degree d. Note in addition that the divergence of a vertex of $\mathcal{D}(f)$ in $\text{Vert}^\circ(\mathcal{D}(f))$ is the degree of the corresponding floor of f. Thanks to Definition 3.8, we can define a map $m : \omega^{\mathbb{T}} \to \mathcal{D}(f) \setminus \text{Source}(\mathcal{D}(f))$ by sending a linear space L in $\omega^{\mathbb{T}}$ to the corresponding elevator or floor of f intersecting L.

Lemma 5.12. *The map $m : \omega^{\mathbb{T}} \to \mathcal{D}(f) \setminus Source(\mathcal{D}(f))$ is a marking of $\mathcal{D}(f)$ for the total order $>>$ on $\omega^{\mathbb{T}}$.*

Proof. Since any floor of f meets a horizontal constraint according to Corollary 5.8, the image of m contains $\text{Vert}^\circ(\mathcal{D}(f))$. Suppose that there exist $L' >> L$ in $\omega^{\mathbb{T}}$ such that $m(L') < m(L)$. Then L' has to be a vertical constraint for f. In particular it meets a floor of f, i.e. $m(L') \in \text{Vert}^\circ(\mathcal{D}(f))$. The horizontal constraint L'' of this floor satisfies $L >> L''$ and $m(L'') = m(L')$. □

Example 5.13. The marked floor diagram corresponding to the tropical line from Examples 3.12 and 5.1 passing through a point and two lines corresponds to the second picture from the left of Figure 2.

In other words, we defined a map $\Phi : f \mapsto (\mathcal{D}(f), m)$ from the set $\mathcal{S}(d,\omega^{\mathbb{T}})$ to the set of floor diagrams of degree d marked by $\omega^{\mathbb{T}}$. Theorem 2.10 is now a corollary of next proposition and Theorem 3.10.

Proposition 5.14. *If $(\mathcal{D}, m)$ is a floor diagram marked by $\omega^{\mathbb{T}}$, then*

$$\sum_{f\in\Phi^{-1}(\mathcal{D},m)} \mu_{\omega^{\mathbb{T}}}(f) = \mu_{\mathbb{C}}(\mathcal{D}, m).$$

Proof. The case $n = 2$ is proved in [BM08], so we suppose that $n \geq 3$. Let Γ be the graph obtained from $\mathcal{D}$ by contracting edges adjacent to a source of $\mathcal{D}$. Given a vertex v in $\text{Vert}^\circ(\mathcal{D})$, we denote by $\text{Edge}^\infty(\mathcal{D}, v)$ the set of edges in $\text{Vert}^\infty(\mathcal{D})$ which are adjacent to v (in other words, $\text{Vert}^\infty(\mathcal{D})$ is the set of edges of $\mathcal{D}$ adjacent to v that are contracted by the map $\mathcal{D} \to \Gamma$). We also define $deg(v)$ to be equal to the divergence of the corresponding vertex of $\mathcal{D}$, and we define the set $\widetilde{\Omega}^{\mathbb{T}}_v$ of complete tropical linear spaces in $\mathbb{R}^{n-1}$ whose element are

- $\pi(\min(m^{-1}(v)))$;
- L'_L if $L \in m^{-1}(v)$ and $L >> \min(m^{-1}(v))$;
- $\widetilde{L}_{m^{-1}(e)}$ if $e \in \text{Edge}^\infty(\mathcal{D}, v)$.

We also define $\Omega^{\mathbb{T}}_v = \widetilde{\Omega}^{\mathbb{T}}_v$ if $\dim(\min(m^{-1}(v))) < n-2$ and by $\Omega^{\mathbb{T}}_v = \widetilde{\Omega}^{\mathbb{T}}_v \setminus \{\pi(\min(m^{-1}(v)))\}$ otherwise. To each edge e of Γ, we associate the complete tropical linear space $L_e = \widetilde{L}_{m^{-1}(e)}$. By construction we have

$$\begin{aligned}\sum_{v\in\text{Vert}(\Gamma)} \sum_{L\in\Omega^{\mathbb{T}}_v} (n-2-\dim(L)) &= \sum_{v\in\text{Vert}(\Gamma)} \sum_{L\in\widetilde{\Omega}^{\mathbb{T}}_v} (n-2-\dim(L)) \\ &= \sum_{v\in\text{Vert}(\Gamma)} \Big[\sum_{L\in m^{-1}(v)}(n-1-\dim(L)) - 1 \\ &\qquad + \sum_{e\in\text{Edge}^\infty(\mathcal{D},v)} \Big(n-2-\sum_{L\in m^{-1}(e)} (n-1-\dim(L))\Big)\Big]\end{aligned}$$

and

$$\sum_{e\in\text{Edge}(\Gamma)} (n-1-\dim(L_e)) = \sum_{e\in\text{Edge}(\Gamma)} \left(n-1-\sum_{L\in m^{-1}(e)} (n-1-\dim(L))\right).$$

In particular, defining $\gamma_j(v)$ to be equal to the number of spaces of dimension j in $\Omega^{\mathbb{T}}_v$, and by $\delta(e)$ to be the dimension of L_e, we see that equality (5) is satisfied with $n-1$ instead of n, since equality (1) is satisfied by $\omega^{\mathbb{T}}$. Denoting by $\mathrm{Vert}_h(\mathcal{D})$ the set of vertices v of $\mathcal{D}$ such that $\dim(\min(m^{-1}(v))) = n-2$, we have by Proposition 4.1

$$\mu(\mathcal{D},m) = \left(\prod_{e\in \mathrm{Edge}(\mathcal{D})} w_e^{1+|m^{-1}(e)|}\right)\left(\prod_{v\in \mathrm{Vert}_h(\mathcal{D})} \mathrm{div}(v)\right) N^{n-1}_{deg}(\Gamma,\gamma_0,\dots,\gamma_{n-3},\delta).$$

Any tropical morphism $f: C\to\mathbb{R}^n$ induces a tropical morphism $\widetilde{f}:\widetilde{C}\to\mathbb{R}^{n-1}$ where $\widetilde{C}$ is the reducible tropical curve obtained from C by declaring the length of all elevators of f to be infinite. We say that f is a lifting of $\widetilde{f}$. As in section 4.2, we define

$$\Omega^{\mathbb{T}} = \bigcup_{v\in\mathrm{Vert}(\Gamma)} \Omega^{\mathbb{T}}_v \bigcup_{e\in\mathrm{Edge}(\Gamma)} \{L_e\},$$

and we denote by $\mathcal{RS}^{\mathbb{T}}(\Gamma,\Omega^{\mathbb{T}})$ the set of all tropical morphisms $f': C'\to\mathbb{R}^{n-1}$ in $\mathcal{R}(\Gamma,deg)$ such that the component v passes through all linear spaces in $\Omega^{\mathbb{T}}_v$ for $v\in\mathrm{Vert}(\Gamma)$, and such that the node e is mapped to L_e by f'. According to Corollary 5.8, we have a natural map

$$\begin{array}{cccc}\Theta: & \Phi^{-1}(\mathcal{D},m) & \longrightarrow & \mathcal{RS}^{\mathbb{T}}(\Gamma,\Omega^{\mathbb{T}}) \\ & f & \longmapsto & \widetilde{f}\end{array}$$

Given v a vertex of Γ, let us denote by C'_v the component of C' corresponding to v. If follows from the fact that $\omega^{\mathbb{T}}$ is a generic (n,d)-decomposing configuration that

$$|\Theta^{-1}(f')| = \prod_{v\in\mathrm{Vert}(\Gamma)} |f'(C'_v)\cap\pi(L_{\min(m^{-1}(v))})|.$$

Indeed, first note that all the sets $f'(C'_v)\cap\pi(L_{\min(m^{-1}(v))})$ are finite by genericity conditions. Next, the only freedom we have in the choice of a lifting $f:C\to\mathbb{R}^n$ of f' is the choice of the length of the elevators of f. But for a choice of an intersection points p_v of $f'(C'_v)$ and $\pi(\min(m^{-1}(v)))$ for all $v\in\mathrm{Vert}(\Gamma)$, there exists exactly one such lifting in $\Phi^{-1}(\mathcal{D},m)$ with $\pi(f(C)\cap\min(m^{-1}(v)))=p_v$ since the choice of p_v fixes the intersection point of the floor of C corresponding to v with the horizontal constraint $\min(m^{-1}(v))$.

Let us now compare the multiplicity of f and $\widetilde{f}$. To avoid any ambiguity, we denote by $w_L\Lambda_L$ the tropical space defined by L in $\mathcal{M}_\alpha$, and by $\widetilde{w}_L\widetilde{\Lambda}_L$ the tropical space defined by L in $\mathcal{M}_{\widetilde{\alpha}}$. A choice of a root vertex of C and of an ordering of edges of C induces coordinates on $\mathcal{M}_\alpha$, as well as on $\mathcal{M}_{\widetilde{\alpha}}$. Moreover we have the natural projection $\pi_{\alpha,\widetilde{\alpha}}:\mathcal{M}_\alpha\to\mathcal{M}_{\widetilde{\alpha}}$ forgetting the length of elevators of f.

Let e be an elevator of f (or equivalently an edge of $\mathcal{D}$) adjacent to a vertex v of C. Clearly, f is in Λ_L if and only if $\widetilde{f}$ is in $\widetilde{\Lambda}_{\widetilde{L}'_{m^{-1}(e)}}$. Since all constraints are tropical linear spaces, we can choose a $\mathbb{Z}$-basis of $\widetilde{L}'_{m^{-1}(e)}$ which lifts to a $\mathbb{Z}$-basis for any constraint in $m^{-1}(e)$. Therefore we have

$$\pi_{\alpha,\widetilde{\alpha}}\left(\prod_{L\in m^{-1}(e)} w_L\Lambda_L\right) = w_{f,e}^{|m^{-1}(v)|}\widetilde{w}_{\widetilde{L}'_{m^{-1}(e)}}\widetilde{\Lambda}_{\widetilde{L}'_{m^{-1}(e)}}.$$

If L is a vertical constraint of dimension j, let us choose $(U_n,e_1,\dots,e_{j-1})$ a $\mathbb{Z}$-basis of L (recall that $U_n=(0,\dots,0,1)\in\mathbb{R}^n$). Then, if v is a vertex adjacent to the edge e of C meeting L, by Proposition 3.11 the equation in $\mathcal{M}_\alpha$ defining the structure on $w_L\Lambda_L$ is

$$U_n\wedge e_1\wedge\dots\wedge e_{j-1}\wedge w_{f,e}u_{f,e}\wedge f(v)=0.$$

This equation is satisfied if and only if

$$\pi(e_1)\wedge\dots\wedge\pi(e_{j-1})\wedge\pi(w_{f,e}u_{f,e})\wedge\pi(f(v))=0. \tag{7}$$

Since L is a tropical linear space, we have that $(\pi(e_0),\dots,\pi(e_{j-1}))$ is a $\mathbb{Z}$-basis of $\pi(L)$. Moreover we have that $\pi(w_{f,e}u_{f,e}) = w_{\widetilde{f},e}u_{\widetilde{f},e}$ so the tropical structure on $\widetilde{w}_{L'}\widetilde{\Lambda}_{L'}$ is precisely given by Equation (7). In other words, we have

$$\pi_{\alpha,\widetilde{\alpha}}\left(w_L\Lambda_L\right) = \widetilde{w}_{L'}\widetilde{\Lambda}_{L'}.$$

Let L be a horizontal constraint meeting a floor $\mathcal{F}$ of f. In the system of equations defining the tropical space $w_l\Lambda_L$, exactly one equation involves the last coordinate of $f(v)$. We denote this equation by $\eta_{\mathcal{F}}$. Recall that we choose a root vertex v_1 on C in order to put coordinates on $\mathcal{M}_\alpha$, and that this choice also induces a choice of a root vertex for $\widetilde{C}$. We re-arrange coordinates on $\mathcal{M}_\alpha$ as follows: the first coordinates are also coordinates for $\widetilde{f}$, then we have the coordinate x_n, and then all coordinates l_e for e a bounded elevators of C. These last $|\mathrm{Edge}(\Gamma)|$ coordinates are ordered as follows: the last elevator e is an end of Γ, the penultimate elevator is an end of $\Gamma\setminus\{e\}$, and so on... If $\mathcal{F}$ is a floor of f not containing v_1, we denote by $e_{\mathcal{F}}$ the elevator of f adjacent to $\mathcal{F}$ which is in the connected component of $C\setminus\mathcal{F}$ containing v_1. Hence the order on elevators of f induces an order on floors of f not containing v_1. We extend this order on all the floors of f by declaring that the floor containing v_1 is the minimal element. We define $w_{\mathcal{F}} = w_{f,e_{\mathcal{F}}}$ if $\mathcal{F}$ doesn't contain v_1, and $w_{\mathcal{F}} = 1$ otherwise. Since the configuration $\omega^{\mathbb{T}}$ is a (d,n)-decomposing configuration, each floor of f meets one and exactly one horizontal constraint. In consequence we can re-order the lines of the determinant whose absolute value is $\mu_{\omega^{\mathbb{T}}}(f)$ such that the first $|Ve(\mathcal{D})|$ equations are the equations $\eta_{\mathcal{F}}$. Hence all equations coming from intersection of floors of f with horizontal constraints can be written

$$\left(\begin{array}{c|cccc} & a_1 & & & \\ & & a_2 & & 0 \\ * & & * & \ddots & \\ & & & & a_{|Ve(\mathcal{D})|} \\ \hline A & & & 0 & \end{array}\right)$$

where

- A are the equations defining the tropical structures of all the spaces $\widetilde{w}_{\pi(L)}\widetilde{\Lambda}_{\pi(L)}$ with L a horizontal constraint meeting a floor of f;
- $a_{\mathcal{F}} = w_{\mathcal{F}} \times (\widetilde{f}(\widetilde{C}).\pi(L))_{\widetilde{f}(f^{-1}(L))}$ if the corresponding horizontal constraint has dimension $n-2$;
- $a_{\mathcal{F}} = w_{\mathcal{F}}$ if the corresponding horizontal constraint has dimension at most $n-3$.

Hence defining by F_h the set of horizontal constraints in $\omega^{\mathbb{T}}$ of dimension $n-2$ meeting a floor of f, we get that

$$\mu_{\omega^{\mathbb{T}}}(f) = \left(\prod_{e\in\mathrm{Edge}(\mathcal{D})} w_e^{1+|m^{-1}(e)|}\right)\left(\prod_{L\in F_h}(\widetilde{f}(\widetilde{C}).\pi(L))_{\widetilde{f}(f^{-1}(L))}\right)\mu_{\Omega^{\mathbb{T}}}(\widetilde{f}).$$

Hence defining $\mathrm{Vert}_h(\mathcal{D})$ as the set of vertices v of $\mathcal{D}$ such that $\dim(\min(m^{-1}(v))) = n-2$, for any $f' \in \mathcal{RS}^{\mathbb{T}}(\Gamma,\Omega^{\mathbb{T}})$ we have

$$\sum_{f\in\Theta^{-1}(f')}\mu_{\omega^{\mathbb{T}}}(f) = \left(\prod_{e\in\mathrm{Edge}(\mathcal{D})} w_e^{1+|m^{-1}(e)|}\right)\left(\prod_{v\in\mathrm{Vert}_h(\mathcal{D})}\mathrm{div}(v)\right)\mu_{\Omega^{\mathbb{T}}}(f')$$

and by summing over all elements in $\mathcal{RS}^{\mathbb{T}}(\Gamma,\Omega^{\mathbb{T}})$ we get by Theorem 4.5 and Proposition 4.1 that

$$\sum_{f\in\Phi^{-1}(\mathcal{D},m)}\mu_{\omega^{\mathbb{T}}}(f) = \left(\prod_{e\in\mathrm{Edge}(\mathcal{D})} w_e^{1+|m^{-1}(e)|}\right)\left(\prod_{v\in\mathrm{Vert}_h(\mathcal{D})}\mathrm{div}(v)\right)N_{deg}^{n-1}(\Gamma,l_0,\ldots,l_{n-3},\delta) = \mu(\mathcal{D},m)$$

as announced. □

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