MÉMOIRE

Présenté pour obtenir

LE DIPLÔME D'HABILITATION À DIRIGER DES RECHERCHES EN MATHÉMATIQUES DE L'UNIVERSITÉ PIERRE ET MARIE CURIE

par

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TITRE DU MÉMOIRE :

Énumération de courbes complexes, réelles et tropicales

Soutenue le 1er décembre 2014

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Remerciements

Je souhaite remercier vivement Mark Gross, Oleg Viro et Jean-Yves Welschinger d'avoir accepté d'être rapporteurs de ce mémoire. Je suis également très honoré de la présence de Denis Auroux, Charles Favre, Ilia Itenberg, Grigory Mikhalkin, Jean-Jacques Risler et Jean-Yves Welschinger à mon jury. Merci à vous.

Le travail présenté dans ce tapuscrit doit beaucoup à mes collaborateurs. Je les remercie chaleureusement pour tous ces bons moments à faire des mathématiques ensemble, pour tout ce qu'ils m'ont appris, ainsi que pour leur patience devant mes humeurs changeantes.

Aux cours de ces dernière années, j'ai eu la chance d'être accueilli au sein de l'IMJ-PRG et du CMLS. C'est un grand plaisir de remercier ces deux laboratoires pour les conditions de travail exceptionnelles qu'ils m'ont offertes. Je remercie tous les collègues scientifiques et administratifs qui contribuent à y faire régner une ambiance chaleureuse et détendue. Une pensée spéciale à la fort sympathique équipe Analyse Algébrique, et plus particulièrement à GG, II et JJ.

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Chapter 1

Introduction

1.1 Première prise

Le problème principal abordé dans ce mémoire est celui de l'énumération de courbes complexes et réelles dans un variété X, sujettes à des conditions d'incidence avec une configuration $\underline{x}$ de contraintes géométriques. On supposera toujours implicitement que le nombre et les dimensions des éléments de $\underline{x}$ sont tels que le nombre attendu de ces courbes est fini. Plus spécifiquement, nous nous intéresserons aux espaces ambiants X suivants[1]:

1. les surfaces toriques;
2. $\mathbb{C}P^2$ éclaté en des points situés sur une conique lisse;
3. les surfaces de Del Pezzo;
4. les espaces projectifs;
5. les variétés rationnelles symplectiques de dimension 4.

Si l'on compte les courbes complexes, alors la réponse dépend uniquement des classes d'homologies réalisées par les contraintes et les courbes énumérées, ainsi que du genre de ces dernières. Ces nombres sont appelés *invariants de Gromov-Witten (relatifs)* de X. L'invariance par rapport aux représentants spécifiques des éléments de $\underline{x}$ peut être établie en reformulant le problème initial en un calcul de nombre d'intersection dans un certain espace de module canoniquement orienté. Cette observation, remontant au moins aux géomètres du XIXème siècle, est le point de départ de formidables et profonds développements en géométrie énumérative complexe basés sur la théorie de l'intersection. Le lecteur intéressé trouvera dans [Kle76, KV06] d'excellentes introductions à ce sujet, voir aussi [MS12].

En ce qui concerne l'énumération des courbes réelles, une adaptation directe de cette approche basée sur la théorie de l'intersection soulève de nombreuses complications par rapport à la situation complexe. Une de raisons pour cela, et non des moindres, est que de nombreux espaces de modules en géométrie réelles ne sont pas orientables. Il est bien sûr possible de considérer homologie et intersections à coefficients

[1] La structure complexe standard est fixée dans les quatre premiers cas, et peut varier dans le dernier.

dans $\mathbb{Z}/2\mathbb{Z}$, mais une application naïve de cette idée ne conduit qu'à une réduction modulo 2 des invariants de Gromov-Witten correspondants.

La réponse à un problème énumératif réel dépend en général drastiquement du choix de $\underline{x}$, et du choix potentiel d'une structure presque complexe sur X. Le nombre de courbes réelles est clairement majoré par le nombre de courbes complexes, et ces deux nombres ont même parité. On ne peut cependant guère en dire plus au premier abord... Néanmoins, l'existence de meilleures bornes inférieures en géométrie énumérative réelle a été observée depuis un certain temps déjà, nous renvoyons par exemple au calcul de Kharlamov datant des années 70 [DK00, Proposition 4.7.3], ou aux travaux de Gabrielov et Eremenko sur le calcul de Schubert réel [EG02].

Lorsque X est une variété symplectique réelle de dimension 4, Welschinger a proposé dans [Wel03, Wel05a] d'énumérer les courbes réelles *rationnelles* avec un poids ± 1, de telle sorte que la somme pondérée de ces courbes réelles devienne indépendante du choix de $\underline{x}$ et du choix[2] générique d'une structure presque complexe compatible J sur X. Ces *invariants de Welschinger* minorent en particulier le nombre de courbes réelles pour *n'importe quel* choix générique de $\underline{x}$ et J, et fournissent un outil puissant dans l'étude des bornes inférieures en géométrie énumérative réelle. Remarquons que ce type d'invariants en géométrie énumérative réelle est lié aux *invariants de Gromov-Witten ouverts*, c'est à dire au dénombrement de surfaces de Riemann à bord. L'avancée majeure initiée par Welschinger conduisit à la découverte de nouveaux invariants, principalement en genre 0, en géométrie énumérative réelle (e.g. [Wel05b, Wel07, Wel06, Wel11, Wel13, Sol06, PSW08, OT14, FK13, KR13, Geo13, Shu14]), ainsi qu'au développement des méthodes pour les calculer (e.g. [Mik05, IKS03, IKS04, IKS09, IKS13c, IKS13b, IKS13a, Wel07, PSW08, Teh13, HS12, KR13, GZ13, BM07, BM08, BMa, ABLdM11, BP14, Bru14]). Mentionnons aussi les *invariants raffinés* des surfaces projectives complexes introduits par Göttsche et Shende dans [GS12], invariants polynomiaux interpolant conjecturalement entre les invariants de Gromov-Witten et certains invariants de Welschinger.

Il est intéressant de remarquer que l'utilisation d'invariants énumératifs réels peut aussi fournir des bornes supérieures non triviales. Le premier (et unique) exemple porté à notre connaissance est celui de la preuve de Klein [Kle76] du fait qu'au plus un tiers des points d'inflexions complexes d'une courbe algébrique réelle plane non singulière peuvent être réels[3], où cette majoration est déduite de l'invariance d'une quantité définie à partir du nombre de points d'inflexions réels et de bitangentes réelles d'une telle courbe (nous renvoyons aussi à la Section Section 6.1).

Malgré les récent et impressionnants progrès dans la découverte d'invariants énumératifs réels, seule une faible proportion des invariants complexes ont pour l'instant trouvé un pendant réel. Par conséquent, l'adaptation des méthodes de géométrie complexe basées sur la théorie de l'intersection reste une tâche non triviale, ces dernières faisant généralement intervenir des invariants complexes sans analogue réel connu.

Une autre possibilité pour appréhender un problème énumératif est de construire une configuration $\underline{x}$ pour laquelle il est possible d'exhiber *toutes* les solutions. De telles configurations sont dites *effectives*. L'avantage majeur de ce type de configurations est de permettre l'énumération simultanée des courbes complexes et réelles, et ce *sans* supposer une quelconque invariance relativement à $\underline{x}$. Ce dernier point est particulièrement appréciable en géométrie réelle, où, comme mentionné plus haut, les invariants ont tendance à faire défaut. Une autre propriété avantageuse de ces configurations effectives est de mettre en lumière, en plus de permettre leur calcul, certaines propriétés qualitatives des invariants considérés. À titre d'exemple, des résultats concernant le signe des invariants de Welschinger, leur optimalité, leurs propriétés arithmétiques, leur annulation, ainsi que leur comparaison avec les invariants de Gromov-Witten sont obtenus de cette manière dans [IKS03, IKS04, IKS09, IKS13c, IKS13b, IKS13a, Wel07, Kol14, BM07, BM08, BP13a, BP14, Bru14]. Mentionnons aussi les résultats d'optimalité concernant le calcul de Schubert

[2] Dans le cas des surfaces de Del Pezzo, on peut choisir pour J la structure complexe standard sur X.

[3] Dans le cas des quartiques, on peut reformuler abstraitement cet énoncé: au plus 8 des 24 points de Weierstrass d'une courbe algébrique réelle de genre 3 non hyperelliptique peuvent être réels.

réels [Sot97, EG02, Vak06] et les nombres caractéristiques [RTV97], ainsi que la construction d'un espace de Fock pour les degrés de Severi de $\mathbb{C}P^1 \times \mathbb{C}P^1$ [CP12], obtenus en utilisant des configurations effectives.

Parmi les techniques disponibles pour construire des configurations effectives, citons les méthodes basées sur la *géométrie tropicale* [Mik05], la *formule de dégénérescence de Li* [Li02, Li04] dans le cadre algébrique, les *formules de la somme symplectique* dans le cadre symplectique [IP04, LR01, TZ14], ou plus généralement la théorie symplectique des champs [EGH00]. Pour schématiser grossièrement, ces méthodes consistent à dégénérer l'espace ambiant X afin de ramener sa géométrie énumérative à celle d'une (ou plusieurs) union(s) $\cup_i Y_i$ d'espaces Y_i "plus simples". Cette réduction nécessite de considérer les invariants de Gromov-Witten *relatifs* aux diviseurs $E_{i,j} = Y_i \cap Y_j$, autrement dit d'énumérer les courbes soumises à des conditions d'incidences *et* d'intersections avec les diviseurs $E_{i,j}$. Un aspect important de ces méthodes de dégénérescences est que dans certains cas favorables, en particulier lorsque X est de dimension réelle 4, les déformations dans X d'une courbe dans $\cup_i Y_i$ dépendent uniquement de ses intersections avec les diviseurs $E_{i,j}$. Lorsque tel est le cas, on peut par conséquent construire des configurations effectives dans X à partir de configurations effectives dans les Y_i.

Cette stratégie générale étant posée, il est néanmoins généralement non trivial de trouver une dégénérescence appropriée de X permettant de déduire des configurations effectives dans X. L'objectif de ce mémoire est d'illustrer quelques méthodes permettant d'obtenir de telles dégénérescences, en utilisant les trois techniques mentionnées ci-dessus. Les cadres et champs d'applications de ces dernières diffèrent sensiblement, et nous présentons diverses situations où l'emploi de l'une d'entre elles semble préférable aux autres. Bien souvent, le choix d'une technique particulière reste cependant une affaire de goûts.

Les contributions principales présentées dans ce tapuscrit sont regroupées dans trois chapitres:

- **Énumération de courbes via les diagrammes en étages** (Chapitre 3)

  La *technique de décompositions en étages* a été élaborée en collaboration avec Mikhalkin dans [BM07, BM08, BMa], et fournit une méthode efficace pour construire des configurations effectives. Dans le cas où X est de dimension réelle 4, le point de départ est d'observer qu'une configuration d'au plus deux points dans une surface de Hirzebruch (i.e. un fibré holomorphe en $\mathbb{C}P^1$ sur $\mathbb{C}P^1$) est toujours effective. La stratégie est alors de choisir une courbe rationnelle adéquate E dans X, de dégénérer X en l'union de X et d'une chaîne de copies du fibré normal (compactifié) de E dans X, et de choisir une configuration d'au plus deux points dans chacune de ces copies. Dans les cas favorables, l'union de tous ces points peut être déformée en une configuration effective $\underline{x}$ dans X. Lorsque c'est le cas, toutes les courbes complexes et réelles passant par $\underline{x}$ peuvent être codées par des objets purement combinatoires appelés des *diagrammes en étages.* En dimensions supérieures, les diagrammes en étages codent récursivement sur la dimension de l'espace ambiant la construction de configurations effectives $\underline{x}$, ainsi que la reconstruction de toutes les courbes contraintes par $\underline{x}$.

  Nous esquissons plus en détails cette technique à la Section 3.1, ainsi que ses liens avec l'approche proposée par Caporaso et Harris (voir [CH98, Vak00a, Vak00b, GM07a] en rapport avec ce mémoire). Nous explicitons ensuite trois situations où l'énumération de courbes peut être remplacée par le dénombrement combinatoire de diagrammes en étages:

  (a) X est une surface torique et E un diviseur torique satisfaisant une condition de h-transversalité;

  (b) X est $\mathbb{C}P^2$ éclaté en n point sur une conique lisse, et E est la transformée stricte de cette conique;

  (c) $X = \mathbb{C}P^n$ et E est un hyperplan.

  Nous utilisons des méthodes tropicales dans les cas (a) et (c), et la formule de Li dans le cas (b). Remarquons que X peut être singulière dans le cas (a), ce qui rend intéressant l'utilisation de la géométrie tropicale. D'un autre côté, la situation (b) n'est pas torique en général, ce qui rend plus pratique l'utilisation des formules de dégénérescences algébriques ou symplectiques.

Nous terminons ce Chapitre 3 en présentant diverses applications des diagrammes en étages: étude qualitative des invariants de Welschinger, optimalité des bornes supérieures pour les problèmes énumératifs concernant les coniques, polynomialité (par morceaux) des invariants de Gromov-Witten.

Les résultats présentés au Chapitre 3 rendent compte des travaux [BM07, BM08, BMa, ABLdM11, BP13b, Bru14, AB].

- **Trois dégénérescences de surfaces** (Chapitre 4)

  Nous donnons trois exemples d'applications des dégénérescences de surfaces en géométrie énumérative.

  (a) Nous établissons à la Section 4.1 l'annulation d'une grande partie des invariants de Welschinger des variétés symplectiques de dimension 4. Nous relions aussi les invariants de Welschinger d'une variété symplectique rationnelle de dimension 4 donnée équipée de différentes structures réelles. Ces résultats sont obtenus en dégénérant une telle variété vers une variété nodale. Pour modéliser cette dégénérescence, nous présentons la variété symplectique comme somme symplectique d'elle même avec le fibré normal (compactifié) du cycle évanescent Lagrangien réel correspondant.

  (b) Dans la Section 4.2, nous calculons les invariants de Gromov-Witten et de Welschinger des surfaces de Del Pezzo à partir des diagrammes en étages relatifs à une conique introduits à la Section 3.2.2. En première approximation, cette réduction s'obtient en dégénérant une surface de Del Pezzo vers $\mathbb{C}P^2$ éclaté en des points, dont sept se trouvent sur une conique lisse. Afin d'optimiser l'énumération des courbes réelles, il est cependant plus adéquat de considérer un raffinement de cette dégénérescence (voir la Remarque 4.18). Ce travail peut être vu comme une généralisation aux (-3)-courbes de la formule d'Abramovich-Bertram-Vakil [AB01, Vak00a] et de ses versions réelles présentées à la Section 4.1 (voir aussi le Théorème 4.36 appliqué avec $n = 3$). Nous avons choisi ici de travailler dans la catégorie algébrique, et d'utiliser la formule de dégénérescence de Li. On aurait cependant pu travailler dans le cadre symplectique.

  À notre connaissance, les résultats présentés dans cette section fournissent le premier calcul explicite des invariants de Gromov-Witten en tout genre des surfaces de Del Pezzo de degré 1 (voir [CH98, Vak00a, SS13] pour les autres surfaces de Del Pezzo). Un calcul relié des invariants de Welschinger des surfaces de Del Pezzo de degré 2, pour des configurations composées uniquement de points réels, a été indépendamment proposé dans [IKS13a].

  (c) La formule d'Abramovich-Bertram-Vakil [AB01, Vak00a] relie les degrés de Severi des surfaces de Hirzebruch Σ_0 et Σ_2. Nous étendons cette formule à la Section 4.3 au cas des surfaces de Hirzebruch Σ_n et Σ_{n+2}. Notre stratégie généralise celle proposée dans [AB01], à savoir nous dégénérons Σ_n vers Σ_{n+2} et étudions le comportement des courbes énumérées durant cette dégénérescence.

  Les formules de dégénérescences actuellement disponibles ne s'appliquent à priori pas lorsque $n \geq 2$, celles-ci requérant alors de considérer des invariants de Gromov-Witten relatifs à un diviseur *singulier*. Pour contourner cette difficulté, nous utilisons une approche tropicale. Dans un premier temps, nous modélisons la déformation de Kodaira des surfaces de Hirzebruch par une surface tropicale X dans $\mathbb{R}^3$. Puis nous transposons la stratégie d'Abramovich et Bertram dans ce cadre tropical. Notre formule se déduit finalement d'un théorème de correspondance reliant la géométrie énumérative de X avec celle des surfaces de Hirzebruch.

  Les résultats présentés au Chapitre 4 rendent compte des travaux [BP14, Bru14, BM13].

- **Nombres de Hurwitz et caractéristiques tropicaux** (Chapitre 5)

  Les nombres de Hurwitz comptent les revêtements ramifiés d'une surface fermée compacte S, avec un ensemble donné de valeurs critiques et ayant des profils de ramification fixés. Les nombres caractéristiques de $\mathbb{C}P^2$ comptent les courbes planes sujettes à des conditions d'incidence et de tangence, et peuvent être vus comme une généralisation en dimension 2 des nombres de Hurwitz. Rappelons que la technique de décomposition en étages, lorsque disponible, permet de calculer des invariants énumératifs par récurrence sur la dimension de l'espace ambiant. En l'appliquant dans le cas particulier des nombres caractéristiques, il est donc raisonnable d'espérer exprimer ces derniers en fonctions des nombres de Hurwitz. À notre connaissance, il n'existe pas de formule de dégénérescence algébrique ou symplectique permettant de contrôler les tangences avec des courbes s'intersectant les unes les autres. La géométrie tropicale semble ainsi fournir le cadre le plus simple pour obtenir des diagrammes en étage calculant les nombres caractéristiques de $\mathbb{C}P^2$. Remarquons que les relations que nous obtenons ne font pas seulement apparaître les nombres de Hurwitz fermés, mais aussi les nombres de Hurwitz *ouverts*, dénombrant les surfaces possédant potentiellement un bord.

Nous introduisons les nombres de Hurwitz ouverts, et en proposons un calcul tropical à la Section 5.1. Dans la Section 5.2.2, nous identifions les tangences entre courbes tropicales planes, et dégageons un analogue tropical des nombres caractéristiques en genre 0 que nous relions ensuite à leur pendant classique grâce à un théorème de correspondance adapté. Nous consacrons la Section 5.2.3 à l'application de la technique de décomposition en étages développée au Chapitre 3. En particulier, nous exprimons les nombres caractéristiques en genre 0 de $\mathbb{C}P^2$ en termes des nombres de Hurwitz ouverts de $\mathbb{C}P^1$.

Les résultats présentés au Chapitre 5 rendent compte des travaux [BBM11, BBM14].

Nous présentons au Chapitre 6 un bref aperçu de quelques uns de nos autres travaux, sur les points d'inflexions réels des courbes algébriques réelles [BLdM12, ABdLdM14], sur l'approximation des courbes tropicales dans les surfaces tropicales non singulières [BS14], et sur l'approximation des morphismes tropicaux entre courbes tropicales [ABBR13a, ABBR13b]. Ces travaux sont moins directement reliés au thème central de ce mémoire que le trois chapitres précédents. Nous avons néanmoins décidé de les présenter ici car ils ont été partiellement motivés par le développement de la géométrie tropicale en vue de ses applications en géométrie énumérative.

1.2 Take 2

The main problem addressed in this memoir is the enumeration of complex and real curves in a manifold X subject to some incidence conditions with a configuration $\underline{x}$ of geometric constraints. We always implicitly assume that the number and the dimensions of elements of $\underline{x}$ are chosen such that this number of curves is expected to be finite. More specifically, we will be interested in the following ambient spaces[4] X:

1. toric surfaces;
2. $\mathbb{C}P^2$ blown up at points all located on a smooth conic;
3. Del Pezzo surfaces;
4. projective spaces;
5. rational symplectic 4-manifolds.

When counting complex curves, the answer only depends on the homology classes realized by the constraints and the curves under enumeration, as well as on the genus of these latter. These numbers are known as *(relative) Gromov-Witten invariant* of X. The invariance with respect to specific representatives of elements of $\underline{x}$ can be established by reformulating the initial problem into the computation of an

[4] We fix the standard integrable complex structure in the first four cases, and let it vary in the last case.

intersection number in certain canonically oriented moduli space. This latter observation, which can be traced back at least to algebraic geometers of XIXth century, is the starting point for beautiful and powerful developments in complex enumerative geometry based on intersection theory. The interested reader will find in [Kle76, KV06] very nice introductions to the subject, see also [MS12].

As for the enumeration of real curves, a direct adaptation of this intersection theoretic approach raises many complications compared to the complex situation. One of the reasons for that, not the least of which, is that many moduli spaces in real geometry are not orientable. One can of course use homology and intersection with coefficient in $\mathbb{Z}/2\mathbb{Z}$, however a naive application of this idea only leads to the reduction modulo 2 of the corresponding Gromov-Witten invariants.

The answer to a real enumerative problem usually drastically depends on $\underline{x}$, and on a potential choice of an almost complex structure on X. Clearly, the number of real curves is bounded from above by the number of complex curves, and these numbers are equal modulo two. However one can not say much more in general... Nevertheless it has been observed for some time that stronger lower bounds exist in real enumerative geometry, see for example Kharlamov's computation [DK00, Proposition 4.7.3] from the 70's, or Gabrielov and Eremenko's work on real Schubert calculus [EG02].

In the case when X is a real symplectic 4-manifold, Welschinger provided in [Wel03, Wel05a] a way to count real *rational* curves with a weight ± 1, so that the weighted sum of real curves becomes independent on $\underline{x}$ and on the choice of a generic compatible almost complex structure[5] J on X. In particular, these *Welschinger invariants* bound from below the number of real curves for *any* generic choice of $\underline{x}$ and J, and provide an efficient way to tackle the problem of lower bounds in enumeration of real rational curves. Note that real enumerative invariants are related to *open Gromov-Witten invariants*, i.e. the enumeration of Riemann surfaces with boundary. The breakthrough initiated by Welschinger lead to the discovery of further invariants, mostly in the genus zero case, in real enumerative geometry (e.g. [Wel05b, Wel07, Wel06, Wel11, Wel13, Sol06, PSW08, OT14, FK13, KR13, Geo13, Shu14]), and to the development of methods to compute them (e.g. [Mik05, IKS03, IKS04, IKS09, IKS13c, IKS13b, IKS13a, Wel07, PSW08, Teh13, HS12, KR13, GZ13, BM07, BM08, BMa, ABLdM11, BP14, Bru14]). Let us also mention *refined invariants* of projective complex surfaces introduced by Göttsche and Shende in [GS12], which are polynomials that conjecturally interpolate between Gromov-Witten and some Welschinger invariants.

It is interesting to remark that real enumerative invariants may also provide non-trivial upper bounds. The first (and only) example we are aware of is Klein's proof [Kle76] that at most one third of complex inflections points of a smooth real plane projective curve can be real[6], where this fact is deduced from the invariance of a quantity involving the number of real inflection points and bitangents of the curve (see also Section 6.1).

Despite the impressive recent progress made in the discovery of real enumerative invariants, only a small proportion of complex enumerative invariants have a real analogue yet. As a consequence, adaptation of methods from complex geometry based on intersection theory still remain a non-trivial task, since those latter involve quite often complex invariants with no known real analogues.

Another possible approach to solve an enumerative problem is to construct configurations $\underline{x}$ for which one can exhibit *all* solutions. Such configurations are called *effective*. The main advantage of effective configurations is to provide simultaneous enumeration of both complex and real solutions, furthermore *without* assuming any invariance with respect to $\underline{x}$. This is particularly useful in real enumerative geometry where, as mentioned above, invariants are lacking. Another nice property of effective configurations is that, in addition to allowing computations of enumerative invariants, they often bring out some of their qualitative properties. For example results about the sign of Welschinger invariants, their sharpness, their arithmetical properties, their vanishing, and comparison of real and complex invariants have been obtained

[5] In the case of Del Pezzo surfaces, one can fix J to be the standard integrable complex structure on X.

[6] For quartic curves, this has the following abstract formulation : no more than 8 out of the 24 Weierstrass points of a real algebraic non-hyperelliptic curve of genus 3 can be real.

in this way in [IKS03, IKS04, IKS09, IKS13c, IKS13b, IKS13a, Wel07, Kol14, BM07, BM08, BP13a, BP14, Bru14]. Let us also mention the sharpness results concerning real Schubert calculus [Sot97, EG02, Vak06] and characteristic numbers [RTV97], as well as the construction of a Fock space formalism for Severi degrees of $\mathbb{C}P^1 \times \mathbb{C}P^1$ [CP12], obtained using effective configurations.

Among the available techniques to construct effective configurations, one can cite methods based on *Tropical geometry* [Mik05], on *Li's degeneration formula* [Li02, Li04] in the algebraic setting, on *symplectic sum formulas* in the symplectic setting [IP04, LR01, TZ14], or more generally on *symplectic field theory* [EGH00]. As a rough outline, these methods consist of degenerating the ambient space X in order to reduce its enumerative geometry to the one of a (or several) union(s) $\cup_i Y_i$ of "simpler" spaces Y_i. Note that this reduction requires to consider Gromov-Witten invariants of the spaces Y_i *relative* to the divisors $E_{i,j} = Y_i \cap Y_j$. In other words, one has to enumerate curves satisfying some incidence conditions *and* intersecting the divisors $E_{i,j}$ in some prescribed way. A very convenient feature of these degeneration methods is that in nice cases, in particular when X has real dimension 4, deformations of a curve in $\cup_i Y_i$ to a curve in X only depend on the intersections of the curve with the divisors $E_{i,j}$. As a consequence, if one knows how to construct effective configurations in the spaces Y_i, one can construct effective configurations in X.

Using this general strategy, it nevertheless usually remains a non-trivial task to find a suitable degeneration of a particular variety X, from which one can deduce effective configurations in X. The objective of this memoir is to illustrate a few methods to produce such useful degenerations, using the three above-mentioned techniques. The framework and range of applications of these latter are different, and we present various situations when the use of one of them seems preferable than the others. Nevertheless, the choice of a particular technique remains certainly a matter of taste in a number of instances.

The main contributions presented in this typescript are grouped into three chapters:

- **Enumeration of curves via floor diagrams** (Chapter 3)

  The *floor decomposition technique* has been elaborated in collaboration with Mikhalkin in [BM07, BM08, BMa] and provides a method to construct effective configurations. In the case when X has real dimension 4, the starting observation is that configurations containing at most two points in a Hirzebruch surface (i.e. holomorphic $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$) are effective. Then the strategy is to choose a suitable smooth rational curve E in X, to degenerate X into the union of X and a chain of copies of the (compactified) normal bundle of E, and to choose a configuration of at most two points in each of these copies. In nice situations, the union of all those points can be deformed into an effective configuration $\underline{x}$ in X. When this is the case, all complex and real curves passing through $\underline{x}$ can be encoded into purely combinatorial objects called *floor diagrams*. In higher dimensions, floor diagrams encode recursively on the dimension of the ambient space how to construct effective configurations $\underline{x}$, and how to recover all curves constrainted by $\underline{x}$.

  We give in Section 3.1 a more detailed outline of this technique, together with its relation to Caporaso and Harris approach (see [CH98, Vak00a, Vak00b, GM07a] in relation with this memoir). Next we give three instances when the enumeration of algebraic curves can be reduced to the combinatorial enumeration of floor diagrams:

  (*a*) X is a toric surface, and E is a toric divisor satisfying some h-transversality condition;

  (*b*) X is $\mathbb{C}P^2$ blown up in n points located on a smooth conic, and E is the strict transform of the conic;

  (*c*) $X = \mathbb{C}P^n$, and E is a hyperplane.

  We use tropical methods in cases (*a*) and (*c*), and Li's degeneration formula in case (*b*). Note that X may be singular in case (*a*), which makes worthwhile the use of tropical geometry. On the other hand, the situation in (*b*) is non-toric, which makes more adapted the use of algebraic or symplectic degeneration formulas.

We conclude Chapter 3 by presenting several applications of floor diagrams: qualitative study of Welschinger invariants, sharpness of upper bounds in real enumerative problems involving conics, (piecewise-)polynomiality behavior of Gromov-Witten invariants.

Results presented in Chapter 3 are contained in [BM07, BM08, BMa, ABLdM11, BP13b, Bru14, AB].

- **Three surface degenerations** (Chapter 4)

  We give three examples of applications of surface degenerations to enumerative geometry.

  (a) We prove in Section 4.1 the vanishing of a large part of Welschinger invariants of symplectic 4-manifolds. We also relate Welschinger invariants of a given symplectic 4-manifolds equipped with different real structures. Those results are obtained by degenerating a symplectic 4-manifold to a nodal variety. To model this degeneration, we present the symplectic manifold as the symplectic sum of itself with the (compactified) normal bundle of the corresponding real Lagrangian vanishing cycle.

  (b) In Section 4.2, we compute Gromov-Witten and Welschinger invariants of Del Pezzo surfaces out of floor diagrams relative to a conic introduced in Section 3.2.2. In first approximation, this reduction is obtained by degenerating a Del Pezzo surface to $\mathbb{C}P^2$ blown up at points, seven of them lying on a smooth conic. However, it is more suitable to consider a refinement of this degeneration for the purpose of enumeration of real curves (see Remark 4.18). This work can be seen as a generalization to (-3)-curves of Abramovich-Bertram-Vakil's formula [AB01, Vak00a] and its real versions from Section 4.1 (see also Theorem 4.36 applied with $n = 3$). We chose in this section to work in the algebraic category, and to use Li's degeneration formula. Nevertheless one could also have used the symplectic framework.

  To the best of my knowledge, results presented in this section provide the first explicit computation of Gromov-Witten invariants in any genus of the Del Pezzo surface of degree 1 (see [CH98, Vak00a, SS13] for similar computations in other Del Pezzo surfaces). An independent but related computation of Welschinger invariants of Del Pezzo surface of degree 2, for configuration of real points, have been proposed in [IKS13a].

  (c) Abramovich-Bertram-Vakil's formula [AB01, Vak00a] relates Severi degrees of the Hirzebruch surfaces Σ_0 and Σ_2. We extend this formula in Section 4.3 to the Hirzebruch surfaces Σ_n and Σ_{n+2}. Our strategy generalizes the one from [AB01], namely we degenerate Σ_n to Σ_{n+2} and study how curves under enumeration behave during this degeneration.

  The currently available degeneration formulas do not apply when $n \geq 2$, since it would require to consider Gromov-Witten invariants of Hirzebruch surfaces relative to a *singular* divisor. To avoid this difficulty, we use a tropical approach. First we model tropically Kodaira's deformation of Hirzebruch surfaces by some tropical surface X in $\mathbb{R}^3$. Then we transpose Abramovich and Bertram strategy to this tropical setting. Finally we prove our formula thanks to a suitable correspondence theorem relating enumerative geometry in X and in Hirzebruch surfaces.

  Results presented in Chapter 4 are contained in [BP14, Bru14, BM13].

- **Tropical Hurwitz and characteristic numbers** (Chapter 5)

  Hurwitz numbers count ramified coverings of a compact closed oriented surface S having a given set of critical values with given ramification profiles. Characteristic numbers of $\mathbb{C}P^2$ count plane curves subject to some incidence and tangency conditions, and can be seen as a 2-dimensional generalization of Hurwitz numbers. Recall that the floor decomposition technique, when available, allows to compute enumerative invariants by induction of the dimension of the ambient space. Hence by applying it, one reasonably expects to obtain relations among Hurwitz and characteristic numbers. However we are not aware of any symplectic nor algebraic degeneration formula that allows to keep track of tangencies with curves intersecting each other, and tropical geometry seems to be the easiest framework to get floor diagrams computing characteristic numbers. Note that the relation we obtain involves not only closed Hurwitz numbers but also *open* ones, which enumerate surfaces possibly with boundary.

We introduce open Hurwitz numbers, and compute them tropically in Section 5.1. In Section 5.2.2, we identify tropical tangencies between plane tropical curves, and we define tropical analogues of characteristic numbers in genus 0 that we relate to their classical counterpart thanks to suitable correspondence theorem. Section 5.2.3 is devoted to the application of the floor decomposition technique developed in Chapter 3. In particular we express genus 0 characteristic numbers of $\mathbb{C}P^2$ in terms of open Hurwitz numbers of $\mathbb{C}P^1$.

Results presented in Chapter 5 are contained in [BBM11, BBM14].

We present in Chapter 6 a brief overview of some other of our works, on real inflection points of real algebraic curves [BLdM12, ABdLdM14], on the approximation of tropical curves in tropical surfaces [BS14], and on the approximation of tropical morphisms between tropical curves [ABBR13a, ABBR13b]. These works are less directly related to the central theme of this memoir than the three above chapters. Since they were partly motivated by the development of tropical geometry with a view towards applications to enumerative geometry, we decided nevertheless to give a short presentation at the end of this memoir.

Presented works

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[ABBR13b] O. Amini, M. Baker, E. Brugallé, and J. Rabinoff. Lifting harmonic morphisms II: tropical curves and metrized complexes. arXiv:1404.3390, 2013.

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[Bru14] E. Brugallé. Floor diagrams of plane curves relative to a conic and GW-W invariants of del pezzo surfaces. arXiv:1404.5429, 2014.

[BS14] E. Brugallé and K. Shaw. Obstructions to approximating tropical curves in surfaces via intersection theory. To appear in Canadian Journal of Mathematics, 2014.

Chapter 2

Preliminary definitions

In this chapter we fix notations and definitions that we use throughout the text.

2.1 Notations and conventions

1. **Real varieties.** A *real algebraic manifold* $X_{\mathbb{R}} = (X, \tau)$ is a complex algebraic manifold X equipped with an antiholomorphic involution τ. Complex projective spaces are always considered equipped with their standard real structure given by the complex conjugation.

   A *real symplectic manifold* $X_{\mathbb{R}} = (X, \omega, \tau)$ is a symplectic manifold (X, ω) equipped an antisymplectic involution τ. We say that an almost complex structure J tamed by ω is τ-compatible if τ is J-antiholomorphic, i.e. $J \circ d\tau = -d\tau \circ J$.

   In both cases, the *real part* of $X_{\mathbb{R}}$, denoted by $\mathbb{R}X$, is by definition the fixed point set of τ.

2. The connected sum of $1+k$ copies of $\mathbb{R}P^2$ is denoted by $\mathbb{R}P^2_k$.

3. The normal bundle of a submanifold E of a manifold X is denoted by $\mathcal{N}_{E/X}$.

4. **Graphs.** Given a finite graph Γ (i.e. Γ has a finite number of edges and vertices) we denote by $\mathrm{Vert}(\Gamma)$ the set of its vertices, and by $\mathrm{Edge}(\Gamma)$ the set of its edges. By definition, the valency of a vertex $v \in \mathrm{Vert}(\Gamma)$, denoted by $\mathrm{val}(v)$, is the number of edges in $\mathrm{Edge}(\Gamma)$ adjacent to v.

   A *weighted graph* is a graph Γ equipped with a function $w : Edge(\Gamma) \to \mathbb{Z}_{>0}$. The weight allows one to define the *divergence* at the vertices of an oriented graph. Namely, for a vertex $v \in Vert(\Gamma)$ we define the divergence $\mathrm{div}(v)$ to be the sum of the weights of all incoming edges minus the sum of the weights of all outgoing edges.

5. Given $a \in \mathbb{Z}$ and $\{a_i\}_{i\in I}$ a finite set of integer numbers, we define the multinomial coefficient

$$\binom{a}{\{a_i\}_{i\in I}} = \frac{a!}{(a - \sum_{i\in I} a_i)! \prod_{i\in I} a_i!}.$$

   If $I = \{1, \dots, k\}$, we also use the notation $\binom{a}{a_1, \dots, a_k}$.

Recall also that $(2k)!! = (2k-1)(2k-3)\dots 1$.

6. Given a vector $\alpha = (\alpha_i)_{i\geq 1} \in \mathbb{Z}_{\geq 0}^{\infty}$, we use the notation
$$|\alpha| = \sum_{i=1}^{\infty} \alpha_i, \quad I\alpha = \sum_{i=1}^{\infty} i\alpha_i, \quad \text{and} \quad I^{\alpha} = \prod_{i=1}^{\infty} i^{\alpha_i}.$$
The vector α is said to be *odd* if $\alpha_{2i} = 0$ for all $i \geq 1$. Given two vectors α and β in $\mathbb{Z}_{\geq 0}^{\infty}$, we write $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for all i. The vector in $\mathbb{Z}_{\geq 0}^{\infty}$ whose all coordinates are equal to 0, except the ith one which is equal to 1, is denoted by u_i.

   If $\alpha, \alpha_1, \dots, \alpha_l$ are vectors in $\mathbb{Z}_{\geq 0}^{\infty}$, then we set
$$\begin{pmatrix} \alpha \\ \alpha_1, \dots, \alpha_l \end{pmatrix} = \prod_{i=1}^{\infty} \begin{pmatrix} \alpha_i \\ (\alpha_1)_i, \dots, (\alpha_l)_i \end{pmatrix}.$$

2.2 Welschinger invariants of real symplectic 4-manifolds

Let $X_{\mathbb{R}} = (X, \omega, \tau)$ be a real symplectic manifold of dimension 4. Let C be an immersed real rational J-holomorphic curve in X for some τ-compatible almost complex structure J, and denote by L the connected component of $\mathbb{R}X$ containing the 1-dimensional part $\widetilde{\mathbb{R}C}$ of $\mathbb{R}C$. Fix also a τ-invariant class F in $H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$. Any half of $C \setminus \widetilde{\mathbb{R}C}$ defines a class $\underline{C}$ in $H_2(X, L; \mathbb{Z}/2\mathbb{Z})$ whose intersection number modulo 2 with F, denoted by $\underline{C} \cdot F$, is well defined and does not depend on the chosen half. We further denote by $m(C)$ the number of nodes of C in L with two τ-conjugated branches, and we define the F-mass of C as
$$m_F(C) = m(C) + \underline{C} \cdot F.$$

Choose a connected component L of $\mathbb{R}X$, a class $d \in H_2(X; \mathbb{Z})$, and $r, s \in \mathbb{Z}_{\geq 0}$ such that
$$c_1(X) \cdot d - 1 = r + 2s.$$
Choose a configuration $\underline{x}$ made of r points in L and s pairs of τ-conjugated points in $X \setminus \mathbb{R}X$. Given a τ-compatible almost complex structure J, we denote by $\mathcal{C}(d, \underline{x}, J)$ the set of real rational J-holomorphic curves C in X realizing the class d, passing through $\underline{x}$, and such that L contains $\widetilde{\mathbb{R}C}$.

The following theorem constituted a breakthrough in real enumerative geometry.

Theorem 2.1 (Welschinger, [Wel05a, IKS13b]) *For a generic choice of J, the set $\mathcal{C}(d, \underline{x}, J)$ is finite, and the integer*
$$W_{X_{\mathbb{R}},L,F}(d, s) = \sum_{C \in \mathcal{C}(d,\underline{x},J)} (-1)^{m_F(C)}$$
depends neither on $\underline{x}$, J, nor on the deformation class of $X_{\mathbb{R}}$

We call these numbers the *Welschinger invariants of* $X_{\mathbb{R}}$. When $F = [\mathbb{R}X \setminus L]$, we simply write $W_{X_{\mathbb{R}},L}(d, s)$ instead of $W_{X_{\mathbb{R}},L,[\mathbb{R}X\setminus L]}(d, s)$. If $\mathbb{R}X$ is connected, we further write $W_{X_{\mathbb{R}}}(d, s)$ instead of $W_{X_{\mathbb{R}},L}(d, s)$. Note that Welschinger invariants are non-trivial to compute only in the case of rational manifolds.

Remark 2.2 *Welschinger originally considered in [Wel05a] only the case when $F = [\mathbb{R}X \setminus L]$. In this case $m_F(C)$ is the number of solitary nodes of $\mathbb{R}C$. Later, Itenberg, Kharlamov, and Shustin observed in [IKS13b] that Welschinger's proof extends literally to arbitrary τ-invariant classes in $H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$. See also [Geo13] for a related discussion.*
Note that our convention differs from [IKS13b], where the sign of a curve in $\mathcal{C}(d, \underline{x}, J)$ depends on the parity of $m(C) + \underline{C} \cdot (F + [\mathbb{R}X \setminus L])$ instead of $m(C) + \underline{C} \cdot (F)$.

Among the results presented in this memoir, we prove that $W_{X_{\mathbb{R}},L,F}(d, s)$ vanishes for many choices of F, and we compute Welschinger invariants of several Del Pezzo surfaces.

2.3 Tropical curves

We use tropical methods at several places in the text. Since many definitions in tropical geometry differ from an author to another, we fix here the definitions of tropical curves and morphisms we use in this typescript. We also illustrate applications of tropical geometry to enumerative geometry with Mikhalkin's seminal Correspondence Theorem.

2.3.1 Abstract tropical curves

In this memoir, we identify a graph and any of its topological realization. A *metric graph* is the data of a graph C equipped with a complete metric on

$$C \setminus \mathrm{Vert}^\infty(C),$$

where $\mathrm{Vert}^\infty(C)$ is a subset of 1-valent vertices of C. In particular, vertices in $\mathrm{Vert}^\infty(C)$ are at the infinite distance from all the other points of C. Next definition is essentially [BBM11, Definition 2.1]. Tropical curves with boundary will be needed in Chapter 5.

Definition 2.3 *A* tropical curve *C with boundary is a metric graph equipped with a map*

$$\begin{array}{ccc} \mathit{Vert}(C) \setminus \{1-\textit{valent vertices of } C\} & \longrightarrow & \mathbb{Z}_{\geq 0} \\ v & \longmapsto & g_v \end{array}$$

such that any 2-valent vertex v of C satisfies $g_v \geq 1$. A 1-valent vertex of C not in $\mathit{Vert}^\infty(C)$ is called a boundary component *of C.*

The tropical curve C is said to be irreducible *if it is connected. The* Euler characteristic *of C is*

$$\chi_{\mathrm{trop}}(C) = b_0(C) - b_1(C) - \sum g_v$$

where $b_i(C)$ is the ith Betti number of C. The genus *of C with no boundary component is defined to be*

$$g(C) = 1 - \chi_{\mathrm{trop}}(C).$$

The integer g_v is called the genus of v. The tropical curve C is explicit *if g_v is identically 0.*

Our definition of tropical curves with boundary extends the definition of tropical curves with stops introduced by Nishinou in [Nis12]. We denote by ∂C the set of boundary components of C, and by $\mathrm{Edge}^\infty(C)$ the set of edges adjacent to a vertex in $\mathrm{Vert}^\infty(C)$. A tropical curve C is said to be *closed* if $\partial C = \emptyset$.

Two tropical curves C_1 and C_2 are said to be *isomorphic* if there exists an isometry $\phi : C_1 \to C_2$ such that $g_{\phi(v)} = g_v$ for any k-valent vertex of C with $k \geq 2$.

A *punctured tropical curve* C' is given by $C \setminus \mathcal{P}$ where C is a tropical curve, and $\mathcal{P}$ is a subset of $\mathrm{Vert}^\infty(C)$. Note that C' has a tropical structure inherited from C. We set $\mathrm{Edge}^\infty(C') = \mathrm{Edge}^\infty(C)$. We define the genus of a punctured tropical curve C' without boundary as $g(C') = g(C)$ and its Euler characteristics as

$$\chi_{\mathrm{trop}}(C') = \chi_{\mathrm{trop}}(C) - |\mathcal{P}|.$$

2.3.2 Tropical morphisms

Given e an edge of a tropical curve C, we choose a point p in the interior of e and a unit vector u_e of the tangent line to C at p (recall that C is equipped with a metric). Of course, the vector u_e depends on the choice of p and is well-defined only up to multiplication by -1, but this will not matter in the following. In cases we will need u_e to have a prescribed direction, we will then specify this direction. The standard inclusion of $\mathbb{Z}^n$ in $\mathbb{R}^n$ induces a standard inclusion of $\mathbb{Z}^n$ in the tangent space of $\mathbb{R}^n$ at any point of $\mathbb{R}^n$. A vector in $\mathbb{Z}^n$ is said to be *primitive* if the greatest common divisor of its coordinates equals 1.

Definition 2.4 *Let C be a punctured tropical curve. A continuous map $f : C \to \mathbb{R}^n$ is a* tropical morphism *if*

- *for any edge e of C, the restriction $f_{|e}$ is a smooth map with $df(u_e) = w_{f,e}u_{f,e}$ where $u_{f,e} \in \mathbb{Z}^n$ is a primitive vector, and $w_{f,e}$ is a positive integer;*
- *for any vertex v in Vert(C) whose adjacent edges are $e_1, \ldots, e_k$ with $k \geq 2$, one has the balancing condition*
$$\sum_{i=1}^{k} w_{f,e_i} u_{f,e_i} = 0$$
where u_{f,e_i} is chosen so that it points away from v.

The integer $w_{f,e}$ is called the *weight of the edge e with respect to f*. When no confusion is possible, we simply speak about the weight of an edge, without referring to the morphism f. Note that a morphism f is proper and $\mathrm{Vert}^\infty(C) = \emptyset$. The morphism f is called an *immersion* if it is a topological immersion, i.e. if f is a local homeomorphism on its image.

Remark 2.5 *Definition 2.4 is a simplify and rather coarse definition of a tropical morphism, in particular because one should allow edges to have weight* 0. *We made this simplification for the sake of shortness, since in all tropical enumerative problems considered in this typescript, only tropical morphisms with positive weights finally appear as solutions.*

More important, one can easily construct tropical morphisms from a positive genus tropical curve which are superabundant, i.e. whose space of deformation has a strictly bigger dimension that the expected one (see [Mik05, Section 2]). In particular, when the corresponding situation in classical geometry is regular (i.e. with no superabundancy phenomenon) as in the case of reduced projective plane curves, such a superabundant tropical morphism is unlikely to be presented as the tropical limit of a family of holomorphic maps (see for example [Mik06, Section 6] or [BBM14, Section 6] for the definition of a tropical limit). One may refine Definition 2.4, still using pure combinatoric, to get rid of many of these superabundant tropical morphisms, see Section 6.2.

Definition 2.4 will be extended to morphisms between any tropical curves in Definition 5.4.

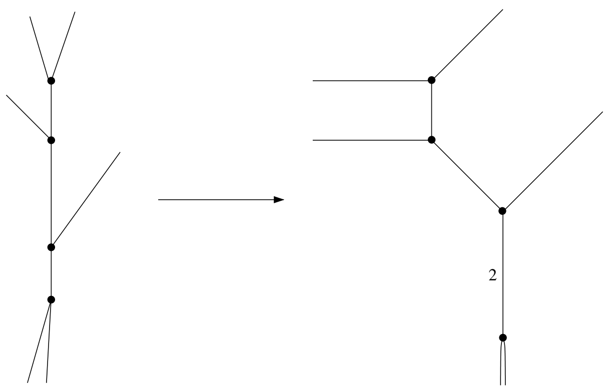


Figure 2.1: Example of a morphism: a plane conic

Example 2.6 *In Figure 2.1 we depicted a plane conic, which is the image in $\mathbb{R}^2$ of a morphism from a trivalent punctured curve with four vertices. We label the image of an edge with the corresponding weight if it exceed 1. This allows us to omit the source of the morphism which is then implicit.*

Two tropical morphisms $f_1 : C_1 \to \mathbb{R}^n$ and $f_2 : C_2 \to \mathbb{R}^n$ are said to be *isomorphic* if there exists a tropical isomorphism $\phi : C_1 \to C_2$ such that $f_1 = f_2 \circ \phi$. In this text, we consider tropical morphisms up to isomorphism.

Given a closed tropical morphism $f : C \to \mathbb{R}^n$, we define its *degree* and its *Newton fan*.

The degree of f is defined as

$$d = \sum_{e \in \mathrm{Vert}^\infty(C)} w_{f,e} \; \max_{j=1}^n \{0, u_{f,e,j}\}, \tag{2.1}$$

where $u_{f,e} = (u_{f,e,j})_{j=1}^n$ is oriented in the unbounded direction of e. Note that if f has degree d, then $\mathrm{Edge}^\infty(C)$ contains at most $(n+1)d$ edges. If equality holds f is said to be *generic at infinity.*

The multiset $\delta(f) = \{u_{f,e}\}_{e \in \mathrm{Vert}^\infty(C)}$ is called the *Newton fan* of f. We use the notation

$$\delta(f) = \{v_1^{m_1}, \ldots, v_k^{m_k}\}$$

to indicate that the vector v_i appears m_i times in δ. Thanks to the balancing condition, we have

$$\sum_{u \in \delta(f)} u = 0.$$

In particular, this implies that any Newton fan δ of vectors in $\mathbb{Z}^2$ has a unique (up to translation) *dual polygon* Π_δ in $\mathbb{R}^2$. We call Π_δ the *Newton polygon* of f.

To any complex algebraic curve C in $(\mathbb{C}^*)^n$, we may associate its Newton fan δ_C as follows: consider the toric compactification $Tor(\Pi_C)$ of $(\mathbb{C}^*)^n$ given by a polytope Π_C such that C does not intersect any boundary components of $Tor(\Pi_C)$ of codimension two or more. Then each puncture p of C corresponds to a facet γ of Π_C. We associate to p the element $w_p v_p$ where v_p is the primitive normal vector to γ oriented outward Π_C, and w_p is the order of contact at p of C with the toric divisor corresponding to γ in the toric variety $Tor(\Pi_C)$. The choice of Π_C is clearly not unique however δ_C does not depend on this choice. Note that if $n = 2$, a canonical choice for Π_C is of course Π_{δ_C}.

2.3.3 An example of Correspondence Theorem

Here we illustrate the use of tropical geometry in complex and real geometry by giving Mikhalkin's original Correspondence Theorem [Mik05] to enumerate curves in toric surfaces. This Theorem constitutes a cornerstone of tremendous developments in tropical and enumerative geometry during the last decade (e.g. [GM07b, NS06, CJM11, Tyo12, IM13, GS14]). We present two generalizations of Mikhalkin's Correspondence Theorem in this memoir.

A *lattice polygon* Δ is a convex polygon in $\mathbb{R}^2$ whose vertices are in $\mathbb{Z}^2$. We denote by Δ_d the lattice polygon with vertices $(0,0)$, $(d,0)$, and $(0,d)$.

Recall that a lattice polygon Δ defines a complex toric surface $Tor(\Delta)$ equipped with a polarization $d_\Delta \in H_2(Tor(\Delta); \mathbb{Z})$. Note that $Tor(\Delta)$ is naturally equipped with a real structure induced by the complex conjugation on $(\mathbb{C}^*)^2$. We denote by $Tor_{\mathbb{R}}(\Delta)$ the corresponding real algebraic surface. We define $N(\Delta, g)$ to be the number of irreducible complex algebraic curves in $Tor(\Delta)$ of genus g, realizing the class d_Δ, and passing through a generic configuration of $\aleph_{\Delta,g} = |\partial\Delta \cap \mathbb{Z}^2| + g - 1$ points in $Tor(\Delta)$. Equivalently, $N(\Delta, g)$ is the number of complex algebraic curves of genus g in $(\mathbb{C}^*)^2$, with Newton polygon Δ, and passing through a generic configuration of $\aleph_{\Delta,g}$ points in $(\mathbb{C}^*)^2$. These numbers are known as *(irreducible) Severi degrees* of the surface $Tor(\Delta)$. When $Tor(\Delta)$ is a Del Pezzo surface, the number $N(\Delta, g)$ is a Gromov-Witten invariants of $Tor(\Delta)$.

Mikhalkin's Correspondence Theorem reduces the enumeration of complex and real curves in $Tor(\Delta)$ to the enumeration of tropical curves in $\mathbb{R}^2$. For the sake of shortness, we only explain here how to compute Welschinger invariants of real Del Pezzo toric surfaces when $\underline{x}$ does not contain any pairs of complex conjugated points. We refer to [Shu06] for a tropical computations of Welschinger invariants of real Del Pezzo toric surfaces for *any* real configurations $\underline{x}$. See also [BM08, Section 4] for another presentation.

Choose a lattice polygon Δ in $\mathbb{R}^2$, an integer $g \geq 0$, and a configuration $\underline{x}$ of $\aleph_{\Delta,g}$ points in $\mathbb{R}^2$. We denote by $\mathbb{T}\mathcal{C}(\Delta, g, \underline{x})$ the set of irreducible closed tropical morphisms $f : C \to \mathbb{R}^2$ of genus g, Newton polygon Δ, and such that $\underline{x} \subset f(C)$.

Proposition 2.7 (Mikhalkin, [Mik05]) *For a generic configuration $\underline{x}$, the set $\mathbb{T}\mathcal{C}(\Delta, g, \underline{x})$ is finite. Moreover any element $f : C \to \mathbb{R}^2$ of $\mathbb{T}\mathcal{C}(\Delta, g, \underline{x})$ is an immersion, generic at infinity, from a trivalent tropical curve*

Choose a generic configuration $\underline{x}$, and an element $f : C \to \mathbb{R}^2$ of $\mathbb{T}\mathcal{C}(\Delta, g, \underline{x})$. Let $v \in \mathrm{Vert}(C)$ and e_1 and e_2 be two of its adjacent edges. We define the following numbers

$$\mu^{\mathbb{C}}(v) = w_{f,e_1} w_{f,e_2} |\det(u_{f,e_1}, u_{f,e_2})|, \quad \text{and} \quad \mu^{\mathbb{R}}(v) = \begin{cases} 0 & \text{if } \mu^{\mathbb{C}}(v) = 0 \bmod 2 \\ 1 & \text{if } \mu^{\mathbb{C}}(v) = 1 \bmod 4 \\ -1 & \text{if } \mu^{\mathbb{C}}(v) = 3 \bmod 4 \end{cases}.$$

As v is trivalent, the balancing condition implies that these two numbers do not depend on the choice of e_1 and e_2.

Definition 2.8 *The* complex and real multiplicities *of $f : C \to \mathbb{R}^2$ are respectively defined as*

$$\mu^{\mathbb{C}}(f) = \prod_{v \in Vert(C)} \mu^{\mathbb{C}}(v) \quad \text{and} \quad \mu^{\mathbb{R}}(f) = \prod_{v \in Vert(C)} \mu^{\mathbb{R}}(v).$$

Enumerations of algebraic and tropical curves are related by the following fundamental theorem.

Theorem 2.9 (Mikhalkin's Correspondence Theorem, [Mik05]) *For any lattice polygon Δ, any genus g, and any generic configuration $\underline{x}$ of $\aleph_{\Delta,g}$ points in $\mathbb{R}^2$, one has*

$$N(\Delta, g) = \sum_{f \in \mathbb{T}\mathcal{C}(\Delta, g, \underline{x})} \mu^{\mathbb{C}}(f).$$

If in addition the surface $Tor(\Delta)$ is Del Pezzo and $g = 0$, then one has

$$W_{Tor_{\mathbb{R}}(\Delta)}(d_\Delta, 0) = \sum_{f \in \mathbb{T}\mathcal{C}(\Delta, g, \underline{x})} \mu^{\mathbb{R}}(f).$$

Historically, Theorem 2.9 provided the first systematic computations of Welschinger invariants, and has been generalized in [Shu06] to compute all Welschinger invariants of real Del Pezzo toric surfaces. There also exist generalizations of Theorem 2.9 to enumerate rational curves in higher dimensional spaces [NS06, Mik, BMa]. Note that it follows from Theorem 2.9 that the right-hand side of the two above identities do not depend on $\underline{x}$. A direct proof of this fact (i.e. not using Theorem 2.9) has been given in [GM07b] and [IKS09]. These proofs have been extended in [IM13] to prove the existence of *tropical refined Severi invariants* of tropical toric surfaces, in relation with refined Severi degrees introduced in [GS12].

Example 2.10 *All irreducible tropical curves of genus 0 and Newton polygon Δ_3 in $\mathbb{T}\mathcal{C}(\Delta, g, \underline{x})$ for the configuration $\underline{x}$ of 8 points depicted in Figure 2.2a are depicted in Figure 2.2b, ..., j. We verify that*

$$N(\Delta_3, 0) = 12 \quad \text{and} \quad W_{\mathbb{R}P^2}(3, 0) = 8.$$

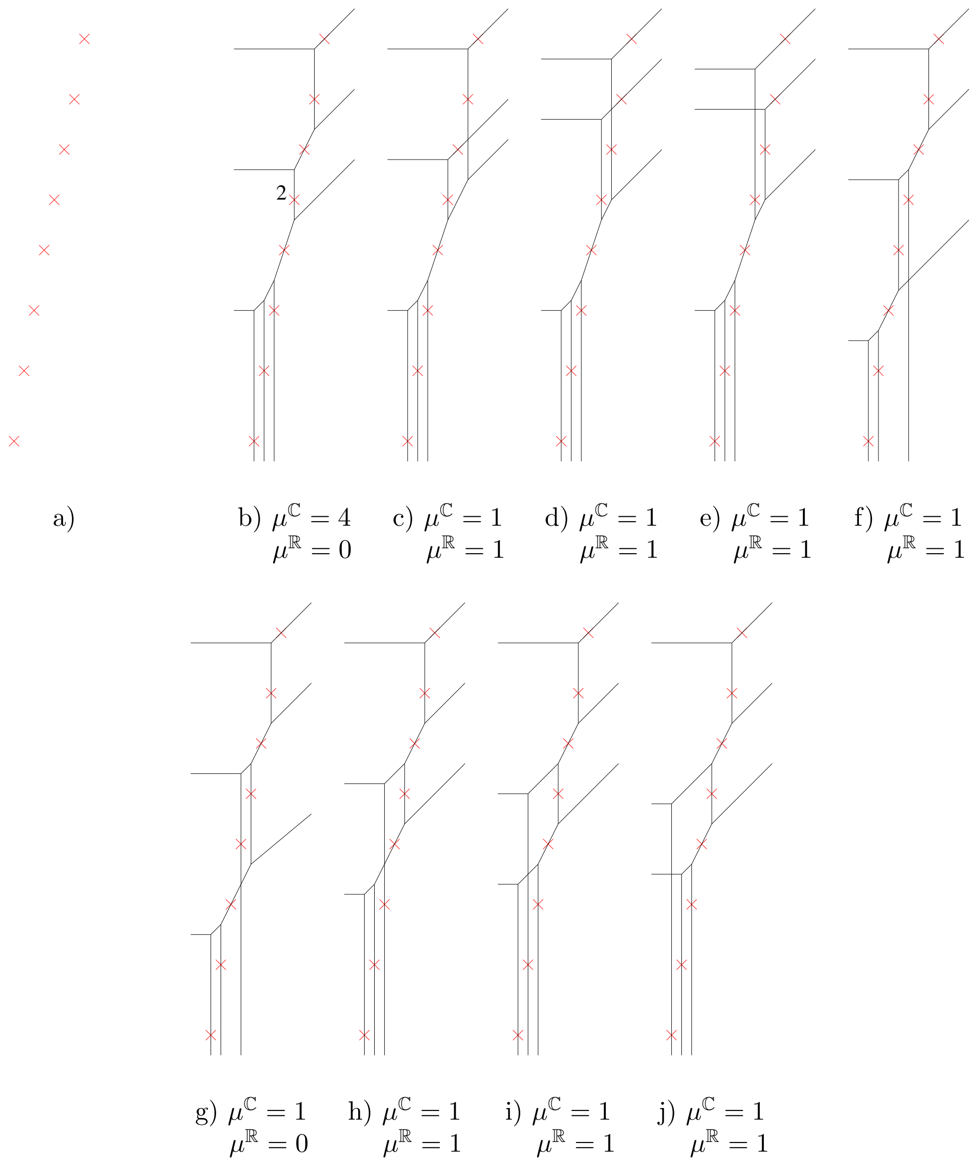


Figure 2.2: $N(\Delta_3, 0) = 12$ and $W_{\mathbb{R}P^2}(3, 0) = 8$.

The proof of Theorem 2.9 goes by constructing effective configurations $\underline{x}'$ of points in $(\mathbb{R}^*)^2$ out of a tropical configuration $\underline{x}$ of points in $\mathbb{R}^2$. In particular it implies the following proposition, which will be used in Section 3.3.3. Given an integer a, we denote by $[a]_2$ its value modulo 2.

Proposition 2.11 (Mikhalkin, [Mik05]) *If $\underline{x}$ is generic, then there exists a generic configuration $\underline{x}'$ of $\aleph_{\Delta,g}$ points in $(\mathbb{R}^*)^2$ such that at least*

$$\sum_{f\in\mathbb{T}\mathcal{C}(\Delta,g,\underline{x})} [\mu^{\mathbb{C}}(f)]_2$$

real algebraic curves in $(\mathbb{C}^*)^2$ of genus g and Newton polygon Δ pass through $\underline{x}'$.

2.4 Hirzebruch surfaces

Given $n \geq 0$, we denote by Σ_n the nth Hirzebruch surface, i.e. $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \mathbb{C})$. The group $H_2(\Sigma_n;\mathbb{Z})$ is the free abelian group generated by the classes of a section B of $\mathcal{O}_{\mathbb{C}P^1}(n)$ and a fiber F. An algebraic curve in Σ_n is said to be of *bidegree* (a, b) if it realizes the homology class $a[B] + b[F]$ in $H_2(\Sigma_n;\mathbb{Z})$.

The surface Σ_n is the projective toric surface defined by the polygon with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$, and $(n + 1, 0)$. In particular it is obtained by taking two copies of $\mathbb{C} \times \mathbb{C}P^1$ glued by the biholomorphism

$$\begin{array}{ccc} \mathbb{C}^* \times \mathbb{C}P^1 & \longrightarrow & \mathbb{C}^* \times \mathbb{C}P^1 \\ (x_1, y_1) & \longmapsto & (\frac{1}{x_1}, \frac{y_1}{z_1^n}) \end{array}$$

The coordinate system (x_1, y_1) in the first chart is called *standard*. An algebraic curve of bidegree (a, b) in Σ_n is defined in a standard coordinate system by a polynomial with Newton polygon contained in the trapeze with vertices $(0, 0)$, $(0, a)$, (b, a), and $(an + b, 0)$, with equality for a generic curve when $a, b \geq 0$.

Chapter 3

Enumeration of curves via floor diagrams

In this chapter we present the main ideas underlying the floor decomposition technique, and implement this strategy in three cases: for toric surfaces together with a toric divisor, for $\mathbb{C}P^2$ blown up at n point on a conic together with this conic, and for projective spaces together with a hyperplane. We end this chapter with a few examples of application of floor diagrams. In particular, we illustrate with conics that it provides an efficient tool in the study of maximality of real enumerative problems.

We will present in Chapter 5 another application of floor decompositions, relating characteristic numbers of $\mathbb{C}P^2$ to open Hurwitz numbers of $\mathbb{C}P^1$. This requires much more preliminary work than in the three cases presented here, and will require the whole Chapter 5.

3.1 Basic strategy, relations with Caporaso-Harris approach

We start by giving the general ideas underlying the floor decomposition technique. For the sake of simplicity, we restrict to the problem of counting curves of a given genus, realizing a given homology class in $H_2(X;\mathbb{Z})$, and passing through a generic configuration $\underline{x}$ of points on a (maybe singular) complex algebraic surface X (the cardinality of $\underline{x}$ being such that the number of curves is expected to be finite). All ideas developed here have a natural generalization in higher dimensions, at the cost of much heavier notations. We refer to Section 3.2.3 for the case of projective spaces, and to Section 5.2.3 for the case of characteristic numbers of $\mathbb{C}P^2$. The floor decomposition technique is related to the approach proposed by Caporaso and Harris in [CH98] to compute Gromov-Witten invariants of $\mathbb{C}P^2$, and it is natural to start with an outline of this latter.

The paradigm underlying a Caporaso-Harris type formula is the following. Choose a suitable irreducible curve E in X, and specialize points in $\underline{x}$ one after the other to E. After the specialization of sufficiently many points, one expects that curves under consideration degenerate into reducible curves having E as a component. By forgetting this component, one is reduced to an enumerative problem in X concerning curves realizing a "smaller" homology class. With a certain amount of optimism, one can then hope to solve the initial problem by induction.

This method has been first proposed and successfully applied by Caporaso and Harris [CH98] in the case of $\mathbb{C}P^2$ together with a line, and has been since then applied in several other situations. Directly related to this text, one can cite the work of Vakil [Vak00a, Vak00b] and the generalization of [Vak00a] by Shoval and Shustin [SS13] As a very nice fact, it turns out that this approach also provides a way to compute certain Welschinger invariants for configuration $\underline{x}$ only composed of real points [IKS13c, IKS13b, IKS13a].

When both X and E are smooth, Ionel and Parker observed in [IP98, Section 5] that the method proposed by Caporaso and Harris could be interpreted in terms of degeneration of the target space X. We present below the algebro-geometric version of this interpretation [Li04, Section 11]. The ideas underlying symplectic interpretation are similar, however the two formalisms are quite different. We particularly refer to [Li04] for an introduction to this degeneration technique in enumerative geometry. Given X and E as above, denote by $\mathcal{N}_E = \mathbb{P}(\mathcal{N}_{E/X} \oplus \mathbb{C})$, and do the following:

1. degenerate X into a reducible surface $Y = X \cup \mathcal{N}_E$, and specialize exactly one point of $\underline{x}$ to $\mathcal{N}_E$ during this degeneration;
2. determine all possible degenerations in Y of the enumerated curves;
3. for each such limit curve in Y, compute the number of curves of which it is the limit.

This method produces recursive formulas à la Caporaso-Harris if all limit curves in Y can be recovered by solving separate enumerative problems in its components X and $\mathcal{N}_E$.

The basic idea of floor diagrams is to get rid of any recursion, which implicitly refers to some invariance property of the enumerative problem under consideration. To do so, one considers a single degeneration of X into the union Y_{max} of X and a chain of copies of $\mathcal{N}_E$, and specializes exactly one element of $\underline{x}$ to each copy of $\mathcal{N}_E$. Floor diagrams then correspond to dual graphs of the limit curves in Y_{max}, and the way they meet the points in $\underline{x}$ is encoded in a *marking*. In good situations, all limit curves in Y_{max} can be completely recovered only from the combinatoric of marked floor diagrams. In particular, effective configurations in X can be deduced from effective configurations in $\mathcal{N}_E$.

This method have been first successfully applied in collaboration with Mikhalkin in [BM07, BM08, BMa], in the case when X is a toric surface and E is a toric divisor satisfying some *h-transversality* condition, or in higher dimension when X is a projective space and E is a hyperplane. We used methods from tropical geometry, which in particular allowed us to get rid of the smoothness assumption on X required in algebraic and symplectic degeneration formulas currently available. Note that when both floor diagrams and Caporaso-Harris type formulas are available, it follows from the above description that these two methods provide two different, although equivalent, ways of clustering curves under enumeration. Passing from one presentation to the other does not present any difficulty other than technical, see [ABLdM11] or Section 3.3.2.

When both X and E are chosen to be real, floor diagrams can also be adapted to enumerate real curves passing through a real configuration of r real points and s pairs of complex conjugated points:

(1') degenerate X to the union Y'_{max} of X and a chain of $r+s$ copies of $\mathcal{N}_E$, specializing exactly one real point or one pair of complex conjugated points of $\underline{x}$ to each copy of $\mathcal{N}_E$;

(2') determine real curves in step (2) above;

(3') adapt computations of step (3) above to determine real curves converging to a given real limit curve.

As in the complex situation, one can associate floor diagrams to real limit curves in Y'_{max}, each of them being now naturally equipped with an involution induced by the real structure of X. Again, in several situations all necessary informations about enumeration of real curves in Y'_{max} are encoded by the combinatoric of these *real marked floor diagrams*. This is in particular the case in the three situations presented in Section 3.2.

This reduction of an algebraic problem to a purely combinatorial question might not seem so surprising when all points in $\underline{x}$ are real, since then the situation is similar to the complex one. However in the presence of complex conjugated points, we are still puzzled by the many cancellations that allow this reduction.

The floor diagram technique clearly takes advantage over the Caporaso-Harris method when one wants to count real curves satisfying incidence conditions with general real configuration $\underline{x}$. In the enumeration of complex curves, or of real curves when elements of $\underline{x}$ are all real, the use of any of these two methods is certainly a matter of taste. From our own experience, we could notice that floor diagrams provide a more geometrical picture of curve degenerations which helps sometimes to minimize mistakes in practical computations. Finally, it is worth stressing that floor diagrams also led to the discovery of new phenomenons also in complex enumerative geometry, for example concerning the (piecewise-)polynomial behavior of relative Gromov-Witten invariants of complex surfaces, e.g. [FM10, Blo11, AB13, LO14, AB], see also Section 3.3.4.

3.2 Three implementations of the floor decomposition technique

We first give some definitions common to all sections of this chapter.

Definition 3.1 *A* floor diagram of genus g *consists in the data of an acyclic (i.e. without any non-trivial oriented cycle) connected oriented graph $\mathcal{D}$, with first Betti number equal to g, equipped with two disjoint subsets $Vert^{+\infty}(\mathcal{D})$ and $Vert^{-\infty}(\mathcal{D})$ of leaves of $\mathcal{D}$, and a map $w : Edge(\Gamma) \to \mathbb{Z}_{>0}$, such that $div(Vert^{-\infty}(\Gamma)) \subset \mathbb{Z}_{<0}$ and $div(Vert^{+\infty}(\Gamma)) \subset \mathbb{Z}_{>0}$.*

A vertex $v \in Vert(\Gamma) \setminus Vert^{\pm\infty}(\Gamma)$ is called a floor of $\mathcal{D}$.

A floor diagram is said to be simple *if $|w(e)| = 1$ for all edge e adjacent to a leaf in $Vert^{\pm\infty}(\Gamma)$.*

We denote by $\text{Vert}^\circ(\mathcal{D})$ the set of floors of $\mathcal{D}$, and by $\text{Edge}^{\pm\infty}(\Gamma)$ the set of edges adjacent to a leaf in $\text{Vert}^{\pm\infty}(\Gamma)$. We provide many examples of floor diagrams in this chapter, and here are the convention we use to depict them : floors are represented by white ellipses; vertices in $\text{Vert}^{\pm\infty(\mathcal{D})}$ are not depicted; edges of $\mathcal{D}$ are represented by vertical lines, and the orientation is implicitly from down to up. We specify the weight of an edge only if this latter is at least 2.

Definition 3.2 *An isomorphism of floor diagrams is a graph isomorphism $\phi : \mathcal{D} \to \mathcal{D}'$ such that $w = w' \circ \phi$ and $\phi\left(Vert^{\pm\infty}(\mathcal{D})\right) = Vert^{\pm\infty}(\mathcal{D}')$.*

Floor diagrams will be mostly considered together with a *marking*, which is an increasing map $m : \mathcal{P} \to \mathcal{D}$ from some partially ordered set $\mathcal{P}$. Note that a floor diagram inherits a partial ordering from the orientation of its underlying graph, and that a map m between two partially ordered sets is said to be *increasing* if

$$m(i) > m(j) \Longrightarrow i > j.$$

3.2.1 Floor diagrams for curves in toric surfaces

In this section we compute complex and real invariants of $Tor(\Delta)$ via floor diagrams, for a suitable class of polygons Δ. Methods presented in this section have recently been used in [BG14] to compute tropical refined Severi degrees of some tropical toric surfaces. It particular, Block and Göttsche proved that these latter indeed interpolate between Severi degrees and Tropical Welschinger invariants (see [IKS09] for a definition) as conjectured in [GS12].

3.2.1.1 Planar floor diagrams and their markings

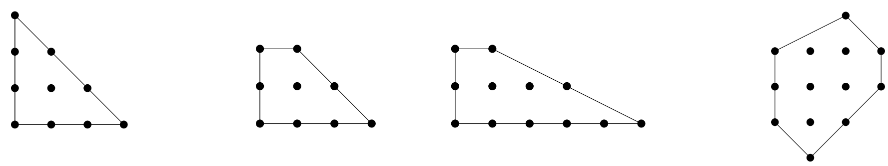
Given Δ a lattice polygon in $\mathbb{R}^2$, we define

$$\partial_l \Delta = \{p \in \partial\Delta \mid \forall t > 0,\ p + (-t, 0) \notin \Delta\},$$

$$\partial_r \Delta = \{p \in \partial\Delta \mid \forall t > 0,\ p + (t, 0) \notin \Delta\}.$$

A lattice polygon Δ is said to be *h-transverse* if any primitive vector parallel to an edge of $\partial_l\Delta$ or $\partial_r\Delta$ is of the form $(\alpha, \pm 1)$ with α in $\mathbb{Z}$. In this case, we define the *left directions* (resp. *right directions*) of Δ, denoted by $d_l(\Delta)$ (resp. $d_r(\Delta)$), as the unordered list that consists of the elements α repeated $l(e)$ times for all edge vectors $e = \pm l(e)(\alpha, -1)$ of $\partial_l\Delta$ (resp. $\partial_r\Delta$). If Δ has a bottom (resp. top) horizontal edge e then we set $d_-(\Delta) = l(e)$ (resp. $d_+(\Delta) = l(e)$) and $d_-(\Delta) = 0$ (resp. $d_+(\Delta) = 0$) otherwise. We call the cardinality $|d_l(\Delta)|$ the *height* of Δ.

Example 3.3 *Some h-transverse polygons are depicted in Figure 3.1. By abuse of notation, we write unordered lists within brackets* $\{\}$.



a)		*b)*		*c)*		*d)*	
	$d_l = \{0,0,0\}$		$d_l = \{0,0\}$		$d_l = \{0,0\}$		$d_l = \{1,0,0,-2\}$
	$d_r = \{1,1,1\}$		$d_r = \{1,1\}$		$d_r = \{2,2\}$		$d_r = \{-1,-1,0,1\}$
	$d_- = 3$		$d_- = 3$		$d_- = 5$		$d_- = 0$
	$d_+ = 0$		$d_+ = 1$		$d_+ = 1$		$d_+ = 0$

Figure 3.1: Examples of h-transverse polygons

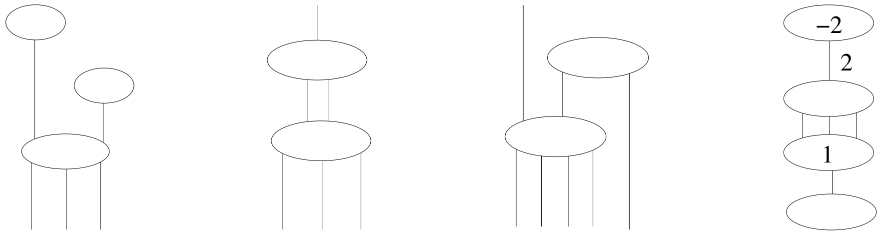
Remark 3.4 *If Δ is a lattice polygon in $\mathbb{R}^2$ and if v is a primitive integer vector such that for any edge e of Δ we have $|det(v, e)| \leq l(e)$, then Δ is a h-transverse polygon after a suitable change of coordinates in $SL_2(\mathbb{Z})$.*

Definition 3.5 *Let Δ be an h-transverse lattice polygon. A simple planar floor diagram with Newton polygon Δ is the data of a simple floor diagram $\mathcal{D}$ equipped with a map $\theta : Vert(\mathcal{D}) \to \mathbb{Z}$ which satisfy the following conditions:*

- *there are exactly $d_\pm(\Delta)$ edges in $Edge^{\pm\infty}(\mathcal{D})$;*
- *the (unordered) collection of numbers $\theta(v)$, where v goes through vertices of $\mathcal{D}$, coincides with $d_l(\Delta)$;*
- *the (unordered) collection of numbers $\theta(v) + div(v)$, where v goes through vertices of $\mathcal{D}$, coincides with $d_r(\Delta)$.*

Example 3.6 *Figure 3.2 depicts an example of simple planar floor diagram for each h-transverse polygon depicted in Figure 3.1. We precise $\theta(v)$ inside the ellipse representing v only if $\theta(v) \neq 0$.*

Example 3.7 *We depicts in Figure 3.3 all simple floor diagrams with Newton polygon Δ_4.*



a) $g=0$ b) $g=1$ c) $g=0$ d) $g=2$

Figure 3.2: Examples of simple planar floor diagrams whose Newton polygon are depicted in Figure 3.1

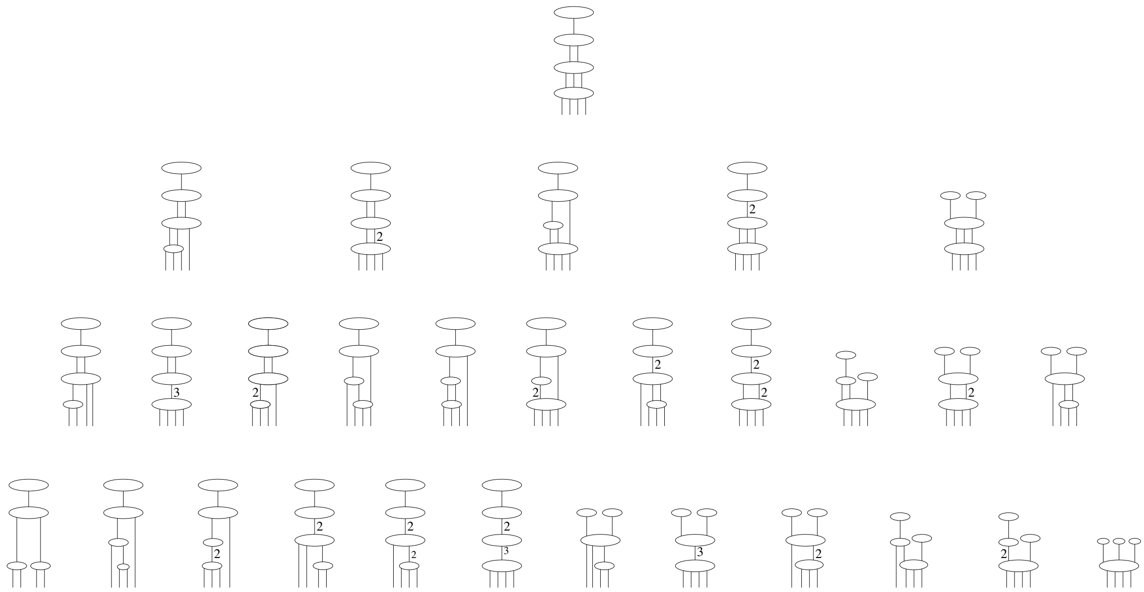


Figure 3.3: Simple floor diagrams with Newton polygon Δ_4

Recall that we have defined $\aleph_{\Delta,g} = |\partial\Delta \cap \mathbb{Z}^2| + g - 1$ in Section 2.3.3. Euler's formula implies that for any simple floor diagram $\mathcal{D}$ of genus g with Newton polygon Δ, we have

$$|\mathrm{Vert}^\circ(\mathcal{D})| + |\mathrm{Edge}(\mathcal{D})| = \aleph_{\Delta,g}.$$

Definition 3.8 *A marking of a simple planar floor diagram $\mathcal{D}$ of genus g with Newton polygon Δ is an increasing bijection $m : \{1, \dots, \aleph_{\Delta,g}\} \to \mathcal{D} \setminus Vert^{\pm\infty}(\mathcal{D})$.*

A planar floor diagram enhanced with a marking is called a *marked planar floor diagram* and is said to be marked by m.

Definition 3.9 *Two marked planar floor diagrams $(\mathcal{D}, m)$ and $(\mathcal{D}', m')$ are called* equivalent *if there exists an isomorphism of floor diagrams $\phi : \mathcal{D} \to \mathcal{D}'$ such that $\theta = \theta' \circ \phi$, and $m = m' \circ \phi$.*

From now on, we consider marked planar floor diagrams up to equivalence. To any (equivalence class of) marked planar floor diagram, we assign a sequence of non-negative integers called *multiplicities* : a *complex* multiplicity, and some *r-real* multiplicities.

3.2.1.2 Enumeration of complex curves

Definition 3.10 *The complex multiplicity of a simple marked planar floor diagram $(\mathcal{D}, m)$, denoted by $\mu^{\mathbb{C}}(\mathcal{D}, m)$, is defined as*

$$\mu^{\mathbb{C}}(\mathcal{D}, m) = \prod_{e \in Edge(\mathcal{D})} w(e)^2$$

Note that the complex multiplicity of a marked planar floor diagram depends only on the underlying floor diagram. The numbers $N(\Delta, g)$ introduced in Section 2.3.3 can be computed only using of marked floor diagrams thanks to next theorem.

Theorem 3.11 ([BM08, Theorem 3.6]) *For any h-transverse polygon Δ and any genus g, one has*

$$N(\Delta, g) = \sum \mu^{\mathbb{C}}(\mathcal{D}, m)$$

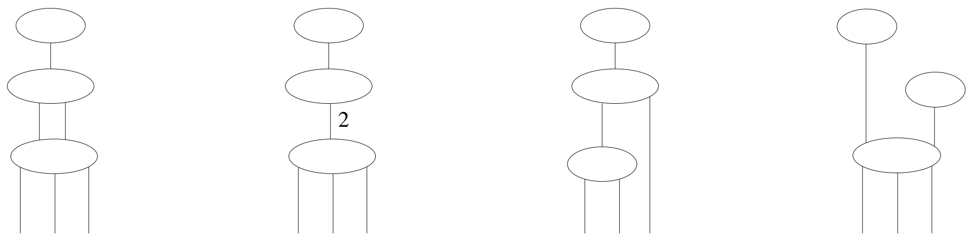
where the sum is taken over all marked simple floor diagrams of genus g and Newton polygon Δ.

Example 3.12 *Using floor diagrams depicted in Figures 3.4 and 3.3, we verify that*

$N(\Delta_3, 1) = 1$, *and* $N(\Delta_3, 0) = 12$ *(see Figure 3.4).*

$N(\Delta, 0) = 93$, *where Δ is the polygon depicted in Figure 3.1c.*

$N(\Delta_4, 3) = 1$, $N(\Delta_4, 2) = 27$, $N(\Delta_4, 1) = 225$, *and* $N(\Delta_4, 0) = 640$, *(see Figure 3.3).*



a) $\mu^{\mathbb{C}} = 1$, 1 marking b) $\mu^{\mathbb{C}} = 4$, 1 marking c) $\mu^{\mathbb{C}} = 1$, 5 markings d) $\mu^{\mathbb{C}} = 1$, 3 markings

Figure 3.4: Simple floor diagrams of genus 1 and 0, with Newton polygon Δ_3

3.2.1.3 Enumeration of real curves

Let us now turn to the enumeration of real curves via floor diagrams. First of all, we have to define the notion of real marked floor diagrams. Choose two integers $r, s \geq 0$ such that $\aleph_{\Delta,0} = r + 2s$, and $\mathcal{D}$ a floor diagram of genus 0 and Newton polygon Δ, marked by a map m.

The set $\{i, i+1\}$ is a called *s-pair* if $i = \aleph_{\Delta,0} - 2k + 1$ with $1 \leq k \leq s$. Denote by $\Im(m, s)$ the union of all the s-pairs $\{i, i+1\}$ where $m(i)$ is not adjacent to $m(i+1)$. Let $\rho_s : \{1, \ldots, \aleph_{\Delta,0}\} \to \{1, \ldots, \aleph_{\Delta,0}\}$ be the bijection defined by $\rho_s(i) = i$ if $i \notin \Im(m, s)$, and by $\rho_s(i) = j$ if $\{i, j\}$ is a s-pair contained in $\Im(m, s)$. Note that ρ_s is an involution, and that $\rho_0 = Id$.

We define o_s to be half of the number of vertices v of $\mathcal{D}$ in $m(\Im(m, s))$ with odd divergence, and we set $A = \text{Edge}(\mathcal{D}) \cap m(\{r+1, \ldots, \aleph_{\Delta,0}\})$.

Definition 3.13 *A simple marked planar floor diagram $(\mathcal{D}, m)$ is called s-real if the two marked floor diagrams $(\mathcal{D}, m)$ and $(\mathcal{D}, m \circ \rho_s)$ are equivalent.*

The s-real multiplicity of a s-real marked floor diagram, denoted by $\mu_s^{\mathbb{R}}(\mathcal{D}, m)$, is defined as

$$\mu_s^{\mathbb{R}}(\mathcal{D}, m) = (-1)^{o_s} \prod_{e \in A} w(e)$$

if all edges of $\mathcal{D}$ of even weight are contained in $m(\Im(m, s))$, and as

$$\mu_s^{\mathbb{R}}(\mathcal{D}, m) = 0$$

otherwise.

Note that $\mu_0^{\mathbb{R}}(\mathcal{D}, m) = 1$ or 0 and is equal to $\mu^{\mathbb{C}}(\mathcal{D}, m)$ modulo 2, and hence does not depend on m. However, $\mu_s^{\mathbb{R}}(\mathcal{D}, m)$ depends on m as soon as $s \geq 1$. Next theorem is the second main result of this section.

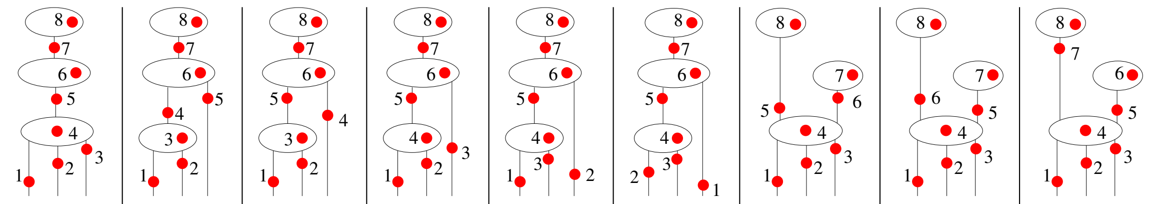
Theorem 3.14 ([BM08, Theorem 3.9]) *Let Δ be a h-transverse polygon such $Tor(\Delta)$ is a Del Pezzo surface. Then for any two integers $r, s \geq 0$ such that $\aleph_{\Delta,0} = r + 2s$, one has*

$$W_{Tor_{\mathbb{R}}(\Delta)}(d_\Delta, s) = \sum \mu_s^{\mathbb{R}}(\mathcal{D}, m)$$

where the sum is taken over all s-real simple marked floor diagrams of genus 0 and Newton polygon Δ.

Example 3.15 *All marked floor diagrams of genus 0 and Newton polygon Δ_3 are depicted in Table 3.1 together with their real multiplicities when they are defined. The first floor diagram has an edge of weight 2, but we didn't mention it in the picture to avoid confusion. Thanks to Theorem 3.14 we compute*

$$W_{\mathbb{C}P^2}(3, s) = 8 - 2s.$$



$\mu^{\mathbb{C}}$	4	1	1	1	1	1	1	1	1
$\mu_0^{\mathbb{R}}$	0	1	1	1	1	1	1	1	1
$\mu_1^{\mathbb{R}}$	0	1	1	1	1	1	-	-	1
$\mu_2^{\mathbb{R}}$	0	1	1	1	1	1	-1	-1	1
$\mu_3^{\mathbb{R}}$	0	1	-	-	1	1	-1	-1	1
$\mu_4^{\mathbb{R}}$	0	1	-	-	-	-	-1	-1	1

Table 3.1: Computation of $W_{\mathbb{C}P^2}(3, s)$

Example 3.16 *Thanks to Figure 3.3, we compute*

s	0	1	2	3	4	5
$W_{\mathbb{C}P^2}(4, s)$	240	144	80	40	16	0

3.2.1.4 Elements of the proof of Theorems 3.11 and 3.14

In order to avoid as much as possible purely technical details, we sketch the proof of Theorem 3.14 only in the case when $s = 0$. Let Δ be a lattice polygon in $\mathbb{R}^2$, and E an edge of Δ. Recall that Δ also defines a tropical toric variety $Tor_{\mathbb{T}}(\Delta)$. The deformation of $Tor(\Delta) \cup \mathcal{N}_E$ to $Tor(\Delta)$ mentioned in Section 3.1 is modeled tropically by the tropical sum of $Tor_{\mathbb{T}}(\Delta)$ and the tropical normal bundle of E. We refer to [Sha11, Section 3.2] for the tropical sum construction. As a consequence, to implement in the tropical setting the strategy described in Section 3.1, one simply has to consider configurations of points in $\mathbb{R}^2$ contained in a strip of the form $[a; b] \times \mathbb{R}$, and where all points are very far one from the others. The fact that this strategy works in the case of a h-transverse polygon is ensured by Proposition 3.17 and Corollary 3.18. Recall that the set $\mathbb{T}\mathcal{C}(\Delta, g, \underline{x})$ has been defined in Section 2.3.3.

Proposition 3.17 ([BM08, Proposition 5.4]) *Let $I = [a; b]$ be a bounded interval of $\mathbb{R}$, and suppose that Δ is h-transverse. Then, if $\underline{x}$ is a subset of $I \times \mathbb{R}$, then any vertex of any curve in $\mathbb{T}\mathcal{C}(\Delta, g, \underline{x})$ is mapped to $I \times \mathbb{R}$.*

An *elevator* of a tropical morphism $f : C \to \mathbb{R}^2$ is an edge with $u_{f,e} = \pm(0, 1)$. A *floor* of f is a connected component of C with all elevators removed.

Corollary 3.18 ([BM08, Corollary 5.5]) *Let I be a bounded interval of $\mathbb{R}$, and suppose that Δ is h-transverse. If $\underline{x}$ is a subset of $I \times \mathbb{R}$ and if the points of $\underline{x}$ are far enough one from the others, then the image of any floor, and of any elevator of any curve in $\mathbb{T}\mathcal{C}(\Delta, g, \underline{x})$ contains exactly one point in $\underline{x}$.*

Suppose that the hypothesis of Corollary 3.18 are satisfied, and that $\underline{x} = \{p_1, \dots, p_{\aleph_{\Delta,g}}\}$ with the second coordinate of p_i less than the one of p_{i+1}. To a tropical morphism $f : C \to \mathbb{R}^2$ in $\mathbb{T}\mathcal{C}(\Delta, g, \underline{x})$, we construct a marked planar floor diagram $\Phi(f) = (\mathcal{D}(f), m_f)$ with Newton polygon Δ as follows: floors (resp. elevators) of C are in a natural one to one correspondence with floors (resp. edges) of $\mathcal{D}(f)$; given a floor ε of C with a (unique) leaf e with $u_{f,e} = (-1, -\alpha)$ where $u_{f,e}$ points to infinity, we define $\theta(\varepsilon) = \alpha$; Corollary 3.18 implies that $\underline{x}$ induces a marking m_f of $\mathcal{D}(f)$.

Theorems 3.11 and 3.14 now follows from next proposition.

Proposition 3.19 *The map Φ is a bijection. Moreover, for any element f in $\mathbb{T}\mathcal{C}(\Delta, g, \underline{x})$, one has $\mu^{\mathbb{C}}(f) = \mu^{\mathbb{C}}(\Phi(f))$ and $\mu^{\mathbb{R}}(f) = \mu_0^{\mathbb{R}}(\Phi(f))$.*

Example 3.20 *Applying the above procedure to the tropical cubics depicted in Figure 2.2, we obtain all marked floor diagrams depicted in Table 3.1 together with their complex and 0-real multiplicities.*

3.2.2 Floor diagrams with respect to a conic

Here we use floor diagrams to enumerate complex and real curves in the blow up of $\mathbb{C}P^2$ at n points lying on a smooth conic E. We denote this surface by $\widetilde{X}_n$. Note that enumerating real rational curves in $\widetilde{X}_n$ à la Welschinger does not produce an invariant as soon as $n \geq 6$, due to the non-genericity of the complex structure on $\widetilde{X}_n$.

The results presented in this section will be used in Section 4.2 to deduce explicit computations of Gromov-Witten and Welschinger invariants of Del Pezzo surfaces.

As mentioned above, Itenberg, Kharlamov, and Shustin used the Caporaso-Harris approach to study Welschinger invariants in the case of configurations of real points. In a series of paper [IKS13c, IKS13b, IKS13a], they thoroughly studied the case of all real structures on Del Pezzo surfaces of degree at least two. Due to methods presenting some similarities, this section, Section 4.2, and [IKS13b, IKS13a] contain some results in common, nevertheless obtained independently and more or less simultaneously.

3.2.2.1 Enumeration of curves in $\widetilde{X}_n$

The strict transform of E in $\widetilde{X}_n$ is still denoted by E. We denote by $E_1, \dots, E_n$ the exceptional divisors of the n blow ups, and by D the strict transform of a line not passing through any of these n points. The group $H_2(\widetilde{X}_n;\mathbb{Z})$ is the free abelian group generated by $[D], [E_1], \dots, [E_n]$, and we have

$$c_1(\widetilde{X}_n) = 3[D] - \sum_{i=1}^{n}[E_i] \quad \text{and} \quad [E]^2 = 4 - n.$$

Let $d \in H_2(\widetilde{X}_n;\mathbb{Z})$ with $d \neq l[E_i]$ with $l \geq 2$, and $\alpha, \beta \in \mathbb{Z}_{\geq 0}^{\infty}$ such that

$$I\alpha + I\beta = d \cdot [E].$$

Choose a configuration $\underline{x} = \underline{x}^\circ \sqcup \underline{x}_E$ of points in $\widetilde{X}_n$, with $\underline{x}^\circ$ a configuration of $d \cdot [D] - 1 + g + |\beta|$ points in $\widetilde{X}_n \setminus E$, and $\underline{x}_E = \{p_{i,j}\}_{0 \leq j \leq \alpha_i, i \geq 1}$ a configuration of $|\alpha|$ points in E. Let $\mathcal{C}^{\alpha,\beta}(d, g, \underline{x})$ be the set of algebraic curves C of genus g in $\widetilde{X}_n$ such that:

- C realizes the homology class d in $\widetilde{X}_n$;
- $\underline{x} \subset C$;
- E is not a component of C;
- C has order of contact i with E at each point $p_{i,j}$;
- C has order of contact i with E at exactly β_i distinct points in $E \setminus \underline{x}_E$.

It follows from [SS13, Proposition 2.1] that for a generic choice of $\underline{x}$, the set $\mathcal{C}^{\alpha,\beta}(d, g, \underline{x})$ is finite and its cardinal does not depend on $\underline{x}$. The Gromov-Witten invariant $GW^{\alpha,\beta}_{\widetilde{X}_n}(d, g)$ of $\widetilde{X}_n$ relative to E is defined by

$$GW^{\alpha,\beta}_{\widetilde{X}_n}(d, g) = \left|\mathcal{C}^{\alpha,\beta}(d, g, \underline{x})\right|$$

for a generic choice of $\underline{x}$. When $\alpha = 0$ and $\beta = (d \cdot [E])u_1$, we use the shorter notation $GW_{\widetilde{X}_n}(d, g)$.

Suppose in addition that E is a smooth real conic in $\mathbb{C}P^2$ and that $\widetilde{X}_n$ is obtained by blowing up $n - 2\kappa$ points on $\mathbb{R}E$ and κ pairs of complex conjugated points on E. In the case $n = 2\kappa$, we furthermore assume[1] that $\mathbb{R}E \neq \emptyset$. The real structure on $\widetilde{X}_n$ induced by the real structure on $\mathbb{C}P^2$ via the blow up map is denoted by $\widetilde{X}_n(\kappa)$. In particular $\mathbb{R}\widetilde{X}_n(\kappa) = \mathbb{R}P^2_{n-2\kappa}$.

Definition 3.21 *A real configuration $\underline{x}^\circ$ in $\widetilde{X}_n(\kappa)$ is said to be (E, s)-compatible if $\underline{x}^\circ \cap E = \emptyset$ and $\underline{x}^\circ$ contains s pairs of complex conjugated points. If L is a connected component of $\mathbb{R}\widetilde{X}_n(\kappa) \setminus \mathbb{R}E$, we say that $\underline{x}^\circ$ is (E, s, L)-compatible if $\mathbb{R}\underline{x}^\circ$ is in addition contained in L.*

Given $\alpha^{\Re}, \alpha^{\Im} \in \mathbb{Z}_{\geq 0}^{\infty}$, a real configuration $\underline{x}_E = \{p_{i,j}\}_{0 \leq j \leq \alpha_i^{\Re},\ i \geq 1} \sqcup \{q_{i,j}, \overline{q_{i,j}}\}_{0 \leq j \leq \alpha_i^{\Im},\ i \geq 1}$ in $E \setminus \bigcup_{i=1}^{n} E_i$ is said to be of type $(\alpha^{\Re}, \alpha^{\Im})$ if $\{p_{i,j}\} \subset \mathbb{R}E$ and $\{q_{i,j}\} \subset E \setminus \mathbb{R}E$.

Choose $d \in H_2(\widetilde{X}_n;\mathbb{Z})$ so that $d \neq l[E_i]$ with $l \geq 2$, choose $r, s \in \mathbb{Z}_{\geq 0}$, and $\alpha^{\Re}, \beta^{\Re}, \alpha^{\Im}, \beta^{\Im} \in \mathbb{Z}_{\geq 0}^{\infty}$ such that

$$d \cdot [D] - 1 + g + |\beta^{\Re}| + 2|\beta^{\Im}| = r + 2s \quad \text{and} \quad I\alpha^{\Re} + I\beta^{\Re} + 2I\alpha^{\Im} + 2I\beta^{\Im} = d \cdot [E].$$

Choose a generic real configuration $\underline{x} = \underline{x}^\circ \sqcup \underline{x}_E$ of points in $\widetilde{X}_n$, with $\underline{x}^\circ$ a (E, s)-compatible configuration of $d \cdot [D] - 1 + g + |\beta^{\Re}| + 2|\beta^{\Im}|$ points, and $\underline{x}_E \subset E$ a configuration of type $(\alpha^{\Re}, \alpha^{\Im})$. Denote by

[1] This is a just formal restriction: all the technique used in [Bru14] adapt immediately to the case $\mathbb{R}E = \emptyset$, however the floor diagrams that arise are slightly different from the ones presented here

$\mathbb{R}\mathcal{C}^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}(d,s,\underline{x})$ the set of real curves C in $\mathcal{C}^{\alpha^{\Re}+2\alpha^{\Im},\beta^{\Re}+2\beta^{\Re}}(d,0,\underline{x})$ such that for any $i \geq 1$, the curve C has exactly $\beta_i^{\Re}$ real intersection points (resp. $\beta_i^{\Im}$ pairs of conjugated intersection points) with E of multiplicity i and disjoint from $\underline{x}_E$. Given $F = \mathbb{R}\widetilde{X}_n(\kappa)$ or $F = L$, we denote by $\widetilde{m}_F(C)$ the number of solitary nodes of C in F. Then we define the following number:

$$W_{\widetilde{X}_n(\kappa)}^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}(d,s,\underline{x}) = \sum_{C\in\mathbb{R}\mathcal{C}^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}(d,s,\underline{x})} (-1)^{\widetilde{m}_{\mathbb{R}\widetilde{X}_n(\kappa)}(C)}.$$

Suppose now that $n = 2\kappa$, in particular $\mathbb{R}E$ disconnects $\mathbb{R}\widetilde{X}_n(\kappa)$. Given L a connected component of $\mathbb{R}\widetilde{X}_n(\kappa) \setminus \mathbb{R}E$, and a (E,s,L)-compatible configuration $\underline{x}$, denote by $\mathbb{R}\mathcal{C}_L^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}(d,s,\underline{x})$ the set of elements of $\mathbb{R}\mathcal{C}^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}(d,s,\underline{x})$ such that $L \cup \mathbb{R}E$ contains the 1-dimensional part of $\mathbb{R}C$. Given $F = \mathbb{R}\widetilde{X}_n(\kappa)$ or $F = L$, define:

$$W_{\widetilde{X}_n(\kappa),L,F}^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}(d,s,\underline{x}) = \sum_{C\in\mathbb{R}\mathcal{C}_L^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}(d,s,\underline{x})} (-1)^{\widetilde{m}_F(C)}.$$

Note that these three series of numbers may vary with the choice of $\underline{x}$.

3.2.2.2 Floor diagrams and their markings

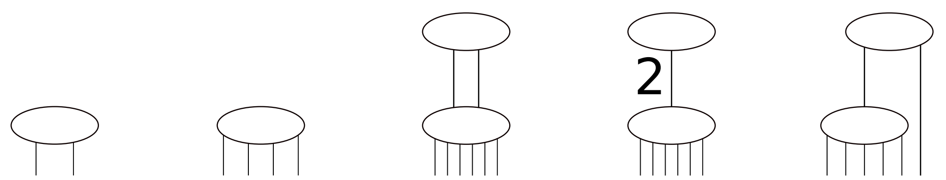
Definition 3.22 *A* floor diagram of degree $d_{\mathcal{D}}$ with respect to a conic *is a floor diagram $\mathcal{D}$ satisfying the following conditions:*

- *$Vert^{+\infty}(\mathcal{D}) = \emptyset$;*
- *$div(v) = 2$ or 4 for any $v \in Vert^{\circ}(\mathcal{D})$;*
- *if $div(v) = 2$, then v is a sink;*
- *one has*

$$\sum_{v\in Vert^{-\infty}(\mathcal{D})} div(v) = -2d_{\mathcal{D}}.$$

A vertex $v \in Vert^{\circ}(\mathcal{D})$ is called a floor of degree $\frac{div(v)}{2}$.

Example 3.23 *Figure 3.5 depicts all simple floor diagrams with respect to a conic of degree 1, 2 and 3.*



a) $d = 1$, $g = 0$ b) $d = 2$, $g = 0$ c) $d = 3$, $g = 1$ d) $d = 3$, $g = 0$ e) $d = 3$, $g = 0$

Figure 3.5: Examples of simple floor diagrams with respect to a conic

Definition 3.24 *Choose two non-negative integers n and g, a homology class $d \in H_2(\widetilde{X}_n;\mathbb{Z})$, and two vectors $\alpha, \beta \in \mathbb{Z}_{\geq 0}^{\infty}$ such that*

$$I\alpha + I\beta = d \cdot [E].$$

Let $A_0, A_1, \ldots, A_n$ be some disjoint sets such that $|A_i| = d \cdot [E_i]$ for $i = 1, \ldots, n$, and

$$A_0 = \{1, \ldots, d \cdot [D] - 1 + g + |\alpha| + |\beta|\}$$

A d-marking of type (α, β) of a floor diagram $\mathcal{D}$ with respect to a conic of genus g and degree $d \cdot [D]$ is a map $m : \bigcup_{i=0}^{n} A_i \to \mathcal{D}$ such that

1. *the map m is injective and increasing, with no floor of degree 1 of $\mathcal{D}$ contained in the image of m;*
2. *for each vertex $v \in Vert^{-\infty}(\mathcal{D})$ adjacent to the edge $e \in Edge^{-\infty}(\mathcal{D})$, exactly one of the two elements v and e is in the image of m;*
3. *$m\left(\bigcup_{i=1}^{n} A_i\right) \subset Vert^{-\infty}(\mathcal{D})$;*
4. *for each $i = 1, \ldots, n$, a floor of $\mathcal{D}$ is adjacent to at most one edge adjacent to a vertex in $m(A_i)$;*
5. *$m\left(\{1, \ldots, |\alpha|\}\right) = m(A_0) \cap Vert^{-\infty}(\mathcal{D})$;*
6. *for $1 \leq k \leq \alpha_j$, the edge adjacent to $m(\sum_{i=1}^{j-1} \alpha_i + k)$ is of weight j;*
7. *exactly β_j edges in $Edge^{-\infty}(\mathcal{D})$ of weight j are in the image of $m_{|A_0}$.*

These conditions imply that all edges in $m\left(\bigcup_{i=1}^{n} A_i\right)$ are of weight 1. A floor diagram enhanced with a d-marking m is called a *d-marked floor diagram* and is said to be marked by m.

Definition 3.25 *Let $\mathcal{D}$ be a floor diagram with respect to a conic equipped with two d-markings*

$$m : A_0 \cup \bigcup_{i=1}^{n} A_i \to \mathcal{D} \quad \text{and} \quad m' : A_0 \cup \bigcup_{i=1}^{n} A'_i \to \mathcal{D}.$$

The markings m and m' are called equivalent if there exists an isomorphism of floor diagrams $\phi : \mathcal{D} \to \mathcal{D}$ and a bijection $\psi : A_0 \cup \bigcup_{i=1}^{n} A_i \to A_0 \cup \bigcup_{i=1}^{n} A'_i$, such that

- *$\psi_{|A_0} = Id$;*
- *$\psi_{|A_i} : A_i \to A'_i$ is a bijection for $i = 1, \ldots, n$;*
- *$m' \circ \psi = \phi \circ m$.*

In particular, for $i = 1, \ldots, n$, the equivalence class of $(\mathcal{D}, m)$ depends on $m(A_i)$ rather than on $m_{|A_i}$. As usual, marked floor diagrams are considered up to equivalence.

3.2.2.3 Relative Gromov-Witten invariants of $\widetilde{X}_n$

Definition 3.26 *The complex multiplicity of a marked floor diagram $(\mathcal{D}, m)$ of type (α, β) with respect to a conic, denoted by $\mu^{\mathbb{C}}(\mathcal{D}, m)$, is defined as*

$$\mu^{\mathbb{C}}(\mathcal{D}, m) = I^{\beta} \prod_{e \in Edge(\mathcal{D}) \setminus Edge^{-\infty}(\mathcal{D})} w(e)^2.$$

Note that the complex multiplicity of a marked floor diagram only depends on its type and the underlying floor diagram.

Theorem 3.27 ([Bru14, Theorem 3.6]) *For any $d \in H_2(\widetilde{X}_n;\mathbb{Z})$ such that $d \cdot [D] \geq 1$ and $d \neq l[E_i]$ with $l \geq 2$, and any genus $g \geq 0$, one has*

$$GW^{\alpha,\beta}_{\widetilde{X}_n}(d,g) = \sum \mu^{\mathbb{C}}(\mathcal{D},m)$$

where the sum is taken over all d-marked floor diagrams with respect to a conic, of genus g and type (α,β).

Example 3.28 *Theorem 3.27 applied with $n \leq 5$, $\alpha = 0$, and $\beta = (d \cdot [D])u_1$ gives Gromov-Witten invariants of the Del Pezzo surfaces of degree $9-n$. In particular, as a simple application of Theorem 3.27 one can use floor diagrams depicted in Figure 3.5 to verify that*

$$GW_{\mathbb{C}P^2}(1,0) = GW_{\mathbb{C}P^2}(2,0) = GW_{\mathbb{C}P^2}(3,1) = 1 \quad \text{and} \quad GW_{\mathbb{C}P^2}(3,0) = 4+8 = 12.$$

Example 3.29 ([Bru14, Example 3.8]) *Thanks to Theorem 3.27, we compute*

$$GW_{\widetilde{X}_6}(4[D] - \sum_{i=1}^{6}[E_i],0) = 616, \quad \text{and} \quad GW_{\widetilde{X}_6}(6[D] - 2\sum_{i=1}^{6}[E_i],0) = 2002.$$

These numbers have been first computed by Vakil in [Vak00a].

3.2.2.4 Enumeration of real rational curves

Let $(\mathcal{D},m)$ be a d-marked floor diagram of genus 0, and let $\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im} \in \mathbb{Z}^{\infty}_{\geq 0}$ such that

$$I\alpha^{\Re} + I\beta^{\Re} + 2I\beta^{\Im} + 2I\beta^{\Im} = d \cdot [E].$$

Let $\aleph = d \cdot [D] - 1 + |\alpha^{\Re}| + |\beta^{\Re}| + 2|\alpha^{\Im}| + 2|\beta^{\Im}|$, and choose two integers $r,s \geq 0$ satisfying $\aleph = r + 2s + |\alpha^{\Re}| + 2|\alpha^{\Im}|$.

The set $\{i,i+1\} \subset A_0$ is a called *s-pair* if either $i = |\alpha^{\Re}| + 2k - 1$ with $1 \leq k \leq |\alpha^{\Im}|$, or $i = |\alpha^{\Re}| + 2|\alpha^{\Im}| + 2k - 1$ with $1 \leq k \leq s$. Denote by $\Im(m,s)$ the union of all the s-pairs $\{i,i+1\}$ where $m(i)$ is not adjacent to $m(i+1)$. Let $\psi_{0,s} : \{1,\ldots,\aleph\} \to \{1,\ldots,\aleph\}$ be the bijection defined by $\psi_{0,s}(i) = i$ if $i \notin \Im(m,s)$, and by $\psi_{0,s}(i) = j$ if $\{i,j\}$ is a s-pair contained in $\Im(m,s)$. Note that $\psi_{0,s}$ is an involution, and that $\psi_{0,0} = Id$.

Now chose an integer $0 \leq \kappa \leq \frac{n}{2}$ such that $d \cdot [E_{2i-1}] = d \cdot [E_{2i}]$ for $i = 1,\ldots,\kappa$. For $i = 2\kappa+1,\ldots,n$, define $\psi_{i,\kappa}$ to be the identity on A_i. For $i = 1,\ldots,\kappa$, choose a bijection $\psi_{2i-1,\kappa} : A_{2i-1} \to A_{2i}$, and define $\psi_{2i,\kappa} = \psi^{-1}_{2i-1,\kappa}$. Finally define the involution $\rho_{s,\kappa} : \bigcup_{i=0}^{n} A_i \to \bigcup_{i=0}^{n} A_i$ by setting $\rho_{s,\kappa|A_0} = \psi_{0,s}$, and $\rho_{s,\kappa|A_i} = \psi_{i,\kappa}$ for $i = 1,\ldots,n$. Note that $\rho_{0,0} = Id$.

Definition 3.30 *A d-marked floor diagram $(\mathcal{D},m)$ of genus 0 is called (s,κ)-real if the two marked floor diagrams $(\mathcal{D},m)$ and $(\mathcal{D},m \circ \rho_{s,\kappa})$ are equivalent.*

A (s,κ)-real d-marked floor diagram $(\mathcal{D},m)$ is said to be of type $(\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im})$ if

1. *the marked floor diagram $(\mathcal{D},m)$ is of type $(\alpha^{\Re} + 2\alpha^{\Im}, \beta^{\Re} + 2\beta^{\Im})$;*
2. *exactly $2\beta^{\Im}_j$ edges of weight j are contained in $Edge^{-\infty}(\mathcal{D}) \cap m(\Im(m,s))$ for any $j \geq 1$.*

The set of (s,κ)-real d-marked floor diagrams of genus 0 and of type $(\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im})$ is denoted by $\Phi^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}(d,s,\kappa)$. Note that the involution $\rho_{s,\kappa}$ induces an involution, denoted by $\rho_{m,s,\kappa}$, on the underlying floor diagram of a real marked floor diagram.

The set of pairs of floors of $\mathcal{D}$ exchanged by $\rho_{m,s,\kappa}$ is denoted by $\text{Vert}_{\Im}(\mathcal{D})$. The subset of $\text{Vert}_{\Im}(\mathcal{D})$ formed by floors of degree i is denoted by $\text{Vert}_{\Im,i}(\mathcal{D})$. To a pair $\{v,v'\} \in \text{Vert}_{\Im}(\mathcal{D})$, we associate the following numbers:

- o_v is the sum of the degree of v and the number of its adjacent edges which are in their turn adjacent to $m\left(\bigcup_{i=2\kappa+1}^n A_i\right)$;
- o'_v is the number of edges of weight $2+4l$ adjacent to v.

The set of edges of $\mathcal{D}$ which are fixed (resp. exchanged) by $\rho_{m,s,\kappa}$ is denoted by $\mathrm{Edge}_{\Re}(\mathcal{D})$ (resp. $\mathrm{Edge}_{\Im}(\mathcal{D})$). The number of edges contained in $m(\{\aleph-r+1,\ldots,\aleph\})$ is denoted by r_m, and the number of edges contained in $\mathrm{Edge}_{\Re}(\mathcal{D})\cap m(\{1,\ldots,\aleph-r\})$ is denoted by r'_m.

If $n=2\kappa$ and $\epsilon\in\{0,1\}$, the marked floor diagram $(\mathcal{D},m)$ is said to be *ϵ-sided* if any edge in $\mathrm{Edge}_{\Re}(\mathcal{D})$ is of even weight, and, if $\epsilon=1$, any floor of degree 1 is contained in a pair in $\mathrm{Vert}_{\Im}(\mathcal{D})$. It is said to be *significant* if it satisfies the three following additional conditions:

- any edge in $\mathrm{Edge}_{\Im}(\mathcal{D})\setminus m(\bigcup_{i=1}^n A_i)$ is of even weight;
- any edge in $\mathrm{Edge}_{\Re}(\mathcal{D})\setminus \mathrm{Edge}^{-\infty}(\mathcal{D})$ has weight $2+4l$;
- for any $\{v,v'\}\in \mathrm{Vert}_{\Im}(\mathcal{D})$ and any $i=1,\ldots,n$, the vertex v is adjacent to an edge adjacent to $m(A_i)$ if and only if so is v'.

Finally define

$$E(\mathcal{D})=\left(\mathrm{Edge}(\mathcal{D})\setminus \mathrm{Edge}^{-\infty}(\mathcal{D})\right)\cap m(\{1,\ldots,\aleph-r\}) \quad \text{and} \quad \beta^{\Re}_{even}=\sum_{j\geq 0}\beta^{\Re}_{2j}.$$

Definition 3.31 *Let $(\mathcal{D},m)$ be a (s,κ)-real d-marked floor diagram. The (s,κ)-real multiplicity of $(\mathcal{D},m)$, denoted by $\mu^{\mathbb{R}}_{s,\kappa}(\mathcal{D},m)$, is defined by*

$$\mu^{\mathbb{R}}_{s,\kappa}(\mathcal{D},m)=2^{\beta^{\Re}_{even}}\ I^{\beta^{\Im}}\prod_{\{v,v'\}\in Vert_{\Im}(\mathcal{D})}(-1)^{o_v}\prod_{e\in E(\mathcal{D})}w(e)$$

if $m(\Im(m,s))\bigcup Edge^{-\infty}(\mathcal{D})$ contains all edges of $\mathcal{D}$ of even weight, and by

$$\mu^{\mathbb{R}}_{s,\kappa}(\mathcal{D},m)=0$$

otherwise.

If in addition $2\kappa=n$ and $(\mathcal{D},m)$ is ϵ-sided, we define an additional (s,κ)-real multiplicity of $(\mathcal{D},m)$ as follows

$$\nu^{\mathbb{R},\epsilon}_{s}(\mathcal{D},m)=(-1)^{\epsilon|Vert_{\Im,1}(\mathcal{D})|}\ 2^{2r_m-r'_m+\beta^{\Re}_{even}}\ I^{\beta^{\Im}}\prod_{\{v,v'\}\in Vert_{\Im,2}(\mathcal{D})}(-1)^{o'_v}\prod_{e\in E(\mathcal{D})}w(e)$$

if $(\mathcal{D},m)$ is significant, and by

$$\nu^{\mathbb{R},\epsilon}_{s}(\mathcal{D},m)=0$$

otherwise.

Next, given $\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}$, we define

$$FW^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa)}(d,s)=\sum\mu^{\mathbb{R}}_{s,\kappa}(\mathcal{D},m)$$

where the sum is taken over all (s,κ)-real d-marked floor diagrams of type $(\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im})$.

If in addition $n=2\kappa$, we define the following numbers

$$FW^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa),\epsilon}(d,s)=\sum\mu^{\mathbb{R}}_{s,\kappa}(\mathcal{D},m)$$

where the sum is taken over all ϵ-sided (s,κ)-real d-marked floor diagrams of type $(\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im})$, and

$$FW^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa),\epsilon,\epsilon}(d,s)=\sum \nu^{\mathbb{R},\epsilon}_s(\mathcal{D},m)$$

where the sum is taken over all significant ϵ-sided (s,κ)-real d-marked floor diagrams of type $(\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im})$. Note that by definition we have

$$FW^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa)}(d,s)=FW^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa),\epsilon}(d,s)=FW^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa),\epsilon,\epsilon}(d,s)=0$$

if $d\cdot[E]\neq I\alpha^{\Re}+I\beta^{\Re}+2I\alpha^{\Im}+2I\beta^{\Im}$.

Next theorem relates the three series of numbers FW to actual enumeration of real curves in $\widetilde{X}_n(\kappa)$. When $n=2\kappa$, the connected component of $\mathbb{R}\widetilde{X}_n(\kappa)\setminus\mathbb{R}E$ with Euler characteristic ϵ is denoted by $\widetilde{L}_\epsilon$.

Theorem 3.32 ([Bru14, Theorem 3.12]) *Let $\aleph_0,r,s,\kappa\geq 0$ be some integers such that $\aleph_0=r+2s$. Then there exists a generic (E,s)-compatible configuration $\underline{x}^\circ$ of $\aleph_0$ points in $\widetilde{X}_n$ such that:*

1. *for any $d\in H_2(\widetilde{X}_n;\mathbb{Z})$ with $d\cdot[D]\geq 1$ and $d\neq l[E_i]$ with $l\geq 2$, any $\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}\in\mathbb{Z}^{\infty}_{\geq 0}$ such that*

$$d\cdot[D]-1+|\beta^{\Re}|+2|\beta^{\Im}|=\aleph_0\quad\text{and}\quad d\cdot[E]=I\alpha^{\Re}+I\beta^{\Re}+2I\alpha^{\Im}+2I\beta^{\Im},$$

and any generic real configuration $\underline{x}_E\subset E$ of type $(\alpha^{\Re},\alpha^{\Im})$, one has

$$W^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa)}(d,s,\underline{x}^\circ\sqcup\underline{x}_E)=FW^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa)}(d,s).$$

2. *If moreover $n=2\kappa$, and $\underline{x}$ is $(E,s,\widetilde{L}_\epsilon)$-compatible, then*

$$W^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa),\widetilde{L}_\epsilon,\mathbb{R}\widetilde{X}_n(\kappa)}(d,s,\underline{x}^\circ\sqcup\underline{x}_E)=FW^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa),\epsilon}(d,s),$$

and

$$W^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa),\widetilde{L}_\epsilon,,\widetilde{L}_\epsilon}(d,s,\underline{x}^\circ\sqcup\underline{x}_E)=FW^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa),\epsilon,\epsilon}(d,s).$$

Example 3.33 *If $n\leq 5$, the surface $\widetilde{X}_n(\kappa)$ is Del Pezzo, hence Theorem 3.32(1) computes Welschinger invariants of $\widetilde{X}_n(\kappa)$. In particular, applying Theorem 3.32 with $n=0$, we verify again that*

$$W_{\mathbb{C}P^2}(1,s)=W_{\mathbb{C}P^2}(2,s)=1\quad\text{and}\quad W_{\mathbb{C}P^2}(3,s)=8-2s.$$

Example 3.34 ([Bru14, Example 3.14]) *Fix $n=6$ and $\aleph_0=5$. Given $0\leq s\leq 2$, let $\underline{x}^\circ_s$ be a configuration whose existence is attested by Theorem 3.32 with $r=5-2s$. All numbers $W^{0,\beta^{\Re},0,\beta^{\Im}}_{\widetilde{X}_6(\kappa)}(d_k,s,\underline{x}^\circ_s)$ for the classes $d_k=6[D]-2\sum_{i=1}^6[E_i]-k[E]$ with $k=0,1,2$, as well as the numbers $W^{0,\beta^{\Re},0,\beta^{\Im}}_{\widetilde{X}_6(3),\widetilde{L}_\epsilon,\widetilde{L}_\epsilon}(d_0,s,\underline{x}^\circ_3)$, are listed in Tables 3.2 and 3.3. In the case $k=2$, this value is 1 for $(\beta^{\Re},\beta^{\Im})$ given in Table 3.2a, and 0 otherwise. In the case $k=1$, the numbers $W^{0,\beta^{\Re},0,\beta^{\Im}}_{\widetilde{X}_6(\kappa)}(d_1,s,\underline{x}^\circ_s)$ vanish for all values of $\beta^{\Re}$ and $\beta^{\Im}$ not listed in Table 3.2b. In the case $k=0$, all $(s,3)$-real diagrams contributing to $W^{0,0,0,0}_{\widetilde{X}_6(3)}(d_0,s,\underline{x}^\circ_s)$ are ϵ-sided with $\epsilon\in\{0,1\}$, so we have $W^{0,0,0,0}_{\widetilde{X}_6(3)}(d_0,s,\underline{x}^\circ_s)=W^{0,0,0,0}_{\widetilde{X}_6(3),\widetilde{L}_\epsilon,\mathbb{R}\widetilde{X}_6(3)}(d_0,s,\underline{x}^\circ_s)$.*

s	$\beta^{\Re}$	$\beta^{\Im}$
0	$4u_1$	0
1	$2u_1$	u_1
2	0	$2u_1$

a) $W^{0,\beta^{\Re},0,\beta^{\Im}}_{\widetilde{X}_6(\kappa)}(d_2, s, \underline{x}^\circ_s) = 1$

$s\backslash\kappa$		0	1	2	3
0	$\beta^{\Re} = 2u_1$	236	140	76	36
1	$\beta^{\Re} = 2u_1$	80	50	28	14
	$\beta^{\Im} = u_1$	62	28	10	0
2	$\beta^{\Im} = u_1$	74	36	14	0

b) $W^{0,\beta^{\Re},0,\beta^{\Im}}_{\widetilde{X}_6(\kappa)}(d_1, s, \underline{x}^\circ_s)$

Table 3.2:

$s\backslash\kappa,\epsilon$	0	1	2	3	0	1
0	522	236	78	0	160	96
1	390	164	50	0	64	32
2	286	128	50	20	24	8

Table 3.3: $W^{0,0,0,0}_{\widetilde{X}_6(\kappa)}(d_0, s, \underline{x}^\circ_s)$ and $W^{0,0,0,0}_{\widetilde{X}_6(3),\widetilde{L}_\epsilon,\widetilde{L}_\epsilon}(d_0, s, \underline{x}^\circ_s)$

3.2.2.5 Elements of the proof of Theorems 3.27 and 3.32

Again we apply the strategy detailed in Section 3.1. Recall that $\mathcal{N}_E = \mathbb{P}(\mathcal{N}_{E/\widetilde{X}_n} \oplus \mathcal{C})$, and let us define $E_\infty = \mathbb{P}(\mathcal{N}_{E/\widetilde{X}_n} \oplus \{0\})$, and $E_0 = \mathbb{P}(E \oplus \{1\})$.

The degeneration of $\widetilde{X}_n$ performed in step (1) is standard, see [Ful84, Chapter 5] for example. Consider the complex variety $\mathcal{Y}$ obtained by blowing up $\widetilde{X}_n \times \mathbb{C}$ along $E \times \{0\}$. Then $\mathcal{Y}$ admits a natural flat projection $\pi : \mathcal{Y} \to \mathbb{C}$ such that

- $\pi^{-1}(t) = \widetilde{X}_n$ for $t \neq 0$;
- $\pi^{-1}(0) = \widetilde{X}_n \cup \mathcal{N}_E$, the surfaces $\widetilde{X}_n$ and $\mathcal{N}_E$ intersecting transversely along E in $\widetilde{X}_n$, and E_∞ in $\mathcal{N}_E$.

If $\mathcal{E}$ denotes the Zariski closure of $E \times \mathbb{C}^*$ in $\mathcal{Y}$, then $\mathcal{E} \cap \pi^{-1}(0) = E_0$. At this point, it is more convenient to consider elements of $\mathcal{C}^{\alpha,\beta}(d, g, \underline{x}(t))$ as maps $f : C \to \widetilde{X}_n$ from a smooth genus g algebraic curve, rather than as embedded curves.

Choose $\underline{x}^\circ(t)$ (resp. $\underline{x}_E(t)$) a set of $d \cdot [D] - 1 + g + |\beta|$ (resp. $|\alpha|$) holomorphic sections $\mathbb{C} \to \mathcal{Y}$ (resp. $\mathbb{C} \to \mathcal{E}$), and denote $\underline{x}(t) = \underline{x}^\circ(t) \sqcup \underline{x}_E(t)$. Define $\mathcal{C}^{\alpha,\beta}(d, g, \underline{x}(0))$ to be the set $\left\{\overline{f} : \overline{C} \to \widetilde{X}_n \cup \mathcal{N}_E\right\}$ of limits, as stable maps, of maps in $\mathcal{C}^{\alpha,\beta}(d, g, \underline{x}(t))$ as t goes to 0. Recall that $\overline{C}$ is a connected nodal curve with arithmetic genus g such that

- $\underline{x}(0) \subset \overline{f}(\overline{C})$;
- any point $p \in \overline{f}^{\,-1}(\widetilde{X}_n \cap \mathcal{N}_E)$ is a node of $\overline{C}$ which is the intersection of two irreducible components $\overline{C}'$ and $\overline{C}''$ of $\overline{C}$, with $\overline{f}(\overline{C}') \subset \widetilde{X}_n$ and $\overline{f}(\overline{C}'') \subset \mathcal{N}_E$;
- if in addition neither $\overline{f}(\overline{C}')$ nor $\overline{f}(\overline{C}'')$ is entirely mapped to $\widetilde{X}_n \cap \mathcal{N}_E$, then the multiplicity of intersection of E with both $\overline{f}(\overline{C}')$ and $\overline{f}(\overline{C}'')$ at $\overline{f}(\overline{C}' \cap \overline{C}'')$ coincide.

Proposition 3.35 ([Bru14, Section 5.1]) *Suppose that $\underline{x}(0)$ is generic, and that $d \neq l[E_i]$ with $l \geq 2$. Then the set $\mathcal{C}^{\alpha,\beta}(d, g, \underline{x}(0))$ is finite, and only depends on $\underline{x}(0)$. Moreover if $\overline{f} : \overline{C} \to \widetilde{X}_n \cup \mathcal{N}_E$ is an element of $\mathcal{C}^{\alpha,\beta}(d, g, \underline{x}(0))$, then the following hold:*

- *no irreducible component of $\overline{C}$ is entirely mapped to $\widetilde{X}_n \cap \mathcal{N}_E$;*
- *any irreducible component of $\overline{C}$ entirely mapped to E_i is isomorphically mapped to E_i;*
- *all morphisms which converge to $\overline{f}$ as t goes to 0 can be recovered from $\overline{f}$;*
- *if in addition $\widetilde{X}_n$, $\underline{x}(t)$, and $\overline{f}$ are real, then all real morphisms which converge to $\overline{f}$ as t goes to 0 can be recovered from $\overline{f}$.*

The last point in Proposition 3.35 is proved thanks to a suitable adaptation of Proposition 4.12 below. Recall that $\mathcal{N}_E$ admits a natural projection $\mathcal{N}_E \to E_0$, and let us denote by ι the class realized by a fiber in $H_2(\mathcal{N}_E; \mathbb{Z})$.

Corollary 3.36 ([Bru14, Corollary 5.5]) *Let $\underline{x}^\circ_{\mathcal{N}} = \underline{x}^\circ(0) \cap \mathcal{N}_E$, and let $\overline{C}'$ an irreducible component of $\overline{C}$ mapped to $\mathcal{N}_E$. If $|\underline{x}^\circ_{\mathcal{N}} \cap \overline{f}(\overline{C}')| \leq 2$, then $\overline{f}(\overline{C}')$ realizes either the class $d_\iota \iota$ or $[E_\infty] + d_\iota \iota$ in $H_2(\mathcal{N}_E; \mathbb{Z})$. Moreover we have $|\underline{x}^\circ_{\mathcal{N}} \cap \overline{f}(\overline{C}')| \leq 1$ in the former case, and $1 \leq |\underline{x}^\circ_{\mathcal{N}} \cap \overline{f}(\overline{C}')| \leq 2$ in the latter case.*

Now Theorem 3.27 can be proved by a recursive use of Proposition 3.35. The fact that such a recursion is indeed possible is in particular ensured by the fact that any class realized by the image of any irreducible component of $\overline{C}$ is distinct from $l[E_i]$ with $l \geq 2$.

The union Y of finitely many irreducible algebraic varieties $Y_1, \ldots, Y_k$ intersecting transversely is called a *chain* if $Y_i \cap Y_j \neq \emptyset$ only when $|i - j| = 1$. In this case denote by Z_i^+ (resp. Z_i^-) the intersection $Y_i \cap Y_{i+1}$ viewed as a subvariety of Y_i (resp. Y_{i+1}), and write

$$Y = Y_k \ {}_{Z_{k-1}^-}\!\cup_{Z_{k-1}^+} Y_{k-1} \ {}_{Z_{k-2}^-}\!\cup_{Z_{k-2}^+} \cdots \ {}_{Z_1^-}\!\cup_{Z_1^+} Y_1.$$

Assume now that $d \cdot [D] \geq 1$ and $d \cdot [D] - 1 + g + |\beta| \geq 1$, all remaining cases being trivial. Recall that we have chosen two non-negative integers r and s such that

$$d \cdot [D] - 1 + g + |\beta| = r + 2s,$$

and that $s > 0$ implies that $g = 0$. By iterating the degeneration process of $\widetilde{X}_n$ described above, we construct a flat morphism $\pi : \mathcal{Z} \to \mathbb{C}$ such that

- $\pi^{-1}(t) = \widetilde{X}_n$ for $t \neq 0$;
- $\pi^{-1}(0)$ is a chain of X_n and $r + s + 1$ copies of $\mathcal{N}_E$:

$$\pi^{-1}(0) = \widetilde{X}_n \ {}_E\!\cup_{E_\infty} \mathcal{N}_{E,s+r} \ {}_{E_0}\!\cup_{E_\infty} \mathcal{N}_{E,s+r-1} \ {}_{E_0}\!\cup_{E_\infty} \cdots \ {}_{E_0}\!\cup_{E_\infty} \mathcal{N}_{E,0}.$$

Choose $\underline{x}^\circ(t)$ a generic set of $d \cdot [D] - 1 + g + |\beta|$ holomorphic sections $\mathbb{C} \to \mathcal{Z}$ such that $\underline{x}^\circ(0)$ contains exactly one point (resp. two points) in each $\mathcal{N}_{E,i}$ with $i \geq s + 1$ (resp. $1 \leq i \leq s$). Choose $\underline{x}_E(t)$ a generic set of $|\alpha|$ holomorphic sections $\mathbb{C} \to \mathcal{Z}$ such that $\underline{x}_E(t) \in E$ for any $t \in \mathbb{C}^*$. In particular $\underline{x}_E(0)$ is contained in the divisor E_0 of $\mathcal{N}_{E,0}$. Define $\underline{x}(t) = \underline{x}^\circ(t) \sqcup \underline{x}_E(t)$.

Using Proposition 3.35 and Corollary 3.36, one describes any limit $\overline{f} : \overline{C} \to \pi^{-1}(0)$ of maps in $\mathcal{C}^{\alpha,\beta}(d, g, \underline{x}(t))$ as t goes to 0 by a marked floor diagram $(\mathcal{D}(\overline{f}), m_{\overline{f}})$ with respect to a conic. Floors of $\mathcal{D}(\overline{f})$ of degree 2 (resp. of degree 1) correspond to components of $\overline{C}$ realizing a class $[E_\infty] + d_\iota \iota$ in $H_2(\mathcal{N}_{E,i}; \mathbb{Z})$ (resp. realizing a class $[D]$ or $[D] - [E_i]$ in $H_2(\widetilde{X}_n; \mathbb{Z})$). The complex and real multiplicities of $(\mathcal{D}(\overline{f}), m_{\overline{f}})$ follows from Proposition 3.35.

3.2.2.6 Further comments

The methods exposed in this paper adapt without any problem to the case when $r = 0$ or 1 and E has an empty real part. In particular, adapting proof of Theorems 3.32 and 4.20, we prove in Proposition 4.24 that the invariant $W_{X_7^-(4),\mathbb{R}P^2,\mathbb{R}X_7^-(4)}(d,s)$ is sharp when $r \leq 1$.

Recently, Kollár proved in [Kol14] the optimality of some real enumerative invariants of projective spaces of any dimension, by specializing the constraints to a real quadric with an empty real part. It would be interesting to try to generalize Kollár's examples, and to tackle the optimality problem of the invariants defined in [Wel05b, GZ13] via floor diagrams relative to a quadric in $\mathbb{C}P^n$.

3.2.3 Floor diagrams for rational curves in projective spaces

Here we use floor diagrams to enumerate rational curves in projective spaces. The strategy is the same than in Section 3.2.1, however the combinatoric becomes much heavier. This is in particular due to the fact that constraints may have different dimensions.

Given some integer numbers $n \geq 2$ and $l_0, \dots, l_{n-2} \geq 0$ such that

$$\sum_{j=0}^{n-2} l_j(n-1-j) = (n+1)d + n - 3, \tag{3.1}$$

we denote by $GW_{\mathbb{C}P^n}(d; l_0, \dots, l_{n-2})$ the Gromov-Witten invariant of $\mathbb{C}P^n$ counting rational curves of degree d intersecting a generic arrangement of l_i linear subspaces of dimension j for all $j = 0, \dots, n-2$ (see for example [KV06]).

3.2.3.1 Projective floor diagrams

Definition 3.37 *A floor diagram $\mathcal{D}$ of genus 0 is said to be* projective *if $Vert^{+\infty}(\mathcal{D}) = \emptyset$ and all floors if $\mathcal{D}$ have positive divergence.*

The degree of a projective floor diagram $\mathcal{D}$ is the sum of the divergence of its floors.

Remark 3.38 *Projective floor diagrams of genus 0 and degree d in the sense of Definition 3.37, with $div(v) = 1$ for all floors v, are precisely the planar floor diagrams of genus 0 with Newton polygon Δ_d in the sense of Definition 3.5.*

Let $n \geq 2$ and $l_0, \dots, l_{n-2} \geq 0$ be integer numbers subject to equation (3.1) and let

$$\mathcal{P} = \{x_1^{(0)}, \dots, x_{l_0}^{(0)}, \dots, x_1^{(n-2)}, \dots, x_{l_{n-2}}^{(n-2)}\}$$

be a set of $\sum_{j=0}^{n-2} l_j$ distinct elements equipped with some total ordering $<$. We define $\dim(x_k^{(j)}) = j$.

Definition 3.39 *A map $m : \mathcal{P} \to \mathcal{D} \setminus Vert^{-\infty}(\mathcal{D})$ is called a* marking *of $\mathcal{D}$ if it satisfies the following conditions:*

- *$Vert^\circ(\mathcal{D}) \subset m(\mathcal{P})$;*
- *if $q < q'$ and $m(q) > m(q')$, then $m(q') \in Vert^\circ(\mathcal{D})$ and there exists $q'' < q$ such that $m(q'') = m(q')$.*

A simple projective floor diagram $\mathcal{D}$ enhanced with a marking is called a *marked floor diagram*, and is said to be *marked by m*. We also say that $\mathcal{D}$ is an n-dimensional projective floor diagram marked by l_0 points, l_1 lines, ..., l_{n-2} codimension 2 planes.

Definition 3.40 *Two marked floor diagrams $(\mathcal{D}, m)$ and $(\mathcal{D}', m')$ are called equivalent if there exists a isomorphism of floor diagrams $\phi : \mathcal{D} \to \mathcal{D}'$ such that $m = m' \circ \phi$.*

They are called to be of the same combinatorial type *if there exists a bijection $\sigma : \mathcal{P} \to \mathcal{P}$ that preserves the dimension of the constraints and such that $(\mathcal{D}, m)$ is equivalent to $(\mathcal{D}, m \circ \sigma)$.*

As usual, we consider marked projective floor diagrams up to equivalence.

3.2.3.2 Enumeration of complex projective curves

To any simple projective marked floor diagram $(\mathcal{D}, m)$ of genus 0, we assign a integer called its *complex multiplicity*, denoted by $\mu_{\mathbb{C}}(\mathcal{D}, m)$. This multiplicity records the number of complex curves encoded by the diagram.

We first need to associate a integer number to any edge e of $\mathcal{D}$ as follows. Let $\mathcal{D}_{<e}$ be the component of $\mathcal{D} \setminus e$ that contains elements of $\mathcal{D}$ lower than e. Then we define

$$h(e) = \sum_{q \in \mathcal{P},\ m(q) \in \mathcal{D}_{<e}} \big(n - 1 - \dim(q)\big) + 1 - w(e) - (n+1) \sum_{v \in \mathrm{Vert}^\circ(\mathcal{D}) \cap \mathcal{D}_{<e}} \mathrm{div}(v).$$

Given a floor v of $\mathcal{D}$, let us do the following: for the minimum element $x_k^{(j)}$ of $m^{-1}(v)$, we take a linear space of dimension j; for each other element $x_{k'}^{(j')}$ in $m^{-1}(v)$, we take a linear space of dimension $j' - 1$; we take a linear space of dimension $h(e)$ for each edge e outgoing from v; for each edge e incoming to v we take a linear space of dimension

$$n - 1 - h(e) - \sum_{q \in \mathcal{P},\ m(q) \in e} \big(n - 1 - \dim(q)\big).$$

If any of these numbers is outside of the range between 0 and $(n-2)$ then we set

$$\mu^{\mathbb{C}}(v) = 0.$$

Otherwise denote the number of resulting j-dimensional linear spaces with $l_j^{(v)}$ and define

$$\mu^{\mathbb{C}}(v) = \mathrm{div}(v)^{l_{n-2}^{(v)}} GW_{\mathbb{C}P^{n-1}}(\mathrm{div}(v); l_0^{(v)}, \ldots, l_{(n-3)}^{(v)}).$$

Note that we express $\mu^{\mathbb{C}}(v)$ in terms a Gromov-Witten invariant of $\mathbb{C}P^{n-1}$.

Definition 3.41 *The complex multiplicity of a simple projective marked floor diagram $(\mathcal{D}, m)$ of genus 0 is defined as*

$$\mu^{\mathbb{C}}(\mathcal{D}, m) = \prod_{v \in Vert^\circ(\mathcal{D})} \mu^{\mathbb{C}}(v) \prod_{e \in Edge(\mathcal{D})} w(e)^{1 + |(m^{-1}(e)|}.$$

Note that two marked floor diagrams of the same combinatorial type have the same complex multiplicity.

Example 3.42 *When $n = 2$, a projective marked diagram $(\mathcal{D}, m)$ of degree d with non-zero multiplicity is a planar marked floor diagram of genus 0 and Newton polygon Δ_d. In particular $h(e) = 0$ for any $e \in Edge(\mathcal{D})$, and*

$$\mu^{\mathbb{C}}(\mathcal{D}, m) = \prod_{e \in Edge(\mathcal{D})} w(e)^2.$$

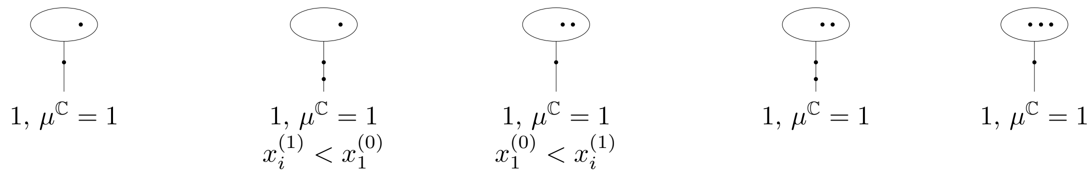


Figure 3.6: 3-dimensional projective marked floor diagrams of genus 0 and degree 1, marked by two points, one point and two lines, and four lines

Example 3.43 *In Figure 3.6, we depict all combinatorial types of* 3*-dimensional floor diagrams of genus 0 and degree* 1 *with non-null multiplicity, marked by either two points, a point and two lines, or four lines. The chosen order on* $\mathcal{P}$ *plays a role only in the case of one point and two lines, for which we depict the combinatorial types corresponding to two different orders.*

For each combinatorial type, we depict the floor diagram together with the image of the marking m*. In addition we write below the number of marked floor diagram of this combinatorial type, and the complex multiplicity of such a floor diagram.*

Example 3.44 *In Figure 3.7, we depict all combinatorial types of* 3*-dimensional floor diagrams of genus 0 and degree* 3*,* 4*, and* 5*, marked by respectively* 6*,* 8*, and* 10 *points and with non-null multiplicity. Note that there does not exist any 3-dimensional marked floor diagram of genus 0 and degree 2 marked by 4 points with non-null multiplicity.*

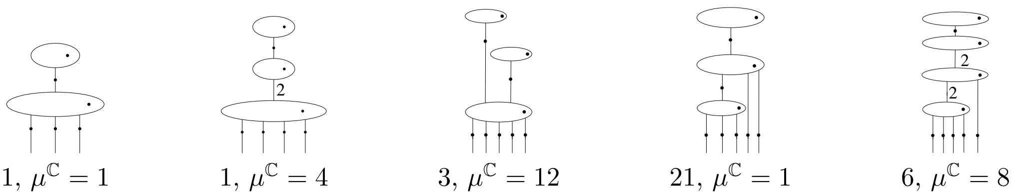


Figure 3.7: 3-dimensional marked floor diagrams of degree 3, 4, and 5 respectively marked by 6, 8, and 10 points

Example 3.45 *In Figure 3.8, we depict all combinatorial types with non-null multiplicity of* 3*-dimensional floor diagrams of degree* 2 *marked by* 8 *lines.*

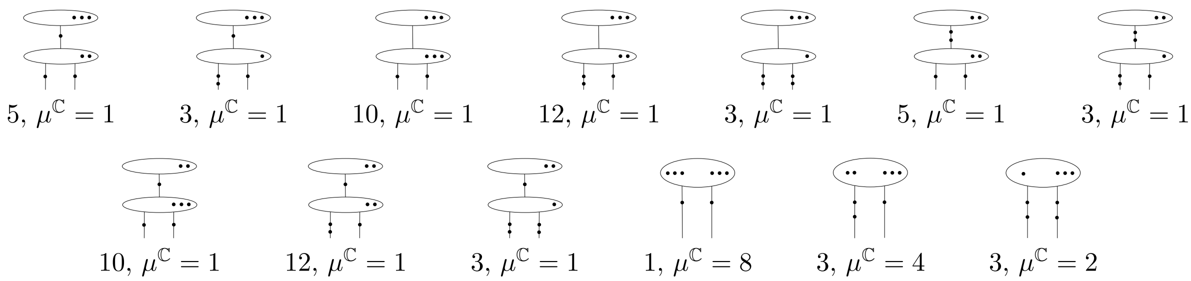


Figure 3.8: 3-dimensional marked floor diagrams of degree 2 marked by 8 lines

Enumeration of marked floor diagrams and of complex rational curves in $\mathbb{C}P^n$ are related by the following theorem.

Theorem 3.46 ([BM07, Theorem 1], [BMa, Theorem 2.10]) *For any $n \geq 2$ and any integer numbers $l_0, \ldots, l_{n-2} \geq 0$ subject to equality (3.1), the number $GW_{\mathbb{C}P^n}(d; l_0, \ldots, l_{n-2})$ is equal to the sum of the complex multiplicity of all n-dimensional floor diagrams of genus 0 and degree d, marked by l_0 points, l_1 lines, ..., l_{n-2} codimension 2 planes.*

Example 3.47 *Using Figures 3.6, 3.7, and 3.8, we verify that*

$$GW_{\mathbb{C}P^3}(1;2,0) = GW_{\mathbb{C}P^3}(1;1,2) = 1, \quad GW_{\mathbb{C}P^3}(1;0,4) = 2, \quad GW_{\mathbb{C}P^3}(2;4,0) = 0,$$

$$GW_{\mathbb{C}P^3}(2;0,8) = 92, \quad GW_{\mathbb{C}P^3}(3;6,0) = 1, \quad GW_{\mathbb{C}P^3}(4;8,0) = 4, \quad GW_{\mathbb{C}P^3}(5;10,0) = 105.$$

3.2.3.3 Welschinger invariants of $\mathbb{R}P^3$

Welschinger also defined invariants for real rational curves in $\mathbb{R}P^3$ passing through points, see [Wel05b]. We denote by $W_{\mathbb{R}P^3}(d)$ the corresponding invariants for curves of degree d passing through a configuration of real points. Note that Mikhalkin observed that, by symmetry reasons, all Welschinger invariants of $\mathbb{R}P^3$ of even degree vanish.

Here we compute the invariants $W_{\mathbb{R}P^3}(d)$ via floor diagrams.

Definition 3.48 *Let $(\mathcal{D}, m)$ be a 3-dimensional projective floor diagram of genus 0 only marked by points. The real multiplicity of $(\mathcal{D}, m)$ is defined by*

$$\mu^{\mathbb{R}}(\mathcal{D}, m) = \prod_{v \in Vert^\circ(\mathcal{D})} W_{\mathbb{R}P^2}(d, 0)$$

if all edges of $\mathcal{D}$ are odd, and by

$$\mu^{\mathbb{R}}(\mathcal{D}, m) = 0$$

otherwise.

Theorem 3.49 ([BMa, Theorem 2]) *For any $d \geq 1$, we have*

$$W_{\mathbb{R}P^3}(d) = (-1)^{\frac{(d-1)(d-2)}{2}} \sum \mu^{\mathbb{R}}(\mathcal{D}, m)$$

where the sum ranges over all 3-dimensional projective floor diagrams of genus 0 and degree d marked by $2d$ points.

Example 3.50 *Using marked floor diagrams depicted in Figures 3.6, and 3.7, we compute*

$$W_{\mathbb{R}P^3}(1) = 1, \quad W_{\mathbb{R}P^3}(2) = 0, \quad W_{\mathbb{R}P^3}(3) = -1, \quad W_{\mathbb{R}P^3}(5) = 45.$$

Note that one can deduce that $W_{\mathbb{R}P^3}(2k) = 0$ also from Theorem 3.49.

3.2.3.4 Elements of the proof of Theorems 3.46 and 3.49

As in the case of planar floor diagrams, the basic ingredient for the floor diagram technique to work in our situation is to stretch the configuration of constraints in the vertical direction. A *complete tropical linear space* in $\mathbb{R}^n$ is a tropical intersection of finitely many tropical hyperplane.

Let us fix some integers $d \geq 1$, $n \geq 2$, $l_0 \geq 0$, ..., $l_{n-2} \geq 0$ subject to equality (3.1), and let us choose a generic configuration $\underline{x}$ of complete tropical linear spaces in $\mathbb{R}^n$ containing exactly l_j spaces of dimension j. We denote by $\mathbb{T}\mathcal{C}(d, \underline{x})$ is the set of all rational closed tropical morphisms $f : C \to \mathbb{R}^n$ of degree d such that $f(C)$ intersects all tropical linear spaces in $\underline{x}$. Let $\mathrm{Vert}(L)$ be the set of vertices of the complete tropical linear space L, and let us fix a hypercube $\mathcal{H}_{n-1}$ in $\mathbb{R}^{n-1}$ such that the cylinder $\mathcal{H}_{n-1} \times \mathbb{R}$ contains the set $\cup_{L \in \underline{x}} \mathrm{Vert}(L)$.

Lemma 3.51 ([BMa, Proposition 5.7]) *For any tropical morphism $f : C \to \mathbb{R}^n$ in $\mathbb{T}\mathcal{C}(d, \underline{x})$, we have $f(Vert(C)) \subset \mathcal{H}_{n-1} \times \mathbb{R}$.*

Given two points $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ in $\mathbb{R}^n$, we define $|p - q|_n = |p_n - q_n|$. Finally, we define $R_{\mathcal{H}}$ to be the euclidean length of the edges of $\mathcal{H}_{n-1}$, and

$$R(\underline{x}^{\mathbb{T}}) = \min_{L \neq L' \in \underline{x},\ p \in \mathrm{Vert}(L),\ q \in \mathrm{Vert}(L')} |p - q|_n.$$

An *elevator* of a tropical morphism $f : C \to \mathbb{R}^n$ is an edge with $u_{f,e} = \pm(0, 0, \ldots, 0, 1)$. A *floor* of f is a connected component of C with all elevators removed. Analogously, the wall (resp. floor) of a complete tropical linear space L is the union of all faces of L which contain (resp. do not contain) the direction $(0, \ldots, 0, 1)$. An element L of $\underline{x}$ is called a vertical (resp. horizontal) constraint for $f \in \mathbb{T}\mathcal{C}(d, \underline{x})$ if $f(C) \cap L$ lies in the wall (resp. floor) of L.

Corollary 3.52 ([BMa, Proposition 5.8]) *There exists a real number $D(n, d)$, depending only in n and d, such that if $R(\underline{x}) \geq R_{\mathcal{H}} \times D(n, d)$, then for each morphism $f : C \to \mathbb{R}^n$ in $\mathbb{T}\mathcal{C}(d, \underline{x})$ and for each floor $\mathcal{F}$ of f, $f(\mathcal{F})$ meets one and exactly one horizontal constraint.*

Now as in the case of h-transverse polygons, a projective marked floor diagram $\Phi(f) = (\mathcal{D}(f), m)$ can be naturally associated to an element f of $\mathbb{T}\mathcal{C}(d, \underline{x})$. The map Φ is no longer a bijection when $n \geq 3$, however we have the following proposition, where $\mu^{\mathbb{C}}_{\underline{x}}(f)$ is the complex multiplicity of an element f of $\mathbb{T}\mathcal{C}(d, \underline{x})$ (see [BMa]).

Proposition 3.53 ([BMa, Proposition 5.14]) *If $(\mathcal{D}, m)$ is a floor diagram marked by $\underline{x}$, then*

$$\sum_{f \in \Phi^{-1}(\mathcal{D}, m)} \mu^{\mathbb{C}}_{\underline{x}}(f) = \mu^{\mathbb{C}}(\mathcal{D}, m).$$

3.3 Applications

We present now various applications of floor diagrams.

3.3.1 Qualitative results about Welschinger invariants

3.3.1.1 Logarithmic asymptotic

Floor diagrams may be used to prove several estimate on Welschinger invariants. In the case of surfaces, Itenberg, Kharlamov, and Shustin studied in particular their asymptotic for real algebraic surfaces whose underlying complex variety is $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2$ blown up in at most 7 points. Floor diagrams can be used to give an alternate proof of those results, which can be extended to the 3-dimensional case.

As an example, a proof using floor diagrams of the following statement can be found in [BM08].

Theorem 3.54 (Itenberg, Kharlamov, Shustin, [IKS04]) *The sequence $(W_{\mathbb{R}P^2}(d, 0))_{d \geq 1}$ is increasing, and strictly increasing starting from $W_{\mathbb{R}P^2}(2, 0)$.*

The logarithmic asymptotic of $W_{\mathbb{R}P^2}(d, 0)$ is given by

$$\log(W_{\mathbb{R}P^2}(d, 0)) \sim_{d \to \infty} 3d \log d \sim_{d \to \infty} \log(GW_{\mathbb{C}P^2}(d; 3d - 1)).$$

The proof from [Bru08] extends easily to the case of $W_{\mathbb{R}P^3}(d)$. Recall that $W_{\mathbb{R}P^3}(2k) = 0$.

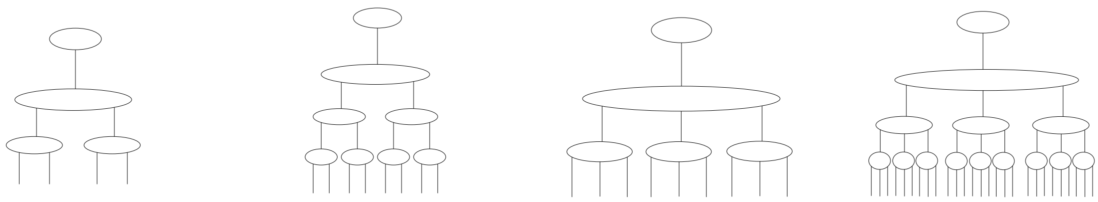
Theorem 3.55 ([BMa]) *The sequence* $(W_{\mathbb{R}P^3}(2k+1))_{k\geq 1}$ *is increasing, and strictly increasing starting from* $W_{\mathbb{R}P^3}(3)$.

The logarithmic asymptotic of $W_{\mathbb{R}P^3}(2k+1)$ *is given by*

$$\log(W_{\mathbb{R}P^3}(2k+1)) \sim_{k\to\infty} 4k\log k \sim_{k\to\infty} \log(GW_{\mathbb{C}P^3}(d;2d,0)).$$

Let us sketch briefly the proof of the logarithmic equivalences. We define the following sequence $(\mathcal{D}_k^n)_{k\geq 1}$ of projective floor diagrams of genus 0:

- $\mathcal{D}_1^n$ is the projective floor diagram of degree 1;
- the floor diagram $\mathcal{D}_k^n$ is obtained from $\mathcal{D}_{k-1}^n$ by attaching to each edge in $\mathrm{Edge}^{-\infty}(\mathcal{D}_{k-1}^n)$ a floor of degree $n-1$ adjacent to n edges, all of them of weight 1.



a) $\mathcal{D}_3^2$ b) $\mathcal{D}_4^2$ c) $\mathcal{D}_3^3$ d) $\mathcal{D}_4^3$

Figure 3.9:

Example 3.56 *The floor diagrams* $\mathcal{D}_3^2$, $\mathcal{D}_4^2$, $\mathcal{D}_3^3$ *and* $\mathcal{D}_4^3$ *are depicted in Figure 3.9.*

Both logarithmic equivalences of Theorems 3.54 and3.55 are now a consequence of the next easy lemma specialized to the case $n = 2$ and 3. We define the integer

$$p_{n,d} = \frac{(n+1)d^n + (n-3)}{n-1}$$

and $\nu(\mathcal{D}_k^n)$ to be the number of n-dimensional markings of $\mathcal{D}_k^n$ by $p_{n,d}$ points.

Lemma 3.57 *The floor diagram* $\mathcal{D}_k^n$ *is of degree* n^{k-1}*, and one has*

$$\log(\nu(\mathcal{D}_k^n)) \sim_{k\to\infty} p^n_{n,n^{k-1}} \log(n^{k-1}).$$

3.3.1.2 Congruences and comparison

It is obvious that real and complex enumerative invariants, when both are defined, are equal modulo 2. Mikhalkin noticed that in the case of toric real Del Pezzo surfaces and $s = 0$, this equality is actually true modulo 4. This observation has been generalized to all maximal Del Pezzo surface of degree at least 2 by Itenberg, Kharlamov, and Shustin [IKS13c, IKS13b, IKS13a]. Combining Theorems 3.27 and 3.32 together with Theorems 4.25 and 4.27, we extend this congruence in Corollary 4.31 to the case of the maximal Del Pezzo surface of degree 1. In their turn, Theorems 3.46 and 3.49 allow to generalize Mikhalkin's congruence to $W_{\mathbb{R}P^3}$.

Proposition 3.58 ([BMa]) *For any* $d \in \mathbb{N}^*$, *one has*

$$|W_{\mathbb{R}P^3}(d)| = GW_{\mathbb{C}P^3}(d;2d,0) \quad mod\ 4.$$

Combining Theorems 3.27 and 3.32 together with Theorems 4.17, 4.20, 4.25, 4.27, and 4.6, we obtain Corollaries 4.21, 4.22, 4.29, 4.30, and Proposition 4.9 on Welschinger invariants of Del Pezzo surfaces. We refer to Chapter 4 for precise statements. These results compare Welschinger invariants for different real structures on the same complex surface, and extend positivity results obtained by Itenberg, Kharlamov, and Shustin in [IKS04, IKS13c, IKS13b, IKS13a].

3.3.1.3 Sharpness of Welschinger invariants

The invariant $W_{(X,\tau),L,L'}(d,s)$ is said to be *sharp* if there exists a real configuration $\underline{x}$ with s pairs of complex conjugated points such that $|\mathbb{R}\mathcal{C}(d,0,\underline{x})| = |W_{(X,\tau),L,L'}(d,s)|$. When $r=0$ or 1, Welschinger proved in [Wel07] the sharpness of $W_{(X,\tau),L}(d,s)$ in some cases. The methods used in the proof of Theorems 3.32 and 4.20 adapt without any problem to the case when $r=0$ or 1 and E has an empty real part. In particular, such adaptations allow one to extend [Wel07, Theorem 1.1] to the real surface $X_7^-(4)$, see Proposition 4.24.

Note that it follows from Example 4.23 that $W_{X_7^+(4),L_0,\mathbb{R}X_7^+(4)}(2c_1(X_7),1)$ is not sharp. This shows that [Wel07, Theorem 1.1] does not extend to all real structures on X_7.

3.3.1.4 Vanishing results

Floor diagrams also provide a simpler proof of the vanishing Theorem 4.4 in some particular instances. Since Theorem 4.4 is more general, we do not develop here this application of floor diagrams. We refer instead to [Bru14] for more details.

3.3.2 Recursive formulas of Caporaso-Harris type

As mentioned in Section 3.1, when both floor diagrams and Caporaso-Harris type formulas are available, these two methods provide two different but equivalent ways of clustering curves under enumeration. The method to pass from floor diagrams to Caporaso-Harris type formulas is explained in [ABLdM11]. In particular we wrote down there such a formula computing Welschinger invariants of $\mathbb{R}P^2$ for any value of r and s. Since it is a rather impressive formula, we do not reproduce it here, and refer instead to [ABLdM11, Theorem 5.2]. As a consequence, we were able to compute for the first time in [ABLdM11] the value $W_{\mathbb{C}P^2}(9,12)$, which turned out to satisfy $0 < W_{\mathbb{C}P^2}(9,12) < W_{\mathbb{C}P^2}(9,13)$. This computation disproved the monotonicity conjecture of the function $s \mapsto W_{\mathbb{C}P^2}(d,s)$ by Itenberg, Kharlamov and Shustin (see [IKS04, Conjecture 6]).

To give a taste of those Caporaso-Harris type formulas, we give below a formula which computes the invariants $W_{\mathbb{R}P^3}(d)$.

Given an integer number $l \geq 0$ and two odd vectors α and β in $\mathbb{Z}_{\geq 0}^{\infty}$, we denote by $\mathcal{S}_3(l,\alpha,\beta)$ the set composed by the vectors $(d_1,\ldots,d_l,\, k_1,\ldots,k_l,\, \alpha_1,\ldots,\alpha_l,\, \beta_1,\ldots,\beta_l) \in \mathbb{Z}_{>0}^{2l} \times (\mathbb{Z}_{\geq 0}^{\infty})^{2l}$ satisfying

- $\forall i, (d_i,k_i,\alpha_i,\beta_i) \leq (d_{i+1},k_{i+1},\alpha_{i+1},\beta_{i+1})$ for the lexicographic order,
- $\sum d_i < I\alpha + I\beta$,
- $\sum \alpha_i \leq \alpha$,
- $\forall i,\ k_i$ is odd,
- $\forall i, \beta_i \geq u_{k_i}$,
- $\sum(\beta_i - u_{k_i}) = \beta$,
- $\forall i, I\alpha_i + I\beta_i = d_i$,
- $l + |\alpha - \sum \alpha_i| = 3d - 3\sum d_i - 2$.

To any element $s = (d_1,\ldots,d_l,k_1,\ldots,k_l,\alpha_1,\ldots,\alpha_l,\beta_1,\ldots,\beta_l)$ of $\mathcal{S}_3(l,\alpha,\beta)$, we associate the equivalence relation $\sim_s$ on the set $\{1,\ldots,l\}$ defined by

$$i \sim_s j \Leftrightarrow (d_i,k_i,\alpha_i,\beta_i) = (d_j,k_j,\alpha_j,\beta_j)$$

For each of the equivalent classes of $\sim_s$, evaluate the factorial of its cardinal, and denote by $\sigma(s)$ the product of these factorials.

Finally, given an integer number $d \geq 1$ and two odd vectors α and β in $\mathbb{Z}^\infty_{\geq 0}$ satisfying $I\alpha + I\beta = d$, we define the numbers $W_3^{\alpha,\beta}(d)$ by the initial value $W_3^{u_1,0}(1) = 1$ and the relation

$$\begin{aligned} W_3^{\alpha,\beta}(d) \quad = \quad & \textstyle\sum_{k \ odd | \beta \geq u_k} W_3^{\alpha+u_k,\beta-u_k}(d) \ + \\ & \sum_{\substack{l\geq 0 \\ s\in\mathcal{S}_3(l,\alpha,\beta)}} \left[\frac{1}{\sigma(s)} W_2\left(d-\textstyle\sum d_i, 0\right) \begin{pmatrix} \frac{3d-|\alpha|+|\beta|}{2} - 1 \\ \frac{3d_1-|\alpha_1|+|\beta_1|}{2}, \dots, \frac{3d_l-|\alpha_l|+|\beta_l|}{2} \end{pmatrix} \right. \\ & \left. \begin{pmatrix} \alpha \\ \alpha_1, \dots, \alpha_l \end{pmatrix} \textstyle\prod_{i=1}^l (\beta_i)_{k_i} W_3^{\alpha_i,\beta_i}(d_i) \right] \end{aligned}$$

The numbers $W_3^{\alpha,\beta}(d)$ and $W_{\mathbb{R}P^3}(d)$ are related by the following theorem.

Theorem 3.59 ([ABLdM11, Theorem 7.1]) *For any $d \geq 1$, one has*

$$W_{\mathbb{R}P^3}(d) = (-1)^{\frac{(d-1)(d-2)}{2}} W_3^{0,du_1}(d).$$

Example 3.60 *We list below the first value of $W_3(d)$ computed using Theorem 3.59 (recall that $W_3(2k) = 0$).*

d	1	3	5	7	9	11	13
$W_3(d)$	1	-1	45	-14589	17756793	-58445425017	426876362998821

3.3.3 Maximal real configurations for real conics

It is an important and difficult question to ask how many solutions of an enumerative problem can be real. When all complex solutions can be real, we say that this enumerative problem is *maximally real*.

To determine whether an enumerative problem is maximally real or not is in general a very difficult task, and not much is known in general. With the help of floor diagrams, we proved in collaboration with Puignau in [BP13b] that problems involving conics in projective spaces subject to simple incidence conditions are maximally real.

Let us first recall the context. Given integer numbers $n \geq 2$ and $l_0, \dots, l_{n-2} \geq 0$ such that

$$\sum_{j=0}^{n-2} l_j(n-1-j) = (n+1)d + n - 3,$$

and given a configuration $\underline{x}$ of l_i *real* linear subspaces of $\mathbb{R}P^n$ of dimension i for all $i = 0 \dots, l_{n-2}$, we denote by $N_n^{\mathbb{R}}(d; l_0, \dots, l_{n-2}, \underline{x})$ the number of real rational curves of degree d in $\mathbb{R}P^n$ intersecting all elements of $\underline{x}$. This number depends on the chosen configuration $\underline{x}$. Clearly, we have the inequality

$$N_n^{\mathbb{R}}(d; l_0, \dots, l_{n-2}, \underline{x}) \leq GW_{\mathbb{C}P^n}(d; l_0, \dots, l_{n-2}) \quad \forall \underline{x}.$$

However, it is unknown in general if there exists a real configuration $\underline{x}$ such that all complex solutions are real. For example, can the 92 complex conics passing through 8 general lines in $\mathbb{R}P^3$ be real?

To stress how difficult these questions are, let us summarize the very few things known in 2014 about the maximality of the enumerative problems defined above. Since the corresponding Gromov-Witten invariant is equal to 1, it is trivial that the problem is maximal in the two following cases:

- $d = 1$, $l_0 \geq 1$;
- $n = 2$, $d = 2$, and $l_0 = 5$.

It is also easy to see that the problem is maximal in the case $n = 2$ and $d = 3$ (and so $l_0 = 8$). The first systematic non-trivial result was obtained by Sottile who proved in [Sot97] that the problem is maximal as soon as $d = 1$ (the so-called problems of "Schubert-type"). Actually problems of Schubert-type involving linear subspaces of any dimension turn out to be maximal by [Vak06]. It is announced in [BM07] that the problem above is maximal for $d = 2$ and $n = 3$. We generalized this observation in [BP13b].

Theorem 3.61 ([BP13b, Theorem 1.1]) *For $n \geq 2$, $l_0, \ldots, l_{n-2} \geq 0$ satisfying (3.1), there exists a generic configuration $\underline{x}$ of real linear subspaces of $\mathbb{R}P^n$ as above such that*

$$N_n^{\mathbb{R}}(2; l_0, \ldots, l_{n-2}, \underline{x}) = GW_{\mathbb{C}P^n}(2; l_0, \ldots, l_{n-2}).$$

Let us say a few words about the proof of Theorem 3.61. Proposition 2.11 generalizes immediately to the case of rational curves in $\mathbb{R}P^n$. Starting from this observation, the idea of Theorem 3.61 is quite simple: we exhibit configurations $\underline{x}$ of complete tropical linear spaces, called *well-ordered totally decomposing configurations* in [BP13b], such that all tropical conics passing through them have tropical multiplicity 1. Since all complete tropical linear spaces are trivially approximable by real linear spaces in $\mathbb{R}P^n$, Theorem 3.61 now follows from (generalized) Proposition 2.11. Floor decomposition of tropical curves is the key ingredient in the construction of well-ordered totally decomposing configurations, since it allows one to construct them by induction on the dimension of the ambient projective space. Note that the proof of Theorem 3.61 also provides a proof of Sottile's Theorem different from the original one, but closely related to Vakil's one.

One could also study maximality of more general real enumerative problems, for example by prescribing tangency conditions with constraints. We refer the interested reader to [RTV97, Ber08, Sot, BBM14] for some partial answers in this direction.

For example, next proposition in proved in [BBM14] by tropical and floor decomposition methods (see also Chapter 5)).

Proposition 3.62 ([BBM14, Proposition 7.3]) *For any $0 \leq k \leq 5$, any $d_1 \ldots, d_{5-k} \geq 1$, and any $g_1 \ldots, g_{5-k} \geq 0$, there exists a generic configuration of k points $p_1, \ldots, p_k$ in $\mathbb{R}P^2$ and $5-k$ immersed real algebraic curves $C_1, \ldots, C_{5-k}$ with C_i of degree d_i and genus g_i such that all conics passing through $p_1, \ldots, p_k$ and tangent to $C_1, \ldots, C_{5-k}$ are real.*

The question of existence of non-trivial lower bounds for the numbers $N_n^{\mathbb{R}}(d; l_0, \ldots, l_{n-2}, \underline{x})$ is also a very important and difficult problem about which not so much is known. Welschinger type invariants provide such non-trivial lower bounds in some cases [Wel05a, Wel05b, Sol06, Geo13]. In the case of enumeration of lines (and more generally in the enumeration of real linear spaces), the existence of some non-trivial lower bounds has been proved by Gabrielov and Eremenko in [EG02]. See also [OT14, FK13] for a related discussion. To our knowledge, the exact determination of the minimal value of $N_n^{\mathbb{R}}(d; l_0, \ldots, l_{n-2}, \underline{x})$ (when non-trivial) is known so far only in the cases $n = 2$, $d = 3$ ([DK00]) and $d = 4$ ([Rey]), and in the cases $d = 1$ and $l_i = 0$ for $i \leq n-3$ ([EG02]).

The technique used in [BP13b] should apply to prove the maximality of other enumerative problems. For example, is it true that enumerative problems involving smooth curves in projective spaces subject to simple incidence conditions are maximally real?

3.3.4 (Piecewise)-polynomiality of Gromov-Witten invariants

Fomin and Mikhalkin used floor diagrams in [FM10] to give a proof of Göttsche conjecture in the case of $\mathbb{C}P^2$: for a fixed δ, the numbers $GW_{\mathbb{C}P^2,\frac{(d-1)(d-2)}{2}-\delta}(d)$ are given by a polynomial for d large enough. This work generated many developments (see for example [Blo11, AB13, BCK13, Liu13, LO14, BG14]), in particular in the case of singular surfaces, which are à priori outside the realm of Göttsche conjecture (see [AB13, LO14]).

Meanwhile, Cavalieri, Johnson, and Markwig used tropical curves in [CJM10, CJM11] to study piecewise-polynomiality of the so-called *double Hurwitz numbers*. Ardila gave in [Ard] an alternative proof of the piecewise-polynomiality of double Hurwitz numbers, based on the De Concini-Procesi-Vergne theory of *remarkable spaces*. This theory turns out in particular to be an efficient tool to prove piecewise-polynomiality of certain functions.

In collaboration with Ardila, we generalized this piecewise-polynomiality of double Hurwitz numbers to *double Gromov-Witten invariant of Hirzebruch surfaces*. The basic observation is that the tropical count of double Hurwitz numbers performed in [CJM10] is nothing but a floor diagram count in dimension 1. This remark being made, it is then immediate that Ardila's approach [Ard] extends to the 2-dimensional case.

In order to give a rigorous statement, we need first to introduce some notations. Let E and B be two non-singular irreducible algebraic curves in Σ_n such that $[B]^2 = -[E]^2 = -n$ and $[B]\cdot[E] = 0$ (note that there is no choice for E if $n > 0$). Let us fix four non-negative integer numbers a, b, n, and g, and four sequences of non-negative integer numbers $\alpha, \widetilde{\alpha}, \beta, \widetilde{\beta} \in \mathbb{Z}_{\geq 0}^{\infty}$ such that

$$I\alpha + I\beta = an + b \quad \text{and} \quad I\widetilde{\alpha} + I\widetilde{\beta} = b.$$

We define $l = 2a + g + I\beta + I\widetilde{\beta} - 1$. Next, let us choose a generic configuration

$$\underline{x} = \{q_1^1, \dots, q_{\alpha_1}^1, \dots, q_1^i, \dots, q_{\alpha_i}^i, \dots, p_1, \dots, p_l, \widetilde{q}_1^1, \dots, \widetilde{q}_{\widetilde{\alpha}_1}^1, \dots, \widetilde{q}_1^i, \dots, \widetilde{q}_{\widetilde{\alpha}_i}^i, \dots\}$$

of $l + I\alpha + I\widetilde{\alpha}$ points in Σ_n such that $q_j^i \in B$, $\widetilde{q}_j^i \in E$, and $p_i \in \Sigma \setminus (B \cup E)$.

We denote by $GW_{\Sigma_n}^{\alpha,\beta,\widetilde{\alpha},\widetilde{\beta}}(a, b, g)$ the number of irreducible complex algebraic curves C in Σ_n of bidegree (a, b) and genus g such that

1. $\underline{x} \subset C$;
2. C has order of contact i with B at q_j^i, and has β_i other (non-prescribed) points with order of contact i with B;
3. C has order of contact i with E at $\widetilde{q}_j^i$, and has $\widetilde{\beta}_i$ other (non-prescribed) points with order of contact i with E.

This number is finite and does not depend on the chosen generic configuration $\underline{x}$. We call this number a *double Gromov-Witten invariant of* Σ_n in reference to double Hurwitz numbers.

Let us encode these invariants in some function. Let us fix a, n, and g as above, and let us also fix two additional non-negative integer numbers n_1 and n_2. Thanks to these data we define

$$\Lambda = \{(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \mid \sum x_i + \sum y_i = an\} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.$$

Given an element $(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2})$ of Λ, we denote by α_i the number of elements x_j equal to i, by β_i the number of elements y_j equal to i, by $\widetilde{\alpha}_i$ the number of elements x_j equal to $-i$, and by $\widetilde{\beta}_i$ the number of elements y_j equal to $-i$. Then we get four sequences $\alpha = (\alpha_i)_{i\geq 1}, \beta = (\beta_i)_{i\geq 1}, \widetilde{\alpha} = (\widetilde{\alpha}_i)_{i\geq 1}$, and $\widetilde{\beta} = (\widetilde{\beta}_i)_{i\geq 1}$ in $\mathbb{Z}_{\geq 0}^{\infty}$. By setting $b = \sum(\widetilde{\alpha}_i + \widetilde{\beta}_i)$, we thus get a function

$$F_{a,n,g,n_1,n_2} : \begin{array}{ccc} \Lambda & \longrightarrow & \mathbb{Z} \\ (x,y) & \longmapsto & GW_{\Sigma_n}^{\alpha,\beta,\widetilde{\alpha},\widetilde{\beta}}(a,b,g) \end{array} .$$

Next theorem is the main result of [AB].

Theorem 3.63 ([AB]) *For any $a, n, g, n_1,$ and n_2, the function F_{a,n,g,n_1,n_2} is piecewise polynomial.*

It would be interesting to generalize Theorem 3.63 to refined Severi degrees.

Chapter 4

Three surface degenerations

In this chapter we provide three examples of applications of surface degenerations to enumerative geometry.

By degenerating a symplectic 4-manifold to a nodal variety, we prove in Section 4.1 the vanishing of a large part of Welschinger invariants of symplectic 4-manifolds (Theorem 4.1). We also give formulas computing Welschinger invariants of a symplectic 4-manifold out of the enumeration of real curves in any of its nodal degeneration (Theorem 4.6). This is a joint work with Puignau which appeared in [BP14].

By degenerating a Del Pezzo surface to $\mathbb{C}P^2$ blown up at points lying on a smooth conic, we reduce in Section 4.2 the computation of absolute invariants of Del Pezzo surfaces to the combinatorial enumeration of floor diagrams relative to a conic introduced in Section 3.2.2 (Theorems 4.17, 4.20, 4.25, and 4.27). Since the complex structure on Del Pezzo surfaces is generic regarding the problems addressed in this memoir, we chose in this section to work in the algebraic category, and to use Li's degeneration formula. Nevertheless the whole section should be easily translated in the symplectic setting using symplectic sum formulas. This work appeared in [Bru14].

By degenerating the Hirzebruch surface Σ_n to Σ_{n+2}, we relate in Section 4.3 complex enumerative invariants of these two surfaces (Theorem 4.36). This generalizes Abramovich-Bertram-Vakil's formula in the case $n = 0$. Our approach is tropical: a tropical counterpart of Kodaira's deformation of Hirzebruch surfaces combined with a suitable correspondence theorem (Theorem 4.40). This is a joint work with Markwig which appeared in [BM13].

4.1 Contraction of real Lagrangian spheres in symplectic 4-manifolds

In [BP14], we established the vanishing of a large part of Welschinger invariants of symplectic 4-manifolds, as well as the sign of $W_{X_{\mathbb{R}},L}(d,s)$ when $\mathbb{R}X$ is disconnected. We also specialized our results to the case of real algebraic rational surfaces, where all necessary homology groups can easily be computed.

Main statements are given in Section 4.1.1. These latter are deduced from more specific statements given in Section 4.1.2. Finally we give some hint about their proofs in Section 4.1.2.

4.1.1 Main statements

Let $X_{\mathbb{R}} = (X, \omega, \tau)$ be a real symplectic 4-manifolds. A *real Lagrangian sphere* of $X_{\mathbb{R}}$ is a Lagrangian sphere globally invariant under τ. Two disjoint surfaces S and S' in X are said to be *connected by a chain* *of real Lagrangian spheres* if there exists real Lagrangian spheres $S_1, \ldots, S_k$ in X such that $S_i \cap S_j = \emptyset$ is $|i-j| \geq 2$, and S_i and S_{i+1} intersect transversely in a single point, as well as S and S_0, and S' and S_k.

Theorem 4.1 ([BP14, Theorem 1]) *Let $X_{\mathbb{R}}$ be a real symplectic 4-manifold, and suppose that $F \in H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$ has a representative which is connected to L by a chain of real Lagrangian spheres.*

1. *If $r \geq 2$, then*
$$W_{X_{\mathbb{R}},L,F}(d,s) = 0.$$

2. *If $r = 1$ and $c_1(X) \cdot d \geq 2$, then*
$$2^{\frac{c_1(X)\cdot d-4}{2}} \mid W_{X_{\mathbb{R}},L,F}(d,s).$$
If in addition $F = [\mathbb{R}X \setminus L]$, then
$$(-1)^{\frac{d^2-c_1(X)\cdot d+2}{2}} W_{X_{\mathbb{R}},L}(d,s) \geq 0.$$

The invariant $W_{X_{\mathbb{R}},L,0}(d,s)$ does not seem to satisfy a vanishing statement analogous to Theorem 4.1(1) (see Examples 4.23 and 4.32, or [IKS13b, IKS13a, Bru14]), implying that the set $\mathcal{C}(d, \underline{x}, J)$ is usually non-empty. Theorem 4.1(2) partially generalizes [Wel07, Theorems 1.1, 2.1, 2.2, and 2.3].

Theorem 4.1 can be specialized to real algebraic rational surfaces, whose classification is well known (see [Sil89, Kol97] for example). A real algebraic rational surface is always implicitly assumed to be equipped with some Kähler form.

Let $\mathcal{G}$ be the subgroup of the τ-invariant classes in $H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$ generated by the kernel of the natural map $H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z}) \to H_2(X; \mathbb{Z}/2\mathbb{Z})$ and by the classes realized by real symplectic curves. We proved in [BP14, Propositions 8 and 9] that $W_{X_{\mathbb{R}},L,F}$ and $W_{X_{\mathbb{R}},L,F'}$ are equal in absolute value if $F - F' \in \mathcal{G}$. Denote by $\mathcal{H}(X_{\mathbb{R}}, L)$ the group of τ-invariant classes in $H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$ quotiented by $\mathcal{G}$. All groups $\mathcal{H}(X_{\mathbb{R}}, L)$ are computed in the case of real algebraic rational surfaces in [BP14, Section 4]. In particular, we prove in [BP14, Proposition 4] that they only depend on a minimal model of $X_{\mathbb{R}}$ and on the choice of L.

Theorem 4.2 ([BP14, Theorem 2]) *Let $X_{\mathbb{R}}$ be a real symplectic 4-manifold equal, up to deformation and equivariant symplectomorphism, to a real algebraic rational surface, and suppose that F is non-zero in $\mathcal{H}(X_{\mathbb{R}}, L)$. Then the conclusions of Theorem 4.1 hold in the following cases:*

- *$X_{\mathbb{R}}$ is obtained from a minimal model by blowing up pairs of complex conjugated points and real points on at most two connected components of $\mathbb{R}X$, one of them being L;*
- *$X_{\mathbb{R}}$ is a Del Pezzo surface;*
- *$F = [\mathbb{R}X \setminus L]$.*

Remark 4.3 *In a burst of enthusiasm, we forgot in [BP13a, Proposition 3.3] the assumption that $X_{\mathbb{R}}$ has to be symplectomorphic/deformation equivalent to a real algebraic rational surface.*

Theorem 4.2 follows from the classification of real algebraic rational surfaces and Theorem 4.1, which in its turn is a direct consequence of Theorem 4.4 and Corollary 4.7 below. As mentioned in the introduction, our strategy is to degenerate $X_{\mathbb{R}}$ into a reducible real symplectic manifold $Y_{\mathbb{R}}$, and to relate enumeration of curves in $Y_{\mathbb{R}}$ and in $X_{\mathbb{R}}$. This degeneration can be thought of as a degeneration of $X_{\mathbb{R}}$ to a real nodal symplectic manifold, and can be described into two equivalent ways:

- the contraction of a real Lagrangian sphere S_V by stretching the neck of a τ-compatible almost complex structure in a neighborhood of S_V, see [EGH00, Wel07];

- the symplectic sum of $X_{\mathbb{R}}$ and the normal bundle of real Lagrangian sphere, see [IP04].

In particular, Corollary 4.7 follows from Theorem 4.6, which can be seen as a real version of Abramovich-Bertram-Vakil formula [AB01, Theorem 3.1.1], [Vak00a, Theorem 4.5]. Another but related treatment of contraction of Lagrangian spheres contained in $\mathbb{R}X$ has previously been proposed by Welschinger in [Wel07]. In this section, we adopt the symplectic sum description of the degeneration of $X_{\mathbb{R}}$ to $Y_{\mathbb{R}}$.

4.1.2 Auxiliary statements

In this section, we denote by $X_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$, by ω_{FS} the Fubini-Study form on $\mathbb{C}P^n$, and by l_1 and l_2 respectively the homology classes $[\mathbb{C}P^1 \times \{0\}]$ and $[\{0\} \times \mathbb{C}P^1]$ in $H_2(X_0;\mathbb{Z})$. Recall that $H_2(X_0;\mathbb{Z})$ is the free abelian group generated by l_1 and l_2. Up to conjugation by an automorphism, there exist four different real structures on $(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} \times \omega_{FS})$, and the class $l_1 + l_2$ is invariant for exactly three of them, see for example [Sil89, Kol97]. These latter are given in coordinate by:

- $\tau_{hy}(z, w) = (\overline{z}, \overline{w})$, $\mathbb{R}X_{hy} = S^1 \times S^1$;
- $\tau_{el}(z, w) = (\overline{w}, \overline{z})$, $\mathbb{R}X_{el} = S^2$;
- $\tau_{em}(z, w) = (-\frac{1}{\overline{z}}, -\frac{1}{\overline{w}})$, $\mathbb{R}X_{em} = \emptyset$.

Note that τ_{hy} and τ_{em} act trivially on $H_2(X_0;\mathbb{Z}/2\mathbb{Z})$, while τ_{el} exchange the classes l_1 and l_2. Note also, with the convention that $\chi(\emptyset) = 0$, that

$$\chi(\mathbb{R}X_{hy}) = \chi(\mathbb{R}X_{em}) = 0, \quad \text{and} \quad \chi(\mathbb{R}X_{el}) = 2.$$

4.1.2.1 Vanishing Lagrangian spheres

A class V in $H_2(X;\mathbb{Z}/2\mathbb{Z})$ is called a *real vanishing cycle* if it can be represented by a real Lagrangian sphere S_V. By stretching the neck of a τ-compatible almost complex structure in a neighborhood of S_V, one decomposes X into the union of $X \setminus S_V$ and T^*S_V. This operation can be thought of as a degeneration of $X_{\mathbb{R}}$ to a real nodal symplectic manifold for which V is precisely the vanishing cycle. Equivalently, the class V is a real vanishing cycle if and only if, up to deformation, $X_{\mathbb{R}}$ can be represented as the real symplectic sum of two real symplectic manifolds (X_1, ω_1, τ_1) and $(X_0, \omega_{FS} \times \omega_{FS}, \tau_0)$ along an embedded symplectic sphere E of self-intersection -2 in X_1 (hence of self-intersection 2 in X_0) where:

- E is real and realizes the class $l_1 + l_2$ in $H_2(X_0;\mathbb{Z})$;
- V is represented by the deformation in X of a representative of the non-trivial class in $H_2(X_0 \setminus E;\mathbb{Z}/2\mathbb{Z})$.

By abuse, we still denote by V the non-trivial class in $H_2(X_0 \setminus E;\mathbb{Z}/2\mathbb{Z})$. We denote by $X_\sharp$ the union of (X_1, ω_1, τ_1) and $(X_0, \omega_{FS} \times \omega_{FS}, \tau_0)$ along E, by $L_\sharp$ the degeneration of L as $X_{\mathbb{R}}$ degenerates to $X_\sharp$, and by $L_i = L_\sharp \cap X_i$. The summand $(X_0, \omega_{FS} \times \omega_{FS}, \tau_0)$ of $X_\sharp$ corresponds to the compactified normal bundle of E in X, i.e. to $\mathbb{P}(\mathcal{N}_{E/X} \oplus \mathbb{C})$. In particular, the homology groups $H_2(X;\mathbb{Z})$ and $H_2(X_1;\mathbb{Z})$ are canonically identified, and we always implicitly use this identification.

Let $\mathcal{F}$ be a τ-invariant representative of a τ-invariant class $F \in H_2(X_\sharp \setminus L_\sharp;\mathbb{Z}/2\mathbb{Z})$, and define $\mathcal{F}_0 = \mathcal{F} \cap X_0$. Note that by construction we have $\partial\mathcal{F}_0 \subset E$. We always assume that $\mathcal{F}$ satisfies the following conditions:

- either $\mathcal{F} \cap \mathbb{R}E = \emptyset$, or there exists a neighborhood U of $\mathbb{R}E$ in $X_\sharp$ such that $\mathcal{F} \cap U \subset \mathbb{R}X_\sharp$;
- one of the two following assumptions holds:

(H_1) $\mathcal{F}_0 \cup L_0$ is a cycle representing the class γV in $H_2(X_0; \mathbb{Z}/2\mathbb{Z})$;

(H_2) $\tau_0 = \tau_{el}$ and $\mathcal{F}_0 \cup L_0 = \mathcal{D} \cup \Gamma$, where $\mathcal{D}$ is a τ-invariant disk with $\partial\mathcal{D} \subset E$, and Γ is a cycle representing the class γV in $H_2(X_0; \mathbb{Z}/2\mathbb{Z})$.

Next theorem is a key ingredient in the proof of Theorem 4.1.

Theorem 4.4 ([BP14, Theorem 3]) *Suppose that L_0 is a disk and that $\mathbb{R}X_0 \subset L_0 \cup \mathcal{F}_0$.*

1. *If $r \geq 2$, then*
$$W_{X_\mathbb{R},L,F}(d,s) = 0.$$

2. *If $r = 1$ and $c_1(X) \cdot d - 1 \geq 2$, then*
$$2^{\frac{c_1(X)\cdot d-4}{2}} \mid W_{X_\mathbb{R},L,F}(d,s) \quad \textit{and} \quad (-1)^{\frac{d^2-c_1(X)\cdot d+2}{2}} W_{X_\mathbb{R},L}(d,s) \geq 0.$$

Note that the assumptions of Theorem 4.4 imply that $\tau_0 = \tau_{el}$ and that $\mathcal{F}$ satisfies (H_2). In the Lagrangian sphere contraction presentation, the condition that L_0 is a disk translates to the condition that $L \cap S_V$ is reduced to a single point, at which the order of intersection is odd.

4.1.2.2 From X_1 to X

Here we reduce the computation of Welschinger invariants of $X_\mathbb{R}$ to enumeration of real J-holomorphic curves in $X_{1,\mathbb{R}}$ for a τ_1-compatible almost complex structure J for which E is J-holomorphic.

Definition 4.5 *Let J be a τ_1-compatible almost complex structure on (X_1, ω_1, τ_1) for which the curve E is J-holomorphic, and let C_1 be an immersed real rational J-holomorphic curve intersecting E transversely. We denote by a the number of points in $\mathbb{R}C_1 \cap \mathbb{R}E$, by b the number of pairs of τ_1-conjugated points in $C_1 \cap E$, and by $m_{L_1,F_1}(C_1)$ the number of intersection points of $\underline{C_1}$ with $L_1 \cup F_1$ (recall that $\underline{C_1}$ has been defined in Section 2.2). Finally, let $k \geq 0$ be an integer.*

1. *If $\mathcal{F}$ satisfies assumption (H_1), then we define*
$$\mu^0_{L_\sharp,\mathcal{F}_0,k}(C_1) = (-1)^{m_{L_1,F_1}(C_1)+\gamma(a+b)} \sum_{k=a_k+2b_k} \binom{a}{a_k}\binom{b}{b_k}$$
and
$$\mu^2_{L_\sharp,\mathcal{F}_0,k}(C_1) = \begin{cases} (-1)^{m_{L_1,F_1}(C_1)+\gamma b}\, 2^b & \textit{if } a = 0 \textit{ and } k = b; \\ 0 & \textit{otherwise.} \end{cases}$$

2. *If $\mathcal{F}$ satisfies assumption (H_2), then we define*
$$\mu_{L_\sharp,\mathcal{F}_0,k}(C_1) = \begin{cases} (-1)^{m_{L_1,F_1}(C_1)} & \textit{if } k = a = b = 0; \\ 0 & \textit{otherwise} \end{cases}$$

Let $d \in H_2(X; \mathbb{Z})$ and $r, s \in \mathbb{Z}_{\geq 0}$ such that
$$c_1(X) \cdot d - 1 = r + 2s.$$

Choose a configuration $\underline{x}$ made of r points in L_1 and s pairs of τ-conjugated points in $X_1 \setminus \mathbb{R}X_1$. Let J be a τ_1-compatible almost complex structure for which E is J-holomorphic.

For each integer $k \geq 0$, we denote by $\mathcal{C}_{1,k}(d, \underline{x}, J)$ the set of all irreducible rational real J-holomorphic curves in (X_1, ω_1, τ_1) passing through all points in $\underline{x}$, realizing the class $d - k[E]$, and such that L_1 contains the 1-dimensional part of $\mathbb{R}C_1$. For a generic choice of J satisfying the above conditions, the set $\mathcal{C}_{1,k}(d, \underline{x}, J)$ is finite, and any curve in $\mathcal{C}_{1,k}(d, \underline{x}, J)$ is nodal and intersects E transversely. Moreover $\mathcal{C}_{1,k}(d, \underline{x}, J)$ is non-empty only for finitely many values of k.

Theorem 4.6 ([BP14, Theorem 4]) *Suppose that $L_1 \neq \emptyset$. Then for a generic choice of J, the two following claims hold.*

1. *If $\mathcal{F}$ satisfies assumption (H_1), then, with the convention that $\chi(\emptyset) = 0$, one has*

$$W_{X_\mathbb{R},L,F}(d,s) = \sum_{k\geq 0} \sum_{C_1 \in \mathcal{C}_{1,k}(d,\underline{x},J)} \mu^{\chi(\mathbb{R}X_0)}_{L_\sharp,\mathcal{F}_0,k}(C_1).$$

2. *If $\mathcal{F}$ satisfies assumption (H_2), then one has*

$$W_{X_\mathbb{R},L,F}(d,s) = \sum_{C_1 \in \mathcal{C}_{1,0}(d,\underline{x},J)} \mu_{L_\sharp,\mathcal{F}_0,0}(C_1).$$

Applying Theorem 4.6(1) with $F = [\mathbb{R}X \setminus L]$, one obtains [BP13a, Theorem 2.2]. Some instances of Theorem 4.6(1) when $\mathbb{R}X_0 = S^1 \times S^1$ have been known for sometimes, e.g. [Bru, Bru11, Kha10, RS]. Since the publication of [BP13a], an algebro-geometric proof of Theorem 4.6(1) appeared in [Bru14] and in [IKS13a] in the particular cases when X is a Del Pezzo surface of degree two or more. Theorem 4.6(2) immediately implies the following corollary.

Corollary 4.7 *Suppose that $V \in H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$ and that $\mathcal{F}$ satisfies assumption (H_2). Then*

$$W_{X_\mathbb{R},L,F}(d,s) = W_{X_\mathbb{R},L,F+V}(d,s).$$

4.1.2.3 Applications of Theorem 4.6(1)

We do not explicitly use Theorem 4.6(1) in the proof of Theorem 4.1, nevertheless its proof is almost contained in the proof of Theorem 4.6(2). Theorem 4.6(1) has many interesting applications, in particular in explicit computations of Welschinger invariants, see [Bru14, IKS13a]. We present here two other consequences.

We first relate some tropical Welschinger invariants to genuine Welschinger invariants of the quadric ellipsoid. We refer to [IKS09] for the definition of tropical Welschinger invariants. The only homology classes of $(X_0, \omega_{FS} \times \omega_{FS}, \tau_{el})$ realized by real curves are of the form $dl_1 + dl_2$ with $d \in \mathbb{Z}_{>0}$. We denote by $W_{\mathbb{T}\Sigma_2}(d)$ the irreducible tropical Welschinger invariant of $\mathbb{T}\Sigma_2$ for rational curves with Newton polygon the triangle with vertices $(0,0), (0,d)$, and $(2d,0)$.

Proposition 4.8 ([BP14, Proposition 1]) *For any $d \in \mathbb{Z}_{>0}$, we have*

$$W_{X_{0,el}}(dl_1 + dl_2) = W_{\mathbb{T}\Sigma_2}(d).$$

It is proved in [IKS04] that given a toric Del Pezzo surface X equipped with its tautological real toric structure and a class $d \in H_2(X;\mathbb{Z})$, we have

$$W_{X_\mathbb{R}}(d,0) \geq W_{X_\mathbb{R}}(d,1).$$

The same idea used in the proof of Proposition 4.8 combined with Theorem 4.6 and Theorem 3.32 provide a natural generalization of this formula in the particular cases when X is a Del Pezzo surface of degree at least three.

Proposition 4.9 ([BP14, Proposition 2]) *Let (X,ω) be a symplectic 4-manifold symplectomorphic/deformation equivalent to a Del Pezzo surface of degree at least three.*

If $X_\mathbb{R} = (X,\omega,\tau_1)$ and $X'_\mathbb{R} = (X,\omega,\tau_2)$ are two real structures on (X,ω), then for any $d \in H_2(X;\mathbb{Z})$ one has

$$W_{X_\mathbb{R},L_1}(d,0) \geq W_{X'_\mathbb{R},L_2}(d,0) \geq 0 \quad \textit{if} \quad \chi(\mathbb{R}X) \leq \chi(\mathbb{R}X').$$

The non-negativity of all Welschinger invariants of Del Pezzo surfaces of degree 3 when $s = 0$ has been first established in [IKS13b]. Note that Proposition 4.9 does not generalize immediately to any symplectic 4-manifold. Indeed, according to [ABLdM11, Section 7.3] one has $W_{\mathbb{C}P^2,\mathbb{R}P^2}(9,12) < W_{\mathbb{C}P^2,\mathbb{R}P^2}(9,13)$, i.e. Proposition 4.9 does not hold in the case of $\mathbb{C}P^2$ blown up in 26 points.

4.1.3 Elements of the proof of Theorems 4.4 and 4.6

4.1.3.1 Symplectic sums

Here we briefly describe a very particular case of the symplectic sum formula from [IP04]. In this section we do not assume that $X_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$.

Let (X_0, ω_0) and (X_1, ω_1) be two symplectic connected compact 4-manifolds, and let $\phi_i : E \to X_i$, $i = 0, 1$, be two symplectic embeddings of a symplectic sphere such that the self-intersection of the $\phi_i(E)$'s in X_i are opposite to each other. By abuse, we still denote by E the image of $\phi_i(E)$ in X_i. The condition on the self-intersections is equivalent to the existence of a symplectic bundle isomorphism ψ between the normal bundle of E in X_0 and the dual of the normal bundle of E in X_1.

Out of these data, one produces a family of symplectic 4-manifolds (Y_t, ω_t) parametrized by a small complex number t in $\mathbb{C}^*$. All those 4-manifolds are deformation equivalent, and are called symplectic sums of (X_0, ω_0) and (X_1, ω_1) along E. Next theorem says that this family can be seen as a symplectic deformation of the singular symplectic manifold $X_\sharp = X_0 \cup_E X_1$ obtained by gluing (X_0, ω_0) and (X_1, ω_1) along E.

Proposition 4.10 ([IP04, Theorem 2.1]) *There exists a symplectic 6-manifold (Y, ω_Y) and a symplectic fibration $\pi : Y \to D$ over a disk $D \subset \mathbb{C}$ such that the central fiber $\pi^{-1}(0)$ is the singular symplectic manifold $X_\sharp$, and $\pi^{-1}(t) = (Y_t, \omega_t)$ for $t \neq 0$.*

Note that the map π provides an identification of all homology groups $H_i(\pi^{-1}(t); \mathbb{Z})$ with $t \in D$.

Let $d \in H_2(Y_t; \mathbb{Z})$, and choose $\underline{x}(t)$ a set of $c_1(X) \cdot d - 1$ symplectic sections $D \to Y$ such that $\underline{x}(0) \cap E = \emptyset$. Choose an almost complex structure J on Y tamed by ω_Y, which restricts to an almost complex structure J_t on Y_t tamed by ω_t on each fiber $\pi^{-1}(t)$, and generic with respect to all choices we made.

Define $\mathcal{C}(d, \underline{x}(0), J_0)$ to be the set $\left\{\overline{f} : \overline{C} \to X_\sharp\right\}$ of limits, as stable maps, of maps in $\mathcal{C}(d, \underline{x}(t), J_t)$ as t goes to 0, and $\mathcal{C}_*(d, \underline{x}(0), J_0) = \{\overline{f}(\overline{C}) \mid \overline{f} \in \mathcal{C}(d, \underline{x}(0), J_0)\}$. If $\overline{C}_1, \ldots, \overline{C}_k$ denote the irreducible components of $\overline{C}$ and if none of them is entirely mapped to E, we define

$$\mu(\overline{f}) = \prod_{p \in \overline{f}^{-1}(E)} \mu_p,$$

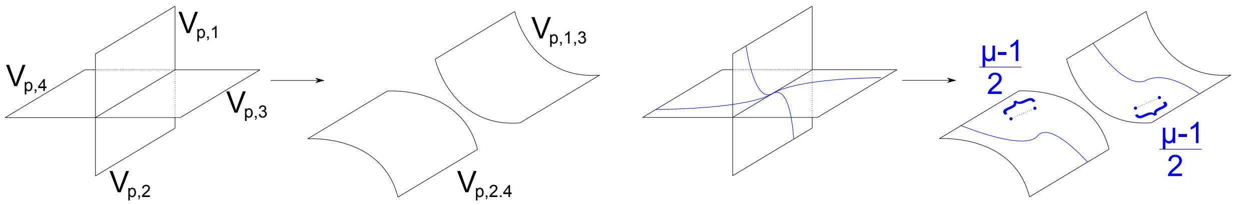
where μ_p is the order of contact of $\overline{C}$ and E at p.

Proposition 4.11 ([BP14, Proposition 6]) *For a generic J_0, the set $\mathcal{C}_*(d, \underline{x}(0), J_0)$ is finite, and only depends on $\underline{x}(0)$ and J_0. Moreover if $\overline{f} : \overline{C} \to X_\sharp$ is an element of $\mathcal{C}(d, \underline{x}(0), J_0)$, then no irreducible component of $\overline{C}$ is entirely mapped to E. If in addition $\overline{f}$ restricted to any irreducible component of $\overline{C}$ is simple, then $\overline{f}$ is the limit of exactly $\mu(\overline{f})$ elements of $\mathcal{C}(d, \underline{x}(t), J_t)$ as t goes to 0.*

Suppose now that the disc D from Proposition 4.10 is equipped with the standard complex conjugation. Let us assume now that (Y, ω_Y) is endowed with a real structure τ_Y such that the map $\pi : Y \to D$ is real, and let us choose the set of sections $\underline{x} : D \to Y$ to be real. Note that each fiber $\pi^{-1}(t)$ comes naturally equipped with a real structure τ_t.

Let us fix a real element $\overline{f}:\overline{C}\to X_\sharp$ of $\mathcal{C}(d,\underline{x}(0),J_0)$. Given a pair $\{q,\tau_0(q)\}$ of elements in $\overline{f}^{\,-1}(E)$, we define $\mu_{\{q,\tau(q)\}}=\mu_q$ (note that $\mu_q=\mu_{\tau(q)}$ so $\mu_{q,\tau(q)}$ is well defined). We denote by ξ_0 the product of the $\mu_{\{q,\tau(q)\}}$ where $\{q,\tau(q)\}$ ranges over all pairs of conjugated elements in $\overline{f}^{\,-1}(E)$.

If $\mathbb{R}E\neq\emptyset$ and given $p\in\mathbb{R}E$, choose a neighborhood U_p of p in $\mathbb{R}X_\sharp$ homeomorphic to the union of two disks. The set $U_p\setminus\mathbb{R}E$ has four connected components $U_{p,1},U_{p,2}\subset\mathbb{R}X_0$ and $U_{p,3},U_{p,4}\subset\mathbb{R}X_1$, labeled so that when smoothing $\mathbb{R}X_\sharp$ to $\mathbb{R}Y_t$ with $t\in\mathbb{R}^*$, the components $U_{p,1}$ and $U_{p,3}$ on one hand hand, and $U_{p,2}$ and $U_{p,4}$ on the other hand, glue together, see Figure 4.1a. Denote respectively by $U_{p,1,3}$ and $U_{p,2,4}$ a deformation of $U_{p,1}\cup U_{p,3}$ and $U_{p,2}\cup U_{p,4}$ in $\mathbb{R}Y_t$. Given $q\in\mathbb{R}\left(\overline{f}^{\,-1}(E)\right)$, denote by $\overline{U}_q$ a



a) b)

Figure 4.1: Real deformations of a real map $\overline{f}:\overline{C}\to X_\sharp$

small neighborhood of q in $\mathbb{R}\overline{C}$. If μ_q is even, define the integer ξ_q as follows:

- if $\overline{f}(\overline{U}_q)\subset U_{\overline{f}(q),1}\cup U_{\overline{f}(q),4}$ or $\overline{f}(\overline{U}_q)\subset U_{\overline{f}(q),2}\cup U_{\overline{f}(q),3}$, then $\xi_q=0$;
- if $\overline{f}(\overline{U}_q)\subset U_{\overline{f}(q),1}\cup U_{\overline{f}(q),3}$ or $\overline{f}(\overline{U}_q)\subset U_{\overline{f}(q),2}\cup U_{\overline{f}(q),4}$, then $\xi_q=2$.

We define $\xi(\overline{f})$ as the product of ξ_0 with all the ξ_q's where q ranges over all points in $\mathbb{R}\left(\overline{f}^{\,-1}(E)\right)$ with μ_q even.

Proposition 4.12 ([BP14, Proposition 7]) *Suppose that the restriction of $\overline{f}$ to any component of $\overline{C}$ is simple. Then the real map $\overline{f}$ is the limit of exactly $\xi(\overline{f})$ real maps in $\mathcal{C}(d,\underline{x}(t),J_t)$. Moreover for each $q\in\mathbb{R}\left(\overline{f}^{\,-1}(E)\right)$, one has*

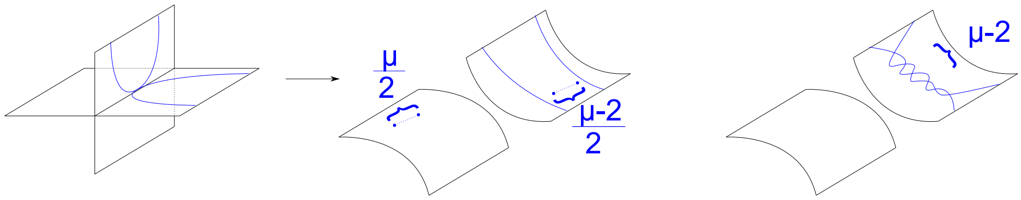
- *if μ_q is odd, then any real deformation of $\overline{f}$ has exactly μ_q-1 solitary nodes in $U_{\overline{f}(q),1,3}\cup U_{\overline{f}(q),2,4}$ (see Figure 4.1b);*
- *if μ_q is even, then half of the real deformations of $\overline{f}$ have exactly $\frac{\mu_q-2}{2}$ solitary nodes in $U_{\overline{f}(q),1,3}$ and $\frac{\mu_q}{2}$ solitary nodes in $U_{\overline{f}(q),2,4}$, while the other half of real deformations of $\overline{f}$ have no solitary nodes in $U_{\overline{f}(q),1,3}\cup U_{\overline{f}(q),2,4}$ (see Figure 4.2).*

Note that the second part of Proposition 4.12 is empty if $\mathbb{R}E=\emptyset$ or $\xi(\overline{f})=0$.

4.1.3.2 Application

Here we apply results from Section 4.1.3.1 to the case exposed in Section 4.1.2.1, i.e. when $(X_0,\omega_0,\tau_0)=(\mathbb{C}P^1\times\mathbb{C}P^1,\omega_{FS}\times\omega_{FS},\tau)$ and $[E]=l_1+l_2$. As explained above, the summand (X_0,ω_0,τ_0) of $X_\sharp$ corresponds to $\mathbb{P}(\mathcal{N}_{E/X}\oplus\mathbb{C})$. Without loss of generality, we may assume that l_2 is the class realized by the compactification of a fiber of $\mathcal{N}_{E/X}$. In particular if $\overline{f}:\overline{C}\to X_\sharp$ is an element of $\mathcal{C}(d,\underline{x}(0),J_0)$, and if C_i is the union of the irreducible components of $\overline{C}$ mapped to X_i, then there exists $k\in\mathbb{Z}_{\geq 0}$ such that

$$\overline{f}_*[C_1]=d-k[E]\quad\text{and}\quad\overline{f}_*[C_0]=kl_1+(d\cdot[E]+k)l_2.\tag{4.1}$$



a)

Figure 4.2: Real deformations of a real map $\overline{f} : \overline{C} \to X_\sharp$, continued

If $\mathcal{F} \cap \mathbb{R}E = \emptyset$, then by perturbing $\mathcal{F}$ if necessary, we may assume that $\overline{f}(\overline{C}) \cap \mathcal{F} \cap E = \emptyset$ for all $\overline{f} \in \mathcal{C}(d, \underline{x}(0), J_0)$

We prove Theorem 4.4 by choosing the sections $\underline{x}(t)$ in such a way that $\underline{x}(0) \cap X_0$ is reduced a single point p_0, and that $\underline{x}(0) \cap X_1 \neq \emptyset$. In this case, Theorem 4.4 follows easily from Theorem 4.12 and next lemma.

Lemma 4.13 ([BP14, Lemma 3]) *Suppose that $|\underline{x}(0) \cap X_0| = 1$ and $\underline{x}(0) \cap X_1 \neq \emptyset$. Let $\overline{f} : \overline{C} \to X_\sharp$ be an element of $\mathcal{C}(d, \underline{x}(0), J_0)$. The image of the irreducible component $\overline{C}'$ of C_0 whose image contains p_0 realizes either a class l_i or the class $l_1 + l_2$.*

- *If this class is l_i, then the curve C_1 is irreducible and $\overline{f}_{|C_1}$ is an element of $\mathcal{C}^{u_1,(d+2k-1)u_1}(d - k[E], \underline{x}(0) \cup \underline{x}_E, J_0)$, where $\underline{x}_E = \overline{f}(\overline{C}') \cap E$.*
- *If this class is $l_1 + l_2$, then $\overline{f}_{|\overline{C}'}$ is an element of $\mathcal{C}^{\alpha,0}(l_1 + l_2, \{p_0\} \cup \underline{x}_E, J_0)$, where $\underline{x}_E = \overline{f}(C_1) \cap E$, and $\alpha = 2u_1$ or $\alpha = u_2$. The curve C_1 has two irreducible components in the former case, and is irreducible in the latter case.*

All the other irreducible components of C_0 realize a class l_i, and $\overline{f}$ restricts to a simple map on each irreducible component of $\overline{C}$.

We prove Theorem 4.6 by choosing the sections $\underline{x}(t)$ in such a way that $\underline{x}(0) \cap X_0 = \emptyset$. In this case, Theorem 4.6 follows easily from Theorem 4.12 and next lemma. We define $\mathcal{C}_{\overline{f}}$ to be the set of elements $\overline{f}' : \overline{C}' \to X_\sharp$ in $\mathcal{C}(d, \underline{x}(0), J_0)$ such that $\overline{f}_{|\overline{C}_1} = \overline{f}'_{|\overline{C}'_1}$.

Lemma 4.14 ([BP14, Lemmas 4 and 5]) *Suppose that $\underline{x}(0) \cap X_0 = \emptyset$. Let $\overline{f} : \overline{C} \to X_\sharp$ be an element of $\mathcal{C}(d, \underline{x}(0), J_0)$. Then the curve C_1 is irreducible, and the image of any irreducible components of C_0 realizes a class l_i. In particular, $\overline{f}$ is the limit of a unique element of $\mathcal{C}(d, \underline{x}(t), J_t)$ as t goes to 0.*
Moreover, if $\overline{f}_*[\overline{C}_1] = d - k[E]$, then $\mathcal{C}_{\overline{f}}$ has exactly $\binom{d \cdot [E] + 2k}{k}$ elements.

4.1.4 Further comments

4.1.4.1 Symplectic sum and real WDVV equations

A real version of the WDVV equations for rational 4-symplectic manifolds have been proposed by Solomon [Sol]. Those equations provide many relations among Welschinger invariants of a given real 4-symplectic manifold, that hopefully reduce the computation of all invariants to the computation of finitely many simple cases. This program has been completed in [HS12] in the case of rational surfaces equipped with a standard real structure, i.e. induced by the standard real structure on $\mathbb{C}P^2$ via the blowing up map.

In a work in progress in collaboration with Solomon, we combine the methods used in [BP14] and real WDVV equations to cover the case of all remaining real rational algebraic surfaces. At the present moment, this project has been completed for all minimal real rational algebraic surfaces, except for the minimal Del Pezzo surface of degree 1. It is worth mentioning that for the minimal Del Pezzo surface of degree 2, Solomon's equations alone are not sufficient to reduce to finitely many simple cases. However the combination of Solomon's equations together with the vanishing ensured by Theorem 4.2 allows this reduction.

As a side remark, we stress that if real WDVV equations are definitely better from a computational point of view than floor diagrams, it seems nevertheless very difficult to extract from them qualitative information about Welschinger invariants. A real WDVV equation in the case of odd dimensional projective spaces has also been proposed by Georgieva and Zinger [GZ13].

4.1.4.2 Sign of Welschinger invariants

The sign of Welschinger invariants seem to obey to some mysterious rule related to the topology of the real part of the ambient manifold. Theorem 4.2 with [Wel07, IKS09, IKS13b, Bru14] explicit this rule in a few cases for $F = [\mathbb{R}X \setminus L]$, namely when $L = T^2$, or S^2 and $r = 0, 1$, when $X = X_8$ and s is very small, or when X is a real rational algebraic surface with disconnected real part and $r = 1$. Unfortunately, the rule controlling the sign of Welschinger invariants in its full generality still remains mysterious.

As an example, let us describe how the signs of Welschinger invariants of $\mathbb{C}P^2$ seem to behave: as r goes from $3d-1$ to 0 or 1, the numbers $W_{\mathbb{C}P^2}(d, s)$ are first positive, and starting from some mysterious threshold, have an alternating sign. This observation has been made experimentally using Solomon's real version of WDVV equations for $\mathbb{C}P^2$.

4.2 Gromov-Witten and Welschinger invariants of Del Pezzo surfaces

In this section, we denote by X_n the complex projective plane blown up at a generic configuration of $n \leq 8$ points. Recall that $\widetilde{X}_n$ denotes the surface $\mathbb{C}P^2$ blown up at n points located on a smooth conic. We further denote by $\widetilde{X}_{n,1}$ the surface $\mathbb{C}P^2$ blown up at n points located on a smooth conic, and at an extra point not on this conic. In the last two cases, we denote by E the strict transform of the conic in $\widetilde{X}_n$ and $\widetilde{X}_{n,1}$.

We reduce in Theorems 4.17 and 4.20 (resp. Theorems 4.25 and 4.27) the computation of absolute invariants of X_7 (resp. X_8) to enumeration of curves in $\widetilde{X}_8$ (resp. $\widetilde{X}_{8,1}$), which in their turn have been reduced to enumeration of floor diagrams in Theorems 3.27 and 3.32 (resp. to a Caporaso-Harris type formula in [SS13, IKS13a]) We detail here only the case of X_7. The case of X_8 follows the same strategy, and we will just mention briefly how to adapt the method used in the case of X_7.

We describe in Section 4.2.1 the degeneration of X_7 we consider, and then sate the main theorems in Sections 4.2.2 and 4.2.3. We finally give some hint of the proofs in Section 4.2.4, and briefly address the case of X_8 in Section 4.2.5. Note that motivated by Theorem 4.2, we only consider the cases of Welschinger invariants with $F = 0$ or $F = [\mathbb{R}X \setminus L]$.

We denote by $X_7(\kappa)$ with $\kappa = 0, \ldots, 3$, and by $X_7^{\pm}(4)$ the surface X_7 equipped with the real structure such that:

$$\mathbb{R}X_7(\kappa) = \mathbb{R}P^2_{7-2\kappa}, \quad \mathbb{R}X_7^-(4) = \mathbb{R}P^2 \sqcup \mathbb{R}P^2, \quad \mathbb{R}X_7^+(4) = S^2 \sqcup \mathbb{R}P^2_1.$$

Recall that these are all real structures on X_7 with a non-orientable real part, and represent half of the possible real structures on X_7. Note that

$$\chi(\mathbb{R}X_7^{\pm}(\kappa)) = -6 + 2\kappa.$$

Recall that $\widetilde{X}_n(\kappa)$ denotes the surface $\widetilde{X}_n$ equipped with the real structure induced by the real structure on $\mathbb{C}P^2$ via the blow up map of $n-2\kappa$ real points and κ pairs of complex conjugated points on a real conic with non-empty real part. Recall also that when $n=2\kappa$, the connected component of $\mathbb{R}\widetilde{X}_n(\kappa)\setminus\mathbb{R}E$ with Euler characteristic $\epsilon\in\{0,1\}$ is denoted by $\widetilde{L}_\epsilon$.

4.2.1 Degeneration of X_7

Theorems 4.17 and 4.20 are obtained by applying Li's degeneration formula, and its real counterpart, to the degeneration of X_7 described in next proposition, together with a set of sections $\underline{x}:\mathbb{C}\to\widetilde{\mathcal{Y}}$ satisfying $\underline{x}(0)\subset\widetilde{X}_6\setminus\widetilde{X}_2$. The point of considering this degeneration is that no non-trivial covering appears.

Proposition 4.15 *There exists a flat degeneration $\pi:\widetilde{\mathcal{Y}}\to\mathbb{C}$ of X_7 with $\pi^{-1}(0)=\widetilde{X}_6\cup\widetilde{X}_2$, where $\widetilde{X}_6\cap\widetilde{X}_2$ is the distinguished curve E in both $\widetilde{X}_6$ and $\widetilde{X}_2$.*

We denote by p_7 and p_8 the points on E corresponding to the two blown up points in $\widetilde{X}_2$. Then one can choose $\pi:\widetilde{\mathcal{Y}}\to\mathbb{C}$ endowed with a real structure such that one of the following holds:

- *$\mathbb{R}\pi^{-1}(0)=\widetilde{X}_6(\kappa)\cup\widetilde{X}_2(0)$ with $0\le\kappa\le3$; in this case the two points p_7 and p_8 are real, and $\mathbb{R}\pi^{-1}(t)=X_7(\kappa)$ for $t\ne0$.*
- *$\mathbb{R}\pi^{-1}(0)=\widetilde{X}_6(\kappa)\cup\widetilde{X}_2(1)$ with $0\le\kappa\le2$; in this case the two points p_7 and p_8 are complex conjugated, and $\mathbb{R}\pi^{-1}(t)=X_7(\kappa+1)$ for $t\ne0$.*
- *$\mathbb{R}\pi^{-1}(0)=\widetilde{X}_6(3)\cup\widetilde{X}_2(1)$ and the component $\widetilde{L}_\epsilon$ of $\widetilde{X}_2(1)$ is glued to the component $\widetilde{L}_0$ of $\widetilde{X}_6(3)$ in the smoothing of $\mathbb{R}\pi^{-1}(0)$; in this case the two points p_7 and p_8 are complex conjugated, and for $t\ne0$ one has $\mathbb{R}\pi^{-1}(t)=X_7^-(4)$ if $\epsilon=1$, and $\mathbb{R}\pi^{-1}(t)=X_7^+(4)$ if $\epsilon=0$.*

4.2.2 Gromov-Witten invariants of X_7

Some additional notation are needed to state Theorem 4.17. Given a graph Γ, denote by $\lambda_{v,v'}$ the number of edges between the distinct vertices v and v' of Γ, by $\lambda_{v,v}$ twice the number of loops of Γ based at the vertex v, and by k_Γ° the number of edges of Γ.

In this section, we consider curves in X_7 and $\widetilde{X}_8$. In order to avoid confusions, let us use the following notation: D denotes the pullback of a generic line in both surfaces, and $E_1,\dots E_7$ (resp. $\widetilde{E}_1,\dots\widetilde{E}_8$) denote the (-1)-curves coming from the presentation of X_7 (resp. $\widetilde{X}_8$) as a blow up of $\mathbb{C}P^2$ (resp. of $\mathbb{C}P^2$ at height points on a conic). Finally, let $V_8\subset H_2(\widetilde{X}_8;\mathbb{Z})\setminus\{0\}$ be the set of classes $d\ne l\widetilde{E}_i$ with $l\ge2$ or $i=7,8$.

Definition 4.16 *A X_7-*graph *is a connected graph Γ together with three quantities $d_v\in V_8$, $g_v\in\mathbb{Z}_{\ge0}$, and $\beta_v=\beta_{v,1}u_1+\beta_{v,2}u_2\in\mathbb{Z}_{\ge0}^\infty$ associated to each vertex v of Γ, such that $I\beta_v=d_v\cdot[E]$.*

An isomorphism between X_7-graphs is an isomorphism of graphs preserving the three quantities associated to each vertex.

An X_7-graphs is always considered up to isomorphism. Given a X_7-graph Γ, define

$$d_\Gamma=\sum_{v\in\mathrm{Vert}(\Gamma)}d_v,\quad\text{and}\quad\beta_\Gamma=\sum_{v\in\mathrm{Vert}(\Gamma)}\beta_v.$$

Given $g,k\in\mathbb{Z}_{\ge0}$ and $d\in H_2(X_7;\mathbb{Z})$ such that $d\cdot[D]\ge1$ (if not the corresponding Gromov-Witten invariants are straightforward to compute), let $\mathcal{S}_7(d,g,k)$ be the set of all pairs (Γ,P_Γ) where

- Γ is a X_7-graph such that
$$\sum_{v\in \mathrm{Vert}(\Gamma)} g_v + b_1(\Gamma) = g$$
and
$$d = (d_\Gamma \cdot [D] + 2k)[D] - \sum_{i=1}^{6} \left(d_\Gamma \cdot [\widetilde{E}_i] + k\right)[E_i] - \left(k_\Gamma^\circ + \beta_{\Gamma,2} + d_\Gamma \cdot ([\widetilde{E}_7] + [\widetilde{E}_8])\right)[E_7];$$

- $P_\Gamma = \bigcup_{v\in \mathrm{Vert}(\Gamma)} U_v$ is a partition of the set $\{1, \ldots, c_1(X_7)\cdot d - 1 + g\}$ such that $|U_v| = d_v \cdot [D] - 1 + g_v + |\beta_v|$.

Given $(\Gamma, P_\Gamma) \in \mathcal{S}_7(d, g, k)$, define
$$k^{\circ\circ} = k - \beta_{\Gamma,2} - k_\Gamma^\circ - d_\Gamma \cdot [\widetilde{E}_7].$$

Denote by $\sigma(\Gamma)$ the number of bijection of $\mathrm{Vert}(\Gamma)$ to itself which are induced by an automorphism of the graph Γ. Define the following complex multiplicities for $(\Gamma, P_\Gamma) \in \mathcal{S}_7(d, g, k)$ and $v \in \mathrm{Vert}(\Gamma)$ (recall that the invariants $GW_{\widetilde{X}_8}$ have been defined in Section 3.2.2.1):
$$\mu^{\mathbb{C}}(v) = \lambda_{v,v}!!\binom{\beta_{v,1}}{\{\lambda_{v,v'}\}_{v'\in \mathrm{Vert}(\Gamma)}} GW_{\widetilde{X}_8}^{0,\beta_v}(d_v, g_v),$$
and
$$\mu^{\mathbb{C}}(\Gamma, P_\Gamma) = \frac{I^{\beta_\Gamma}}{\sigma(\Gamma)}\binom{\beta_{\Gamma,1} - 2k_\Gamma^\circ}{k^{\circ\circ}} \prod_{v\neq v'\in \mathrm{Vert}(\Gamma)} \lambda_{v,v'}! \prod_{v\in \mathrm{Vert}(\Gamma)} \mu^{\mathbb{C}}(v).$$

Note that given d and g, there exists only finitely elements in $\bigcup_{k\geq 0} \mathcal{S}_7(d, g, k)$ with a positive multiplicity. Also given $(\Gamma, P_\Gamma) \in \mathcal{S}_7(d, 0, k)$, we have $\lambda_{v,v'} \leq 1$ (resp. $\lambda_{v,v} = 0$) for each pair of distinct vertices (resp. each vertex) of Γ.

Next theorem reduces the computation of GW_{X_7} to the computation of $GW_{\widetilde{X}_8}$.

Theorem 4.17 ([Bru14, Theorem 6.5]) *Let $g \geq 0$ and $d \in H_2(X_7; \mathbb{Z})$ such that $d \cdot [D] \geq 1$. Then one has*
$$GW_{X_7}(d, g) = \sum_{k\geq 0} \sum_{(\Gamma, P_\Gamma)\in \mathcal{S}_7(d,g,k)} \mu^{\mathbb{C}}(\Gamma, P_\Gamma).$$

Remark 4.18 *We consider the degeneration $\widetilde{\mathcal{Y}}$ having in mind the enumeration real curves, see Section 4.2.3. If one is only interested in the computation of Gromov-Witten invariants of X_7, then it is probably simpler to consider the degeneration of X_7 to $\widetilde{X}_7 \cup \widetilde{X}_1$, the resulting formula being the same. In this perspective, Theorem 4.17 is then analogous to [BM13, Theorem 2.9, Example 2.11], i.e. Theorem 4.36 specialized to the case $n = 1$.*

Example 4.19 ([Bru14, Example 6.7]) *Thanks to Theorems 4.17 and 3.27, we list in table 4.1 the Gromov-Witten invariants $GW_{X_7}(2c_1(X_7), g)$. The value in the rational case has been first computed by Göttsche and Pandharipande in [GP98, Section 5.2]. The cases of higher genus have been first treated in [SS13]. The value $GW_{X_7}(2c_1(X_7), 1) = 204$ corrects the incorrect value announced in [SS13, Example 3.2].*

g	0	1	2	3
$GW_{X_7}(2c_1(X_7), g)$	576	204	26	1

Table 4.1: $GW_{X_7}(2c_1(X_7), g)$

4.2.3 Welschinger invariants of X_7

For $\kappa = 0, \ldots, 3$, define the two involutions τ^0_κ and τ^1_κ on $H_2(\widetilde{X}_8; \mathbb{Z})$ as follows: τ^0_κ (resp. τ^1_κ) fixes the elements $[D]$ and $[\widetilde{E}_i]$ with $i \in \{2\kappa+1, \ldots, 8\}$ (resp. $i \in \{2\kappa+1, \ldots, 6\}$), and exchanges the elements $[\widetilde{E}_{2i-1}]$ and $[\widetilde{E}_{2i}]$ with $i \in \{1, \ldots, \kappa\}$ (resp. $i \in \{1, \ldots, \kappa, 4\}$). These two involutions take into account that for each real structure on $\widetilde{X}_6$, there are two possible real structures on $\widetilde{X}_2$, depending on the real structure on $\pi^{-1}(0)$.

From now on, let us fix $g = 0$, an integer $\kappa \in \{0, \ldots, 3\}$, and a class $d \in H_2(X_7; \mathbb{Z})$. Set $\aleph = c_1(X_7) \cdot d - 1$ and $A = \{1, \ldots, \aleph\}$, and choose two integer $r, s \geq 0$ such that $\aleph = r + 2s$. Define the involution ρ_s on A as follows: $\rho_{s|\{2i-1,2i\}}$ is the non-trivial transposition for $1 \leq i \leq s$, and $\rho_{s|\{2s+1,\aleph\}} = Id$.

Given $\epsilon \in \{0, 1\}$, denote by $\mathbb{R}\mathcal{S}^\epsilon_7(d, k, s, \kappa)$ the set of triples (Γ, P_Γ, τ) where

- $(\Gamma, P_\Gamma) \in \mathcal{S}_7(d, 0, k)$;
- $\tau : \Gamma \to \Gamma$ is an involution such that for any vertex v of Γ, one has $\beta_v = \beta_{\tau(v)}$, $d_{\tau(v)} = \tau^\epsilon_\kappa(d)$, and $\rho_s(U_v) = U_{\tau(v)}$;
- to each vertex v fixed by τ is associated a decomposition $\beta_v = \beta^{\Re}_v + 2\beta^{\Im}_v$ with $\beta^{\Re}_v, \beta^{\Im}_v \in \mathbb{Z}^\infty_{\geq 0}$.

Given $(\Gamma, P_\Gamma, \tau) \in \mathbb{R}\mathcal{S}^\epsilon_7(d, k, s, \kappa)$, denote by $\sigma(\Gamma, \tau)$ the number of bijection of $\mathrm{Vert}(\Gamma)$ to itself which are induced by an automorphism of the graph Γ commuting with τ. Note that $\tau = Id$ if $s = 0$. Denote also by $\mathrm{Vert}_\Im(\Gamma)$ (resp. $k^{\circ,\Im}_\Gamma$) the set of pairs of vertices (resp. the number of pairs of edges) exchanged by τ, and by $\mathrm{Vert}_\Re(\Gamma)$ (resp. $k^{\circ,\Re}_\Gamma$) the set of vertices (resp. the number of edges) fixed by τ. Next, define

$$\beta^{\Re}_\Gamma = \sum_{v \in \mathrm{Vert}_\Re(\Gamma)} \beta^{\Re}_v, \quad \text{and} \quad \beta^{\Im}_\Gamma = \sum_{v \in \mathrm{Vert}_\Re(\Gamma)} \beta^{\Im}_v + \sum_{\{v,v'\} \in \mathrm{Vert}_\Im(\Gamma)} \beta_v.$$

Let us associate different real multiplicities to elements of $\mathbb{R}\mathcal{S}^\epsilon_7(d, k, \kappa)$, accounting all possible smoothings of $\mathbb{R}\pi^{-1}(0)$.

Given $\{v, v'\} \in \mathrm{Vert}_\Im(\Gamma)$, define

$$\mu^{\mathbb{R}}(\{v, v'\}) = (-1)^{d_v \cdot d_{v'}} \binom{\beta_{v,1}}{\{\lambda_{v,v''}\}_{v'' \in \mathrm{Vert}(\Gamma)}} GW^{0,\beta_v}_{\widetilde{X}_8(\kappa)}(d_v, 0).$$

Let $v \in \mathrm{Vert}_\Re(\Gamma)$. Denote respectively by r_v and s_v the number of points in U_v fixed by ρ_s and the number of pairs of points in U_v exchanged by ρ_s. By definition we have $|U_v| = r_v + 2s_v$. Denote also by $k^{\circ,\Im}_v$ the number of pairs of edges of Γ adjacent to v and exchanged by τ. Define (recall that the numbers FW have been defined in Section 3.2.2.4)

$$\mu^{\mathbb{R},\epsilon}_{s,\kappa}(v) = 2^{k^{\circ,\Im}_v} \binom{\beta^{\Re}_{v,1}}{\{\lambda_{v,v'}\}_{v' \in \mathrm{Vert}_\Re(\Gamma)}} \binom{\beta^{\Im}_{v,1}}{\{\lambda_{v,v'}\}_{\{v',v''\} \in \mathrm{Vert}_\Im(\Gamma)}} FW^{0,\beta^{\Re}_v,0,\beta^{\Im}_v}_{\widetilde{X}_8(\kappa+\epsilon)}(d^\epsilon_v, s_v),$$

where $d^0_v = d_v$, and d^1_v is obtained from d_v by exchanging[1] the coefficients of $\widetilde{E}_{2\kappa-1}$ and $\widetilde{E}_7$, and $\widetilde{E}_{2\kappa}$ and

[1]This additional complication is purely formal and comes from the convention used to define the numbers FW in section 3.2.2.4.

$\widetilde{E}_8$. Define also

$$\eta^{\mathbb{R}}_{s,\epsilon}(v) = 2^{k_v^{\circ,\Im}} \binom{\beta^{\Re}_{v,1}}{\{\lambda_{v,v'}\}_{v'\in \mathrm{Vert}_{\Re}(\Gamma)}} \binom{\beta^{\Im}_{v,1}}{\{\lambda_{v,v'}\}_{\{v',v''\}\in \mathrm{Vert}^{\Im}(\Gamma)}} FW^{0,\beta_v^{\Re},0,\beta_v^{\Im}}_{\widetilde{X}_8(4),\epsilon}(d_v, s_v),$$

and

$$\nu^{\mathbb{R}}_{s,\epsilon}(v) = FW^{0,\beta_v^{\Re},0,\beta_v^{\Im}}_{\widetilde{X}_8(4),\epsilon,\epsilon}(d_v, s_v).$$

Let $\mathbb{R}\mathcal{S}^0_{7,m}(d,k,s,\kappa)$ be the subset of $\mathbb{R}\mathcal{S}^0_7(d,k,s,\kappa)$ formed by elements with $\beta^{\Re}_{\Gamma,2} = 0$. Given $(\Gamma, P_\Gamma, \tau) \in \mathcal{R}\mathcal{S}^0_{7,m}(d,k,s,\kappa)$, define the following multiplicity:

$$\mu^{\mathbb{R},0}_{s,\kappa}(\Gamma, P_\Gamma, \tau) = \frac{(-1)^{k_\Gamma^{\circ,\Im}+\beta^{\Im}_{\Gamma,2}}\ I^{\beta^{\Im}_\Gamma}}{\sigma(\Gamma,\tau)} \prod_{v\in \mathrm{Vert}_{\Re}(\Gamma)} \mu^{\mathbb{R},0}_{s,\kappa}(v) \prod_{\{v,v'\}\in \mathrm{Vert}_{\Im}(\Gamma)} \mu^{\mathbb{R}}(\{v,v'\})\times$$

$$\times \sum_{k^{\circ\circ}=r'+2s'} \binom{\beta^{\Re}_{\Gamma,1} - 2k_\Gamma^{\circ,\Re}}{r'} \binom{\beta^{\Im}_{\Gamma,1} - 2k_\Gamma^{\circ,\Im}}{s'}.$$

Let $\mathbb{R}\mathcal{S}^1_{7,m}(d,k,s,\kappa)$ be the subset of $\mathbb{R}\mathcal{S}^1_7(d,k,s,\kappa)$ formed by elements with $\beta^{\Re}_\Gamma = 2k_\Gamma^{\circ,\Re}u_1$ and $k^{\circ\circ} = \beta^{\Im}_{\Gamma,1} - 2k_\Gamma^{\circ,\Im}$. Given $(\Gamma, P_\Gamma, \tau) \in \mathbb{R}\mathcal{S}^1_{7,m}(d,k,s,\kappa)$, define the following multiplicity

$$\mu^{\mathbb{R},1}_{s,\kappa}(\Gamma, P_\Gamma, \tau) = \frac{(-1)^{k_\Gamma^{\circ,\Im}}\ (-2)^{|\beta^{\Im}_\Gamma| - 2k_\Gamma^{\circ,\Im}}}{\sigma(\Gamma,\tau)} \prod_{v\in \mathrm{Vert}_{\Re}(\Gamma)} \mu^{\mathbb{R},1}_{s,\kappa}(v) \prod_{\{v,v'\}\in \mathrm{Vert}_{\Im}(\Gamma)} \mu^{\mathbb{R}}(\{v,v'\}).$$

Note that $\mathbb{R}\mathcal{S}^1_{7,m}(d,k,s,3)$ is composed of elements with $\beta^{\Re}_\Gamma = k_\Gamma^{\circ,\Re} = 0$. Given $(\Gamma, P_\Gamma, \tau) \in \mathbb{R}\mathcal{S}^1_{7,m}(d,k,s,3)$ and $\epsilon \in \{0,1\}$, define the following multiplicity

$$\eta^{\mathbb{R}}_{s,\epsilon}(\Gamma, P_\Gamma, \tau) = \frac{(-1)^{k_\Gamma^{\circ,\Im}}\ (-2)^{|\beta^{\Im}_\Gamma| - 2k_\Gamma^{\circ,\Im}}}{\sigma(\Gamma,\tau)} \prod_{v\in \mathrm{Vert}_{\Re}(\Gamma)} \eta^{\mathbb{R}}_{s,\epsilon}(v) \prod_{\{v,v'\}\in \mathrm{Vert}_{\Im}(\Gamma)} \mu^{\mathbb{R}}(\{v,v'\}).$$

Let $\mathbb{R}\mathcal{S}^1_{7,2}(d,k,s,3)$ (resp. $\mathbb{R}\mathcal{S}^1_{7,3}(d,k,s,3)$) be the subset of $\mathbb{R}\mathcal{S}^1_7(d,k,s,3)$ formed by elements with $k^\circ_\Gamma = \beta_{\Gamma,1} = 0$ (resp. $k^\circ_\Gamma = \beta_{\Gamma,1} = \beta^{\Re}_{\Gamma,2} = 0$). Note that any element of $\mathbb{R}\mathcal{S}^1_{7,2}(d,k,s,3)$ or $\mathbb{R}\mathcal{S}^1_{7,3}(d,k,s,3)$ has a single vertex.

In the following theorem, the connected component of $\mathbb{R}X^+_7(4)$ with Euler characteristic ϵ is denoted by L_ϵ.

Theorem 4.20 ([Bru14, Theorem 6.8]) *Let $d \in H_2(X_7;\mathbb{Z})$ such that $d \cdot [D] \geq 1$, and $r, s \in \mathbb{Z}_{\geq 0}$ such* *that* $c_1(X_7)\cdot d - 1 = r + 2s$. *Then one has*

$$W_{X_7(\kappa)}(d,s) = \sum_{k\geq 0}\ \sum_{(\Gamma,P_\Gamma,\tau)\in\mathbb{R}\mathcal{S}^0_{7,m}(d,k,s,\kappa)} \mu^{\mathbb{R},0}_{s,\kappa}(\Gamma,P_\Gamma,\tau) \quad \text{if } \kappa\in\{0,\dots,3\},$$

$$W_{X_7(\kappa+1)}(d,s) = \sum_{k\geq 0}\ \sum_{(\Gamma,P_\Gamma,\tau)\in\mathbb{R}\mathcal{S}^1_{7,m}(d,k,s,\kappa)} \mu^{\mathbb{R},1}_{s,\kappa}(\Gamma,P_\Gamma,\tau) \quad \text{if } \kappa\in\{0,\dots,2\},$$

$$W_{X_7^-(4),\mathbb{R}P^2,\mathbb{R}X_7^-(4)}(d,s) = \sum_{k\geq 0}\ \sum_{(\Gamma,P_\Gamma,\tau)\in\mathbb{R}\mathcal{S}^1_{7,m}(d,k,s,3)} \eta^{\mathbb{R}}_{s,\epsilon}(\Gamma,P_\Gamma,\tau) \quad \forall\epsilon\in\{0,1\},$$

$$W_{X_7^+(4),L_{2\epsilon},\mathbb{R}X_7^+(4)}(d,s) = \sum_{k\geq 0}\ \sum_{(\Gamma,P_\Gamma,\tau)\in\mathbb{R}\mathcal{S}^1_{7,m}(d,k,s,3)} \eta^{\mathbb{R}}_{s,\epsilon}(\Gamma,P_\Gamma,\tau) \quad \forall\epsilon\in\{0,1\},$$

$$W_{X_7^-(4),L_1,L_1}(d,s) = \sum_{k\geq 0}\ \sum_{(\Gamma,P_\Gamma,\tau)\in\mathbb{R}\mathcal{S}^1_{7,2}(d,k,s,3)} 2^{\beta^{\Re}_{\Gamma,2}+\beta^{\Im}_{\Gamma,2}}\nu^{\mathbb{R}}_{s,1}(v),$$

$$W_{X_7^-(4),L_1,L_1}(d,s) = \sum_{k\geq 0}\ \sum_{(\Gamma,P_\Gamma,\tau)\in\mathbb{R}\mathcal{S}^1_{7,3}(d,k,s,3)} (-2)^{\beta^{\Im}_{\Gamma 2}}\nu^{\mathbb{R}}_{s,0}(v),$$

$$W_{X_7^+(4),L_0,L_0}(d,s) = \sum_{k\geq 0}\ \sum_{(\Gamma,P_\Gamma,\tau)\in\mathbb{R}\mathcal{S}^1_{7,2}(d,k,s,3)} 2^{\beta^{\Re}_{\Gamma,2}+\beta^{\Im}_{\Gamma,2}}\nu^{\mathbb{R}}_{s,0}(v),$$

$$W_{X_7^+(4),L_2,L_2}(d,s) = \sum_{k\geq 0}\ \sum_{(\Gamma,P_\Gamma,\tau)\in\mathbb{R}\mathcal{S}^1_{7,3}(d,k,s,3)} (-2)^{\beta^{\Im}_{\Gamma 2}}\nu^{\mathbb{R}}_{s,1}(v).$$

Combining Theorems 4.20 and 3.32, we obtain the following corollaries.

Corollary 4.21 ([Bru14, Corollary 6.9]) *For any* $d\in H_2(X_7;\mathbb{Z})$, *we have*

$$W_{X_7^+(4),L_0,L_0}(d,0)\geq W_{X_7^+(4),L_2,L_2}(d,0)\geq 0.$$

Moreover both invariant are divisible by $4^{\left[\frac{d\cdot[D]}{2}\right]-\min(d\cdot E_i)-1}$, *as well as* $W_{X_7^-(4),\mathbb{R}P^2,\mathbb{R}P^2}(d,0)$.

The non-negativity of $W_{X_7^+(4),L_0,L_0}(d,0)$ has been first established in [IKS13a].

Corollary 4.22 ([Bru14, Corollary 6.10]) *For any* $d\in H_2(X_7;\mathbb{Z})$, *we have*

$$W_{X_7^+(4),L_0,\mathbb{R}X_7^+(4)}(d,s) = W_{X_7^+(4),L_2,\mathbb{R}X_7^+(4)}(d,s) = W_{X_7^-(4),\mathbb{R}P^2,\mathbb{R}X_7^-(4)}(d,s).$$

Example 4.23 ([Bru14, Example 6.11]) *Theorem 4.20 implies that Welschinger invariants of the real surfaces* $X_7^{\pm}(\kappa)$ *are the one listed in Table 4.2. The invariants* $W_{X_7(\kappa)}(2c_1(X_7),s)$ *have been first computed in [HS12]. In addition to [Bru14], the invariants* $W_{X_7^+(4),L_\epsilon,L_\epsilon}(2c_1(X_7),0)$ *and* $W_{X_7^-(4),\mathbb{R}P^2,\mathbb{R}P^2}(2c_1(X_7),0)$ *have also been computed in [IKS13a].*

Recall that the invariant $W_{(X,\tau),L,L'}(d,s)$ is said to be *sharp* if there exists a real configuration $\underline{x}$ with s pairs of complex conjugated points such that $|\mathbb{R}\mathcal{C}(d,0,\underline{x})| = |W_{(X,\tau),L,L'}(d,s)|$. When $r=0$ or 1, Welschinger proved in [Wel07] the sharpness of $W_{(X,\tau),L}(d,s)$ when L is homeomorphic to either T^2, S^2, or $\mathbb{R}P^2$, with the additional assumption in the latter case that (X,τ) is $\mathbb{C}P^2$ blown up in at most three pairs of complex conjugated points. In the case of $\mathbb{C}P^2$, one possible way to prove this result is by degenerating $\mathbb{C}P^2$ into the union of $\mathbb{C}P^2$ and the normal bundle of a real conic with an empty real part.

It follows from Example 4.23 that $W_{X_7^+(4),L_0,\mathbb{R}X_7^+(4)}(2c_1(X_7),1)$ is not sharp. This shows that [Wel07, Theorem 1.1] does not extend to all real structures on X_7. The methods used in the proof of Theorems 3.32 and 4.20 adapt without any problem to the case when $r=0$ or 1 and E has an empty real part. In particular, such adaptations allow one to extend [Wel07, Theorem 1.1] to the real surface $X_7^-(4)$.

$s\backslash\kappa$	0	1	2	3	4−	4+ $L=\mathbb{R}P^2_1$	4+ $L=S^2$
0	224	128	64	24	0	0	0
1	132	68	28	4	-12	-12	-12

$W_{X_7^\pm(\kappa),L,\mathbb{R}X_7^\pm(\kappa)}(2c_1(X_7),s)$

s	
0	32
1	12

$W_{X_7^-(4),\mathbb{R}P^2,\mathbb{R}P^2}(2c_1(X_7),s)$

$s\backslash\epsilon$	0	2
0	48	16
1	20	4

$W_{X_7^+(4),L_\epsilon,L_\epsilon}(2c_1(X_7),s)$

Table 4.2: Welschinger invariants of X_7 for the class $2c_1(X_7)$

Proposition 4.24 ([Bru14, Proposition 8.1]) *Let $d\in H_2(X_7;\mathbb{Z})$, $r\in\{0,1\}$, and $s\geq 0$ such that $c_1(X_7)\cdot d-1=r+2s$. Then $W_{X_7^-(4),\mathbb{R}P^2,\mathbb{R}X_7^-(4)}(d,s)$ is sharp.*

4.2.4 Elements of the proof of Theorems 4.17 and 4.20

Theorems 4.17 and 4.20 are obtained by applying Li's degeneration formula and its real counterpart to the degenerations $\widetilde{\mathcal{Y}}$ of X_7 describe in Proposition 4.15, together with a set of sections $\underline{x}:\mathbb{C}\to\widetilde{\mathcal{Y}}$ satisfying $\underline{x}(0)\subset\widetilde{X}_6\setminus\widetilde{X}_2$. As mentioned in Section 4.2.1, no non-trivial covering appears during this degeneration.

Let $\widetilde{E}'_1$ and $\widetilde{E}'_2$ be the two (-1)-curves in $\widetilde{X}_2$ intersecting E. Denote respectively by p_7 and p_8 the corresponding intersection points. Let $T_1,\dots,T_6$ be the six rational curves in $\widetilde{X}_2$ such that $T_i^2=T_i\cdot\widetilde{E}'_2=0$, and such that T_i passes through $\widetilde{E}_i\cap E$. Denote also by $\widetilde{E}'_7$ the (-1)-curve in $\widetilde{X}_2$ which does not intersect E. We may further assume that we chose $\widetilde{E}'_1$ and $E_1,\dots,E_7$ such that the seven curves $\widetilde{E}_1\cup T_1,\dots\widetilde{E}_6\cup T_6$, and $\widetilde{E}'_7$ in $\pi^{-1}(0)$ respectively deform to $E_1,\dots,E_7$ in X_7.

Let us choose $\underline{x}(t)$ a generic set of $c_1(X_7)\cdot d-1+g$ sections $\mathbb{C}\to\widetilde{\mathcal{Y}}$ such that $\underline{x}(0)\subset\widetilde{X}_6\setminus\widetilde{X}_2$. For each $t\neq 0$, we denote by $\mathcal{C}(d,g,\underline{x}(t))$ the set of maps $f:C\to X_7$ with C an irreducible curves of geometric genus g, such that $f(C)$ realizes the class d in $H_2(X_7;\mathbb{Z})$, and contains all points in $\underline{x}(t)$. We denote by $\mathcal{C}(d,g,\underline{x}(0))$ the set of limits, as t goes to 0, of elements $\mathcal{C}(d,g,\underline{x}(t))$.

It turns out that the set $\mathcal{C}(d,g,\underline{x}(0))$ is finite, and that its cardinal does not depend on $\underline{x}(0)$ as long as this latter is generic. Moreover if $\overline{f}:\overline{C}\to\pi^{-1}(0)$ is an element of $\mathcal{C}(d,g,\underline{x}(0))$, and if $\overline{C}'$ is an irreducible component of $\overline{C}$ mapped to $\widetilde{X}_2$, then one of the four following situations occurs:

1. $\overline{f}(\overline{C}')$ realizes the class $[D]$, and intersect E in two points determined by $\overline{f}(\overline{C}_{\widetilde{X}_6})$, distinct from p_7 and p_8;
2. $\overline{f}(\overline{C}')$ realizes the class $[D]$, and is tangent to E at a point determined by $\overline{f}(\overline{C}_{\widetilde{X}_6})$, distinct from p_7 and p_8;
3. $\overline{f}(\overline{C}')$ realizes the class $[D]-[\widetilde{E}'_i]$, $i=1,2$, and intersects E in a point determined by $\overline{f}(\overline{C}_{\widetilde{X}_6})$, distinct from p_7 and p_8;
4. $\overline{f}(\overline{C}')$ realizes the class $[\widetilde{E}'_i]$, $i=1,2$;

where $\overline{C}_{\widetilde{X}_6}$ denotes the union of irreducible components of $\overline{C}$ mapped to $\widetilde{X}_6$.

Now one describes easily any element of $\mathcal{C}(d, g, \underline{x}(0))$ by a X_7-graph. The complex and real multiplicities of this latter are deduced from Proposition 4.11 and its real counterpart, an adaptation of Proposition 4.12.

4.2.5 Absolute invariants of X_8

By pushing a bit further the technique used in the proof of Theorems 4.17 and 4.20, we compute Gromov-Witten invariants and some Welschinger invariants of X_8. Here we briefly indicate how to adapt the strategy used in the case of X_7. For more details, we refer to [Bru14, Section 7].

We start with the degeneration $\widetilde{\mathcal{Y}}$ of X_7 described in Proposition 4.15. Choose an additional generic holomorphic section $p_0' : \mathbb{C} \to \widetilde{\mathcal{Y}}$ such that $p_0'(0) \in \widetilde{X}_6 \setminus \widetilde{X}_2$, and denote by $\mathcal{Z}$ the blow up $\widetilde{\mathcal{Y}}$ along the divisor $p_0'(\mathbb{C})$. The map $\pi : \widetilde{\mathcal{Y}} \to \mathbb{C}$ naturally extends to a flat map $\pi : \mathcal{Z} \to \mathbb{C}$, which provides a degeneration of X_8 to $\pi^{-1}(0) = \widetilde{X}_{8,1} \cup \widetilde{X}_2$.

In this framework, Theorem 4.17 extends as follows.

Theorem 4.25 ([Bru14, Theorem 7.2]) *Gromov-Witten invariants of X_8 can be reduced to the enumeration of complex algebraic curves in $\widetilde{X}_{8,1}$.*

All computations in $\widetilde{X}_{8,1}$ needed to compute Gromov-Witten invariants of X_8 are performed in [SS13]. To our knowledge, Theorem 4.25 provides the first explicit computations of Gromov-Witten invariants of X_8 in positive genus.

Example 4.26 ([Bru14, Theorem 7.3]) *Using Theorem 4.25 and [SS13, Theorem 2.1] one computes the values of $GW_{X_8}(2c_1(X_8), g)$ listed in table 4.3. The rational case has been first computed in [GP98, Section 5.2].*

g	0	1	2
$GW_{X_8}(2c_1(X_8), g)$	90	18	1

Table 4.3: $GW_{X_8}(2c_1(X_8), g)$

Theorem 4.25 also has a real counterpart. Denote by $X_8(\kappa)$ with $\kappa = 0, \dots, 3$, and $X_8^{\pm}(4)$ the surface X_8 equipped with the real structure such that

$$\mathbb{R}X_8(\kappa) = \mathbb{R}P^2_{8-2\kappa}, \quad \mathbb{R}X_8^-(4) = \mathbb{R}P^2_1 \sqcup \mathbb{R}P^2, \quad \mathbb{R}X_8^+(4) = S^2 \sqcup \mathbb{R}P^2_2.$$

These real structures on X_8 represent 6 of the 11 deformation classes of real Del Pezzo surfaces of degree 1.

Theorem 4.27 ([Bru14, Theorem 7.5]) *The Welschinger invariants $W_{X_8(\kappa)}(d, 0)$, $W_{X_8^-(4), L_\epsilon}(d, 0)$, and $W_{X_8^+(4), L_\epsilon}(d, 0)$ can be reduced to the enumeration of real algebraic curves in $\widetilde{X}_{8,1}$.*

All enumerations of real curves in $\widetilde{X}_{8,1}$ needed to compute Welschinger invariants using Theorem 4.27 are performed in [IKS13a], by enumerating real curves passing through a configuration of real points in CH position.

Remark 4.28 *Although we do not see any obstruction to enumerate real and complex curves in $\widetilde{X}_{8,1}$ using the floor diagrams techniques, we chose not to do it in [Bru14] for the sake of shortness. As a consequence, Theorem 4.27 computes Welschinger invariants of X_8 only for configurations or real points. For the same shortness reason, we decided to restrict to standard real structures on $\widetilde{X}_{8,1}$. In particular, with some additional effort one should be able to generalize Theorem 4.27 to compute $W_{(X_8,\tau),L,L'}(d,s)$ for $s > 0$ and more real structures on X_8.*

Combining Theorems 4.25 and 4.27 together with Theorems3.27 and 3.32, we obtain the following corollaries.

Corollary 4.29 ([Bru14, Corollary 7.5]) *Given $d \in H_2(X_8;\mathbb{Z})$ with $d \cdot [D] \geq 1$, one has*

$$W_{X_8^-(4),L_\epsilon}(d,0) = W_{X_8^+(4),L_{3\epsilon-1}}(d,0) \quad \forall \epsilon \in \{0,1\}.$$

Corollary 4.30 ([Bru14, Corollary 7.6]) *For any $d \in H_2(X_8;\mathbb{Z})$ and $\kappa \in \{0,\ldots,3\}$, one has*

$$W_{X_8(\kappa)}(d,0) \geq 0 \quad \textit{and} \quad W_{X_8^\pm(4),L_\epsilon,\mathbb{R}X_8^\pm(4)}(d,0) \geq 0.$$

Corollary 4.31 ([Bru14, Corollary 7.7]) *For any $d \in H_2(X_8;\mathbb{Z})$ one has*

$$W_{X_8(0)}(d,0) = GW_{X_8}(d,0) \quad \textit{mod } 4.$$

Example 4.32 ([Bru14, Example 7.9]) *Using Theorem 4.27, one computes the Welschinger invariants of $X_8^\pm(\kappa)$ listed in Table 4.4. The invariants $W_{X_8(\kappa)}(2c_1(X_8),0)$ with $\kappa \leq 3$ have been first computed by Horev and Solomon in [HS12].*

κ	0	1	2	3	$4-$ $L = \mathbb{R}P_1^2$	$4-$ $L = \mathbb{R}P^2$	$4+$ $L = \mathbb{R}P_2^2$	$4+$ $L = S^2$
$W_{X_8^\pm(\kappa),L,\mathbb{R}X_8^\pm(\kappa)}(2c_1(X_8),0)$	46	30	18	10	6	4	6	4

Table 4.4: Welschinger invariants of X_8 for the class $2c_1(X_8)$

4.2.6 Further comments

4.2.6.1 Other Welschinger invariants of X_8

As mentioned above, we do not see any obstruction other than technical to extend Section 3.2.2 to the enumeration of real curves in $\widetilde{X}_{8,1}$ for arbitrary r, s and any real structure on $\widetilde{X}_{8,1}$. In particular Theorem 4.27 should generalize to Welschinger invariant of X_8 for almost all, if not all, real structures. Our method should also apply to compute the invariants recently defined in [Shu14].

The standard methods from [IKS13b, IKS13a, BM08] should also apply here to study logarithmic asymptotic of Welschinger invariants.

4.2.6.2 Relation with tropical Welschinger invariants and refined Severi degrees

Invariance of Gromov-Witten and Welschinger invariants combined with Theorems 4.17, 4.20, 4.25, and 4.27 provide non-trivial relations among marked floor diagrams counted with their various multiplicities. It is not obvious to us how those relations follow from a purely combinatorial study of marked floor diagrams.

Denote by $W^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa)}(d,g,s,\underline{x})$ the straightforward generalization to any genus of the numbers $W^{\alpha^{\Re},\beta^{\Re},\alpha^{\Im},\beta^{\Im}}_{\widetilde{X}_n(\kappa)}(d,s,\underline{x})$ defined in Section 3.2.2.1. In the case when $s=0$, all definitions from Section 3.2.2.4 also make sense for positive genus, and Theorem 3.32 still holds (the proof is exactly the same). If $\underline{x}^\circ$ is a configuration of real points in $\mathbb{R}\widetilde{X}_n(\kappa)$ as in the proof of Theorems 3.27 and 3.32, then one sees easily from the proof of Theorem 3.32 that the numbers $W^{0,\beta_1^{\Re}u_1,0,\beta_1^{\Im}u_1}_{\widetilde{X}_n(\kappa)}(d,g,0,\underline{x}^\circ\sqcup\underline{x}_E)$ do not depend on the position in each copy of $\mathcal{N}$ of the points in $\underline{x}^\circ$.

More surprisingly, the numbers $W^{0,(d\cdot E)u_1,0,0}_{\widetilde{X}_n(\kappa)}(d,g,0,\underline{x}^\circ\sqcup\underline{x}_E)$ we computed on a few examples, with $\underline{x}^\circ$ as in the proof of Theorems 3.27 and 3.32, also satisfy relations analogous to Theorems 4.20 and 4.27 for positive genus. Furthermore in the case of X_3, the numbers I obtained in this way are... the corresponding tropical Welschinger invariants (see [IKS09] for a definition). This observation is certainly in favor of the existence of a more conceptual definition and signification of those tropical Welschinger invariants. To my knowledge, only some tropical Welschinger invariants of the second Hirzebruch surface yet found such an interpretation in Proposition 4.8, where they are shown to correspond to genuine Welschinger invariants of the quadric ellipsoid.

Tropical Welschinger invariants are also related to refined Severi degrees. Still in the case $s=0$, it would have been possible to define and compute analogous polynomials interpolating between real and complex multiplicities of marked floor diagrams relative to a conic. Unfortunately, no relations, even conjectural, are known yet between refined Severi degrees and Welschinger invariants when $s>0$. Since we are interested in this memoir in the computation of Welschinger invariants for any values of s and r, we chose not to develop the refined Severi degree aspect of our computations.

4.3 Deformation of tropical Hirzebruch surfaces and enumerative geometry

In [AB01], Abramovich and Bertram related genus 0 enumerative invariants of $\Sigma_0=\mathbb{C}P^1\times\mathbb{C}P^1$ and the second Hirzebruch surface Σ_2. The strategy of their proof is to understand how algebraic curves on Σ_0 behave when this latter surface degenerates to Σ_2. Later on this method was extended by Vakil in [Vak00a] to enumerative invariants of any genus of Σ_0 and Σ_2, and more generally to relate enumerative invariants of an almost Fano surface and any of its deformations.

In this section, we illustrate the use of tropical techniques by generalizing Abramovich-Bertram-Vakil formula to the case of Σ_n and Σ_{n+2} (Theorem 4.36). The main idea underlying our strategy is to consider a tropical surface X in $\mathbb{R}^3$ which is a suitable tropicalization of Kodaira's deformation of complex Hirzebruch surfaces; see Section 4.3.1.3 for a more detailed description. Next the proof of Theorem 4.36 basically consists of the suitable Correspondence Theorems (see Theorem 4.40 and [BM13, Theorem 3.17]) relating the invariants introduced in Section 4.3.1.1 to their tropical counterparts, and a proof of the tropical version of Theorem 4.36.

We define the enumerative invariants we are interested in, and state our main formula in Section 4.3.1. We also outline there our strategy, and briefly discuss where the difficulties we have to deal with come from. We describe the corresponding tropical problem together with a correspondence theorem in Section 4.3.2, where we also briefly indicate how to deduce Theorem 4.36 from Theorem 4.40. For the sake of shortness, we only present here one of the two Correspondence Theorems proved in [BM13], namely [BM13, Theorem 4.13].

Results presented in this section were obtained jointly with Markwig, and appeared in [BM13]. The two Correspondence Theorems 4.40 and [BM13, Theorem 3.17], and the formula of Theorem 4.36 together with its tropical method of proof should be viewed as the main contributions of [BM13].

4.3.1 Relations among enumerative invariants of Σ_n and Σ_{n+2}

4.3.1.1 Enumerative invariants under consideration

We start by defining enumerative invariants that will be related in Theorem 4.36. Note that contrary to the rest of this typescript, we count *reducible curves* in this section. In particular, it is more natural to consider the Euler characteristic of the curves rather than their genus. Recall that Newton fans have been defined in Section 2.3.2.

Definition 4.33 *Given a Newton fan δ in $\mathbb{Z}^2$ and an integer $\chi \in \mathbb{Z}$, the number of (not necessarily irreducible) algebraic curves in $Tor(\Pi_\delta)$ with Newton fan δ, whose normalization has Euler characteristic χ, and passing through a generic configuration ω of $|\delta| - \frac{\chi}{2}$ points does not depend on ω; we denote this number by $N_\chi(\delta)$.*

Alternatively $N_\chi(\delta)$ is the number of algebraic curves in $(\mathbb{C}^*)^2$ with Newton fan δ, whose normalization has Euler characteristic $\chi - |\delta|$, and passing through a generic configuration of $|\delta| - \frac{\chi}{2}$ points in $(\mathbb{C}^*)^2$.

Example 4.34 *Consider complex curves in $(\mathbb{C}^*)^2$ with Newton fans of the form*

$$\delta = \{(1,n)^a, (0,-1)^{an+b}, (-\alpha_1, \beta_1), \dots, (-\alpha_k, \beta_k)\}$$

with $\alpha_i, \beta_i \geq 0$ for all i. A complex curve with such a Newton fan can naturally be seen as a curve of bidegree (a,b) in Σ_n with a singularity at the point $(0,\infty)$ in the standard coordinates corresponding to Π_δ. In particular, a generic algebraic curve in Σ_n of bidegree (a,b) will have the Newton fan

$$\delta(a,b,n) = \{(1,n)^a, (0,-1)^{an+b}, (-1,0)^a, (0,1)^b\}$$

in standard coordinates.

Theorem 4.36 involves some additional relative enumerative invariants of Hirzebruch surfaces. We give their definition below, and refer to [BM13, Sections 2.3, 3.2, and 5.3] for more details about these invariants and their tropical computation. We denote by E an irreducible non-singular algebraic curve of self-intersection $-n$ in Σ_n (E is unique if $n > 0$).

Let S_0 be a curve of bidegree $(1,1)$ in Σ_n and let $p_0 \in S_0 \setminus E$ be a point. We denote by F_0 the unique curve of bidegree $(0,1)$ passing through p_0 and we choose a (non-standard) local system of coordinates (x_0, y_0) on Σ_n at p_0 such that S_0 has local equation $y_0 = 0$ and F_0 has local equation $x_0 = 0$. Given two integer numbers d_1 and d_2, a (d_1,d_2)-*germ* at p_0 is a curve in Σ_n with local equation $x_0^{d_2'} + cy_0^{d_1'} = 0$ with $c \in \mathbb{C}^*$, $d_1' = \frac{d_1}{\gcd(d_1,d_2)}$, and $d_2' = \frac{d_2}{\gcd(d_1,d_2)}$. Let D be a local branch at p_0 of a reduced algebraic curve in Σ_n containing neither S_0 nor F_0 as an irreducible component. Denote by d_{S_0} (resp. d_{F_0}) the local intersection multiplicity of D with S_0 (resp. F_0) at p_0. Then there exists a unique (d_{S_0}, d_{F_0})-germ at p_0 whose intersection multiplicity with D at p_0 is maximal. We call this curve the *tangent germ* of D at p_0.

Let $n \geq 0$, and $u, \alpha_1, \dots \alpha_r, d_1, \dots d_{r+s} > 0$ be integer numbers such that $\alpha_i \leq d_i - 1$ and $\sum_{i=1}^{r+s} d_i = u(n+1)$, and let S_0, F_0, and p_0 as above. Choose a configuration $\underline{x} = \{G_1, \dots, G_r, p_1, \dots, p_s\}$ of s distinct point $p_1, \dots, p_s$ in $S_0 \setminus (E \cup \{p_0\})$, and r distinct germs at p_0 such that G_i is a $(d_i + \alpha_i(n+1), \alpha_i)$-germ at p_0. We denote by $\mathcal{S}_{S_0,p_0}(\underline{x})$ the set of all algebraic curves S in Σ_n of bidegree $(a,0)$ such that

- S has a smooth branch tangent to S_0 at p_i with intersection multiplicity d_{r+i} for $i = 1, \dots, s$;
- S has exactly r branches $D_1, \dots, D_r$ at p_0, and the local intersection multiplicity of D_i with S_0 (resp. F_0) at p_0 is equal to $d_i + \alpha_i(n+1)$ (resp. α_i);
- G_i is the tangent germ of D_i at p_0;
- S has u connected components, whose normalization are all rational;

- each connected component of S intersect $F_0 \setminus \{p_0\}$ in a single point, with intersection multiplicity 1.

For $\underline{x}$ generic, the set $\mathcal{S}_{S_0,p_0}(\underline{x})$ is finite and its cardinal is independent of S_0, p_0, and $\underline{x}$. We denote it by $\mathcal{N}(u,n,d,\alpha)$. Note that it is independent of the ordering of the pairs $(d_1,\alpha_1),\ldots,(d_r,\alpha_r)$ and of $d_{r+1},\ldots,d_{r+s}$. If $r=0$, we write $\alpha=0$. We refer to [BM13, Example 2.7] for explicit computations of the invariants $\mathcal{N}(u,n,d,\alpha)$ in the cases $n=0,1$, and 2.

Remark 4.35 *Given fixed n and u, there clearly exist finitely many choices for d and α. Furthermore the computation of $\mathcal{N}(u,n,d,\alpha)$ reduces to the computation of the finitely many possible $\mathcal{N}(1,n,d',\alpha')$, and to the combinatorial enumeration of how elements of $\underline{x}$ can be distributed among the u irreducible components of S.*

4.3.1.2 Main formula

Before stating our formula relating enumerative invariants introduced above, we need first to introduce some additional notation. Let δ_0 be the Newton fan

$$\delta_0 = \{(1,n)^a, (0,-1)^{an+b}, (-1,0)^a, (0,1)^b\}.$$

We write $\delta \vdash \delta_0$ if δ is a Newton fan satisfying

$$\begin{aligned}\delta = \quad & \{(1,n+2)^m, (0,-1)^{a(n+1)+b}, (-1,0)^A, \\ & (-\alpha_1,\beta_1),\ldots,(-\alpha_r,\beta_r),(0,\beta_{r+1}),\ldots,(0,\beta_{r+s}),(0,1)^U\}\end{aligned}$$

with

$$0 < m \le a, \quad 0 \le A \le \min\{m,b\},$$

$$\alpha_i,\beta_i > 0 \quad \text{for} \quad i = 1\ldots,r, \quad \text{and} \quad \beta_{r+1},\ldots,\beta_{r+s} > 1.$$

Note that for every $\delta \vdash \delta_0$,

$$\Pi_\delta \subset Conv\{(0,0),(0,a),(b-a,a),(a(n+1)+b,0)\},$$

where the latter is the polygon dual to

$$\{(1,n+2)^a, (0,-1)^{a(n+1)+b}, (-1,0)^a, (0,1)^{b-a}\}.$$

In particular once δ_0 is fixed, the choices of such δ are limited (see Figure 4.3).

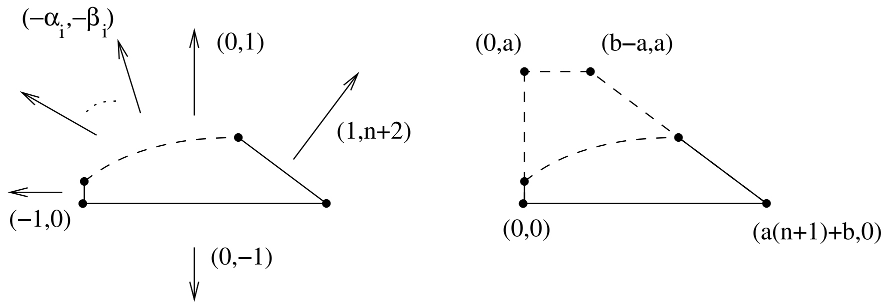


Figure 4.3: Finitely many fans $\delta \vdash \delta_0$.

For a Newton fan $\delta \vdash \delta_0$, we define the following quantities:

- $\chi' = \chi - 2(a+b-m-r-s-A-U)$;

- $d = (\beta_1 + \alpha_1, \dots, \beta_r + \alpha_r, \beta_{r+1}, \dots, \beta_{r+s}, 1^{U+A-b})$;
- $\alpha = (\alpha_1, \dots, \alpha_r)$.

The enumerative invariants of Σ_n and Σ_{n+2} introduced in the preceding section are related in the next theorem.

Theorem 4.36 ([BM13, Theorem 1.2]) *Let $n \geq 0$ and $\chi \in \mathbb{Z}$ be two integers, and let δ_0 be as above the Newton fan*

$$\delta_0 = \{(1,n)^a, (0,-1)^{an+b}, (-1,0)^a, (0,1)^b\}.$$

Then we have the following equation:

$$N_\chi(\delta_0) = \sum_{\delta \vdash \delta_0} \left(\binom{U}{b-A} \cdot \prod_{i=1}^{r} \gcd(\alpha_i, \beta_i) \cdot \prod_{i=1}^{s} \beta_{r+i} \cdot \mathcal{N}(a-m, n, d, \alpha)\ N_{\chi'}(\delta) \right).$$

Note that the factor $N_{\chi'}(\delta)$ indeed counts curves on Σ_{n+2} by Example 4.34. Note also that as pointed out in Remark 4.35, for a fixed n the computations of all numbers $\mathcal{N}(a-m, n, d, \alpha)$ reduces to the computation of finitely many cases. For explicit specialization of Theorem 4.36 to the cases $n = 0, 1, 2$, we refer to [BM13, Examples 1.3, 1.4, and 1.5]

4.3.1.3 Strategy of the proof of Theorem 4.36

Let X be the tropical surface in $\mathbb{R}^3$ defined by the tropical polynomial "$x + y + z$". It consists of three half-planes $\sigma_1 = \{x = y \geq z\}$, $\sigma_2 = \{x = z \geq y\}$ and $\sigma_3 = \{y = z \geq x\}$ meeting along the line $L = \{x = y = z\} = \mathbb{R}(1,1,1)$, see Figure 4.4. We claim that X is a tropicalization of Kodaira's

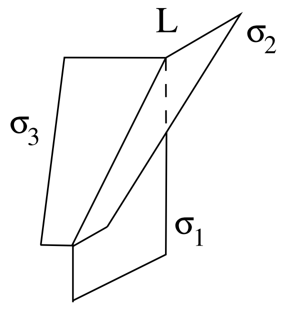


Figure 4.4: The tropical surface X.

deformation of complex Hirzebruch surfaces Σ_n to Σ_{n+2}. Indeed, Kodaira's deformation of complex Hirzebruch surfaces Σ_n to Σ_{n+2k} can be written in coordinates in (a suitable compactification of) $(\mathbb{C}^*)^4$ by the equation (see for example [BM13, Appendix A])

$$z + t(x^k + y) = 0.$$

Considering this 3-fold as a family of complex surfaces indexed by t, its tropicalization in the case $k = 1$ is the tropical hypersurface in $\mathbb{R}^3$ with equation $z +$ "$1 \cdot x + 1 \cdot y$", which is X up to an obvious tropical change of coordinates.

Similarly, a tropicalization of the deformation of Σ_n to Σ_{n+2} together with a configuration of points whose limit, as t goes to $+\infty$, lyes in $(\mathbb{C}^*)^3$ is provided by the tropical surface X together with a configuration of points lying on the face σ_1 very far down from the line L (see [BM13, Appendix A]). In this setting, the tropical curves we are enumerating naturally contain two parts: one corresponding to the face σ_1, and one corresponding to the two upper faces σ_2 and σ_3. The former correspond to curves in $\mathbb{T}\Sigma_{n+2}$, and the latter to curves in $\mathbb{T}\Sigma_n$. This behavior is parallel to Kodaira's deformation in the complex world as we discuss in more detail in Section 4.3.1.4.

Note that this ambient tropical surface X, different from the "usual" $\mathbb{R}^2$, is imposed by our strategy based on deformation of Hirzebruch surfaces. Indeed the presence of a unique exceptional curve on Σ_n, with different self-intersection for different values of n, is an obstruction for any deformation of Σ_n to Σ_{n+2} to be toric. As a consequence, to model this deformation tropically one needs to make use of *tropical modifications* (see [Mik06]) of tropical Hirzebruch surfaces. In particular the tropical model of $\mathbb{T}\Sigma_n$ involved in this deformation is no longer a tropical toric compactification of $\mathbb{R}^2$.

The two main ingredients for our proof of Theorem 4.36 are the enumeration of tropical curves in the tropical surface X and its relation, via Correspondence Theorems 4.40 and [BM13, Theorem 3.17], to the enumeration of algebraic curves in Hirzebruch surfaces. Both aspects have not been much explored in the literature yet. Most papers about tropical enumerative geometry deal with the case of tropical curves in $\mathbb{R}^n$. In the case of curves in X, the tropical inclusion is more subtle than the set theoretic one, as it has already been noticed by several people (see for example [Vig09, BMb, BS14]). In other words a tropical curve C might be set-theoretically contained in X without being tropically contained in X. One has to require extra conditions for this latter inclusion to hold. This phenomenon quite complicates the enumerative geometry of general tropical varieties. However, the surface X we are interested in here is simple enough so that only one extra condition, the so-called *Riemann-Hurwitz condition*, suffices to rule out parasitic tropical curves. The necessity of this condition has been observed earlier by Mikhalkin and the author ([BMb], see also [BBM11], [ABBR13a], and [ABBR13b]). In the other direction, we prove a correspondence theorem for tropical curves in X by reducing to correspondence theorems for tropical curves in $\mathbb{R}^2$ with a multiple point at some fixed point on the toric boundary. Thanks to this reduction, our proofs goes by a mild adaptation of the proofs of Correspondence Theorems from [Mik05] and [Shu12], in the framework of phase-tropical morphisms given in [BBM14].

4.3.1.4 Background, context, and difficulties

As in [AB01], one may try to relate enumerative invariants of Σ_n and Σ_{n+2k} by studying the limit of curves when Σ_n deforms to Σ_{n+2k}. However this analysis gets much more complicated when $n > 0$. The main reason for that is that as soon as $n > 0$ both surfaces Σ_n and Σ_{n+2k} contain an exceptional curve, say E_n and E_{n+2k}, and that curves in Σ_n degenerate to curves in Σ_{n+2k} with singularities at k fixed points on E_{n+2k}. Those latter points may be thought as the "virtual intersection points of E_n and E_{n+2k}" defined by the chosen deformation from Σ_n to Σ_{n+2k}.

Let us explain in details the origin of the complications arising when $n > 0$.

First let us decompose the Kodaira deformation of Σ_n to Σ_{n+2k} into two steps: a deformation of Σ_n to the normal cone of a curve of bidegree $(1, k)$, followed by the blow-down of the Σ_n copy in the special fiber. More precisely, let V be a non-singular curve of bidegree $(1, k)$ in Σ_n, and let Σ be the trivial family $\Sigma_n \times \mathbb{C}$ blown-up along the curve $V \times \{0\}$. The natural projection $\pi : \Sigma \to \mathbb{C}$ defines a flat degeneration of Σ_n into the reducible surface $\pi^{-1}(0) = \Sigma_n \cup \Sigma_{n+2k}$ intersecting transversely along $V \subset \Sigma_n$ and $E_{n+2k} \subset \Sigma_{n+2k}$.

It turns out that the Σ_n copy in $\pi^{-1}(0)$ can be contracted to V by a blow down $\mathrm{bl} : \Sigma \to \Sigma'$, and that the induced projection $\pi' : \Sigma' \to \mathbb{C}$ is precisely the Kodaira deformation of Σ_n to Σ_{n+2k} (see Figure 4.5). With this picture in mind, the "virtual intersection points of E_n and E_{n+2k}" we mentioned above are now simply the intersection points of V and E_n in the Σ_n copy of the central fiber $\pi^{-1}(0) \subset \Sigma$. Relations between enumerative invariants of Σ_n and Σ_{n+2k} should now be derived from a careful analysis of how curves in Σ_n degenerate when this latter surface degenerates to $\Sigma_n \cup \Sigma_{n+2k}$.

At this point, the origin of the complications might appear more clearly with a symplectic point of view on the problem and the methods. On the level of the underlying symplectic manifolds (recall

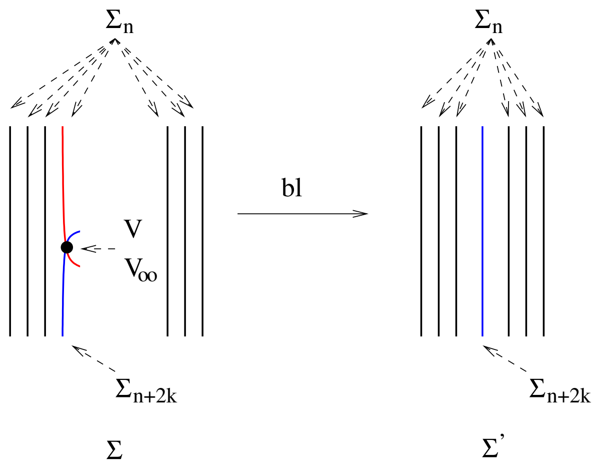


Figure 4.5: The 3-folds Σ and Σ'.

that Σ_n is a Kähler manifold), the above deformation Σ of the reducible surface $\Sigma_n \cup \Sigma_{n+2k}$ to Σ_n can be seen as a symplectic sum of Σ_n and Σ_{n+2k} glued respectively along V and E_{n+2k}. Symplectic sum formulas (e.g. [IP04, LR01, EGH00], or [Li02] in the algebraic setting) relate Gromov-Witten invariants of a symplectic sum with those of the symplectic summands, but these Gromov-Witten invariants are relative to some *smooth* symplectic hypersurface. Symplectic sum formulas provide an alternative proof of the Abramovich-Bertram-Vakil formula, and more generally express Gromov-Witten invariants of a symplectic 4-manifold in term of its Gromov-Witten invariants relative to an embedded symplectic sphere E with self-intersection $-l$. Note however that as soon as $l \geq 2$, these formulas involve enumeration of ramified coverings, which singularly complicates actual computations when $l \geq 3$. Still, one can deduce in this way a relation among enumerative invariants of Σ_1 and Σ_3 equivalent to our Theorem 4.36 specialized to the case $n = 1$. However when $n > 1$, relating enumerative invariants of Σ_n and Σ_{n+2k} requires to consider Gromov-Witten invariants relative to a *singular* symplectic curve. Indeed, the complex structure on the algebraic surface Σ_n is not generic as soon as $n > 1$ and enumerating algebraic curves on Σ_n is the same than computing Gromov-Witten invariants of the underlying symplectic manifold relative to the symplectic divisor E_n. In particular, relating enumerative invariants of Σ_n and Σ_{n+2k} using the deformation Σ can be seen as expressing Gromov-Witten invariants of Σ_n relative to E_n in terms of Gromov-Witten invariants of Σ_n relative to $E_n \cup V$ and of Σ_{n+2k} relative to E_{n+2k}. Since the curves E_n and V intersect in k points, Gromov-Witten invariants relative to a singular divisor show up naturally with the method we intend to apply. Gromov-Witten invariants relative to a singular divisor have been defined only recently (see [Ion13, Par11], or [GS13, AC11] in the algebraic setting) and we are not aware of a general symplectic sum/degeneration formula for those invariants yet.

We view our tropical approach as a tool to overcome these problems. As explained in Section 4.3.1.3, in a suitable tropicalization of the above strategy, the family Σ is replaced by the single tropical surface X, and the study of degenerations of holomorphic curves is replaced by the enumeration of tropical curves in X. We then perform this enumeration in the special case of the degeneration of Σ_n to $\Sigma_n \cup \Sigma_{n+2}$. We obtain in this way Theorem 4.36, which may be seen as such a symplectic sum/degeneration formula in some particular instance of normal crossing divisor. The case of the degeneration of Σ_n to $\Sigma_n \cup \Sigma_{n+2k}$ should also be doable tropically, but requires some additional efforts (see Section 4.3.3).

The aim of the above discussion is to replace our work in the context of current mathematical developments and to explain where the difficulties we have to deal with come from. Having said that, we formulate Theorem 4.36 in the algebraic language without referring explicitly to relative Gromov-Witten invariants. Hence our formula expresses (some) enumerative invariants of Σ_n in terms of (some) enumerative invariants of Σ_{n+2} and some finitely many simple relative enumerative invariants of Σ_n, both of these latter invariants involving curves with a prescribed very singular point.

4.3.2 Correspondence theorem for tropical curves in X

We give in this section the correspondence theorem 4.40 for tropical curves in X, which is one of the main ingredient in the proof of Theorem 4.36. As usual, We start by setting up a tropical enumerative problem in the tropical surface X.

Let us first define tropical morphisms $h : C \to X$ from a tropical curve C to the tropical surface X. As it has already been mentioned, the tropical inclusion is more subtle than the set theoretic inclusion. In the simple situation we deal with here, i.e. X is just made of three faces meeting along the line L, there is only one extra condition we have to impose on h to be tropically contained in X: h should satisfy the so-called Riemann-Hurwitz condition. This condition is based on tropical intersection theory, and we refer for example to [AR10, Sha13] for an account on this latter. More conditions that have to be required for a tropical morphism $h : C \to \widetilde{X}$ to a more general tropical surface $\widetilde{X}$ can be found in [BS14] and [GSW13], see also Section 6.2.

Definition 4.37 *Let $f : C \to \mathbb{R}^n$ be a tropical morphism with $f(C) \subset X$, and let v be a vertex of C mapped to L. The* overvalency *of v is defined by*

$$\mathrm{ov}_v := k_v - (f(C) \cdot L)_v - 2 + 2g_v,$$

where k_v is the number of edges of C adjacent to v and not mapped to L, and $(f(C) \cdot L)_v$ is the tropical intersection number of $f(C)$ and L in X at v.

The map $f : C \to X$ is a tropical morphism to X if $\mathrm{ov}_v \geq 0$ for any vertex v of C with $(f(C) \cdot L)_v > 0$.

This extra condition to be a morphism to X and not only to $\mathbb{R}^n$ is a consequence of the Riemann-Hurwitz formula in complex geometry (see [BM13, Section 5.1] for more details) and is usually referred to as the *Riemann-Hurwitz condition.*

Remark 4.38 *In fact, we need a slightly more general definition of tropical morphisms in [BM13], since C might have* degenerate edges. *However, according to Proposition [BM13, Proposition 4.2], any tropical morphisms which appears in the set $\mathbb{T}\mathcal{C}(\delta, \chi, \underline{x})$ below does not contain any degenerate edge. Hence it is safe in this brief presentation of [BM13] to ignore degenerate edges.*

We describe now a particular kind of tropical enumerative problems concerning tropical morphisms through point conditions in X, and describe properties of the tropical morphisms that are solutions. Recall that the tropical surface X is made of three 2-dimensional cells, $\sigma_1 = \{x = y \geq z\}$, $\sigma_2 = \{x = z \geq y\}$ and $\sigma_3 = \{y = z \geq x\}$, meeting along the line $L = \mathbb{R}(1, 1, 1)$.

Let δ be a Newton fan only containing vectors in X, but no vectors in L, and let $\chi \in \mathbb{Z}$. We denote by δ_i the set of directions of δ in σ_i for $i = 1, 2, 3$, and by d the intersection multiplicity of L with a tropical curves in X with Newton fan δ. For the rest of this section, we assume that any direction in δ_3 has tropical intersection multiplicity 1 with L, i.e. $d = |\delta_3|$.

Given a configuration $\underline{x}$ of $|\delta| - \chi - d$ points in $\sigma_1 \cup \sigma_2$, we denote by $\mathbb{T}\mathcal{C}(\delta, \chi, \underline{x})$ the set of all (maybe reducible) closed tropical morphisms $f : C \to X$ with Newton fan δ, with $\chi_{\mathrm{trop}}(C) = \chi$, and passing through all points in $\underline{x}$. Given a tropical morphism $f : C \to X$, we define $C_i = f^{-1}(\sigma_i)$.

Proposition 4.39 ([BM13, Proposition 4.2]) *For a generic configuration $\underline{x}$, the set $\mathbb{T}\mathcal{C}(\delta, \chi, \underline{x})$ is finite. Moreover any tropical morphism $f : C \to X$ in $\mathbb{T}\mathcal{C}(\delta, \chi, \underline{x})$ satisfies the following properties:*

1. *there is no edge e of C with $f(e) \subset L$;*
2. *any vertex v of C such that $f(v) \notin L$ is 3-valent; furthermore f is an embedding in a neighborhood of v;*
3. *the tropical curve C is explicit;*
4. *C_3 is a union of d ends of C;*
5. *for any vertex v of C such that $f(v) \in L$ one has $ov(v) = 0$, and v is adjacent to exactly one edge of C_1 and C_2;*

If $\underline{x} \subset \sigma_1$ we have in addition:

6 C_2 is a union of $\#\Delta_2$ trees.

Note that it follows from the conditions above that C_3 has exactly d_v ends adjacent to each vertex v such that $f(v) \in L$.

From now on, we consider the following Newton fan

$$\delta = \{(1, -n, 1)^a, (0, 1, 1)^{an+b}, (-1, 0, 0)^a, (0, -1, 0)^b, (0, 0, -1)^{a(n+1)+b}\},$$

in particular we have

$$\begin{aligned}
&\delta_1 = \{(0, 0, -1)^{a(n+1)+b}\},\\
&\delta_2 = \{(1, -n, 1)^a, (0, -1, 0)^b\} \text{ and}\\
&\delta_3 = \{(0, 1, 1)^{an+b}, (-1, 0, 0)^a\}.
\end{aligned}$$

Note that here $d = (n+1)a + b$, and $|\delta_3| = d$. In particular we are in the situation covered by Proposition 4.39. Let us choose an integer $\chi \in \mathbb{Z}$, and a generic configuration $\underline{x}$ of $|\delta| - \chi - d$ points in $\sigma_1 \cup \sigma_2$.

Given an element $f : C \to X$ of $\mathbb{T}\mathcal{C}(\delta, \chi, \underline{x})$, we denote by $\mathrm{Vert}_L(C)$ (resp. $\mathrm{Vert}_{\sigma_i}(C)$) the set of vertices of C mapped to L (resp. $\sigma_i \setminus L$). If $v \in \mathrm{Vert}_{\sigma_1}(C) \cup \mathrm{Vert}_{\sigma_2}(C)$, then it follows from Proposition 4.39 that $\mathrm{val}(v) = 3$. Given $v \in \mathrm{Vert}_{\sigma_i}(C)$, we choose any two of its adjacent edges $e_{v,1}$ and $e_{v,2}$. Note that we have $v_{f,e_{v,j}} = (a_{v,j}, a_{v,j}, b_{v,j})$ if $i = 1$, and $v_{f,e_{v,j}} = (a_{v,j}, b_{v,j}, a_{v,j})$ if $i = 2$ for some $a_{v,j}$ and $b_{v,j}$. A vertex $v \in \mathrm{Vert}_L(C)$ is adjacent to d_v ends mapped to σ_3, say k_v ends with direction $(-1, 0, 0)$ and l_v ends with direction $(0, 1, 1)$ (pointing away from L). Note that $k_v + l_v = d_v$.

We define the multiplicity of a vertex $v \in \mathrm{Vert}_{\sigma_i}(C)$ as

$$\mu(v) = \left|\det \begin{pmatrix} a_{v,1} & a_{v,2} \\ b_{v,1} & b_{v,2} \end{pmatrix}\right|.$$

We define the multiplicity of a vertex $v \in \mathrm{Vert}_L(C)$ as

$$\mu(v) = \binom{k_v + l_v}{k_v}.$$

We define the multiplicity of f as

$$\mu(f) = \prod_{v \in \mathrm{Vert}(C)} \mu_v.$$

We also define the two following number

$$\mathbb{T}N_\chi(\delta, \underline{x}) = \sum_{f \in \mathbb{T}\mathcal{C}(\delta, \chi, \underline{x})} \mu(f).$$

Next Theorem is one of the main results of [BM13].

Theorem 4.40 ([BM13, Theorem 4.13]) *Let δ, χ, and $\underline{x}$ be as above, and let*

$$\delta_0 = \{(1,n)^a, (0,-1)^{an+b}, (-1,0)^a, (0,1)^b\}.$$

Then we have

$$\mathbb{T}N_\chi(\delta, \underline{x}) = N_{2\chi}(\delta_0).$$

A consequence of Theorem 4.40 is that the numbers $\mathbb{T}N_\chi(\delta, \underline{x})$ do not depend on the choice of $\underline{x}$, as long as $\underline{x} \subset \sigma_1 \cup \sigma_2$ is generic. As mentioned earlier, our proof of Theorem 4.40 goes by a mild adaptation of the proofs of Correspondence Theorems from [Mik05] and [Shu12], in the framework of phase-tropical morphisms given in [BBM14].

Remark 4.41 *One can adapt Theorem 4.40 to the enumeration of irreducible curves, see [BM13, Theorem 4.13].*

4.3.2.1 Element of the proof of Theorem 4.36

As explained in Section 4.3.1.3, the proof of Theorem 4.36 can be deduced by applying Theorem 4.40 with a configuration $\underline{x} \subset \sigma_1$ such that points in $\underline{x}$ have very low z-coordinate compared to the x and y-coordinates. In this case, tropical morphisms in $\mathbb{T}\mathcal{C}(\delta, \chi, \underline{x})$ hit the line L in a very particular way. Next lemma is the last key step in the proof of Theorem 4.36 .

Lemma 4.42 ([BM13, Lemma 4.15]) *Suppose that $\underline{x} \subset \sigma_1$ and that the points in $\underline{x}$ have very low z-coordinate compared to the x and y-coordinates. Let $f : C \to X$ be an element of $\mathbb{T}\mathcal{C}(\delta, \chi, \underline{x})$, and v be a vertex of C mapped to the line L. Then all possibilities of how f can look like in a neighborhood of f are depicted in Figure 4.6.*

We deduce from Lemma 4.42 that all elements of $\mathbb{T}\mathcal{C}(\delta, \chi, \underline{x})$ can be recovered only out of the set $\mathbb{T}\mathcal{C}'(\delta, \chi, \underline{x}) = \{f_{|C_1} \mid f \in \mathbb{T}\mathcal{C}(\delta, \chi, \underline{x})\}$. Elements of this latter set are responsible for the $N_{\chi'}$ terms in the formula of Theorem 4.36. The other terms come from the number of possibilities to complete an element of $\mathbb{T}\mathcal{C}'(\delta, \chi, \underline{x})$ into an element of $\mathbb{T}\mathcal{C}(\delta, \chi, \underline{x})$, and from [BM13, Correspondence Theorem 3.17].

4.3.3 Further comments

We discuss some possible extensions of the results and methods presented in this section.

Although Theorem 4.40 assumes that configurations $\underline{x}$ are contained in the two faces σ_1 and σ_2, Theorem 4.36 is obtained just by considering configurations $\underline{x}$ contained in σ_1. It should be possible to generalize Theorem 4.36 for any configuration $\underline{x} \subset \sigma_1 \cup \sigma_2$. This would also require to enlarge the family of $(1,1)$-relative invariants considered here.

It would also be interesting to relate enumerative invariants of Σ_n and Σ_{n+2k} when $k \geq 2$. One possible way would be to study enumerative geometry of the tropical surface X_k in $\mathbb{R}^3$ given by the polynomial "$x^k + y + z$". In this case the assumption we made in Section 4.3.2, i.e. that $d = \#\Delta_i$ for some i, fails. In particular the study of enumerative geometry of X_k requires more care for $k \geq 2$.

Related to the previous paragraph is the question of determining the multiplicity of a tropical morphism to X. In general, the multiplicity of a vertex tropically mapped to the line L should be expressed in terms of *triple Hurwitz numbers* weighted by some binomials coefficients. Although all those numbers are in principle computable, no nice general formula is known yet. In the particular case treated here, the corresponding Hurwitz numbers are very simple: it is the number of rational maps $\mathbb{C}P^1 \to \mathbb{C}P^1$ of degree d with a prescribed pole and zero of maximal order. In particular we could perform easily all computations keeping hidden the Hurwitz numbers aspect. However for more general enumerative problems in X, these Hurwitz numbers will show up naturally.

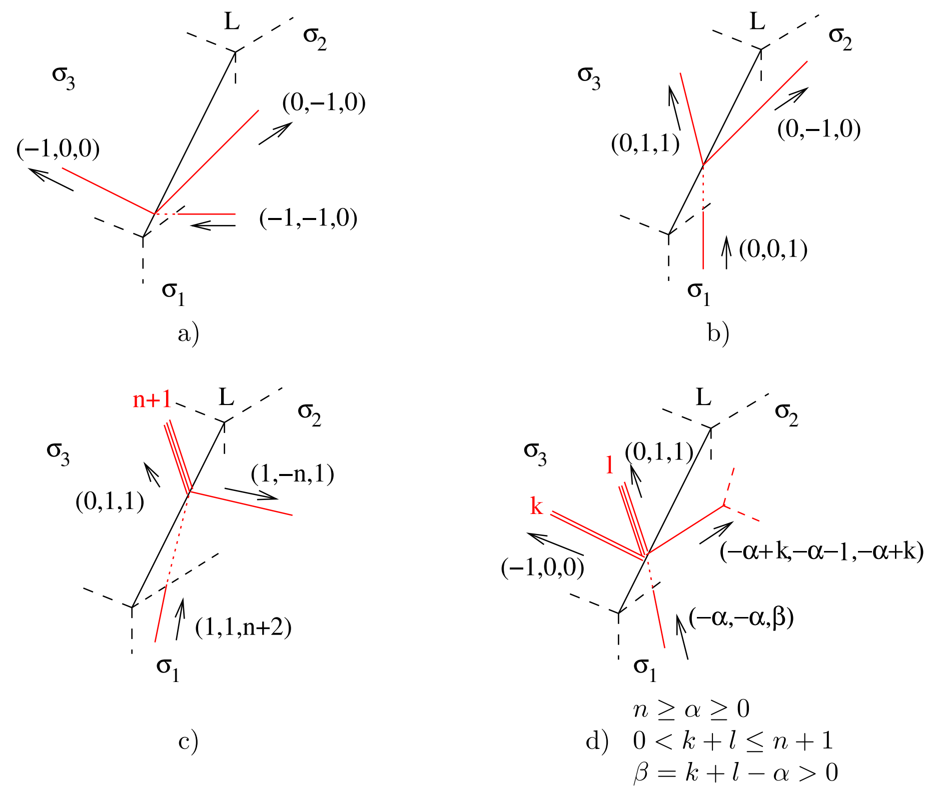


Figure 4.6: Four ways to hit L.

More generally, the study of enumerative geometry of general tropical surfaces, or even tropical varieties of any dimension, is of great interest. So far, little is known about this problem. In this case all Hurwitz numbers will come into the game, not only the triple ones mentioned above.

Chapter 5

Tropical Hurwitz and characteristic numbers

In this chapter we compute Hurwitz numbers, and characteristic numbers of $\mathbb{C}P^2$ via tropical geometry and floor diagrams.

Hurwitz numbers count ramified coverings of a compact closed oriented surface S having a given set of critical values with given ramification profiles. Characteristic numbers of $\mathbb{C}P^2$ count plane curves subject to some incidence and tangency conditions. On the level of maps, tangency conditions are naturally interpreted as ramification conditions, hence Hurwitz numbers can be thought of as 1-dimensional characteristic numbers. One of the main results of this chapter is an expression of genus 0 characteristic numbers of $\mathbb{C}P^2$ in terms of Hurwitz numbers of $\mathbb{C}P^1$ (Theorem 5.38).

This result is obtained first by replacing the computation of Hurwitz and characteristic numbers to the computation of their tropical analogues (Theorems 5.11 and 5.22), and then to apply the floor decomposition technique developed in Chapter 3 (Theorem 5.38). Note that Theorem 5.38 involves not only closed Hurwitz numbers but also *open* ones, which enumerate surfaces possibly with boundary. We introduce these open Hurwitz numbers in Section 5.1.

We stress that we are not aware of any symplectic nor algebraic degeneration formula that allows computations of characteristic numbers. In particular, the floor decomposition technique provides a new insight on these characteristic numbers and their relation to Hurwitz numbers.

All results presented in this chapter were obtained in collaboration with B. Bertrand and G. Mikhalkin. Section 5.1 and Section 5.2 respectively summarize the papers [BBM11] and [BBM14].

5.1 Open Hurwitz numbers

Here we give a tropical interpretation of Hurwitz numbers extending the one discovered in [CJM10]. In addition we treat a generalization of Hurwitz numbers for surfaces with boundary which we call open Hurwitz numbers. This work is motivated by Section 5.2 where the computation of genus 0 characteristic numbers of $\mathbb{C}P^2$ is reduced to enumeration of floor diagrams and computation of genus 0 open Hurwitz numbers.

Hurwitz numbers are defined as the (weighted) number of ramified coverings of a compact closed oriented surface S of a given genus having a given set of critical values with given ramification profiles. These numbers have a long history, and have connections to many areas of mathematics, among which we can mention algebraic geometry, topology, combinatoric, and representation theory (see [LZ04] for example).

In Section 5.1.1, we define a slight generalization of these numbers that we call *open Hurwitz numbers*. To do so, we fix not only points on S and ramification profiles, but also a collection of disjoint circles on S and the behavior of the coverings above each of these circles. Note that the total space of the ramified coverings we consider now is allowed to have boundary components. We extend Definition 2.4 in Section 5.1.2 to tropical morphisms between to tropical curves. In particular, such morphisms have to satisfy a *Riemann-Hurwitz condition* analogous to the one from Definition 4.37. In Section 5.1.3, we define tropical open Hurwitz numbers, and establish a correspondence with their complex counterpart (Theorem 5.11). Theorem 5.11 can be interpreted as a translation in the tropical language of the computation of open Hurwitz numbers by cutting S along a collection of circles.

5.1.1 Classical open Hurwitz numbers

The data we need to define open Hurwitz numbers are

- S an oriented connected closed compact surface;
- $\mathcal{L}$ a finite collection of disjoint smoothly embedded circles in S; we denote by $\overset{\circ}{S}$ the surface $S \setminus \left(\bigcup_{L\in\mathcal{L}} L\right)$;
- $\mathcal{P}$ be a finite collection of points in $\overset{\circ}{S}$;
- a number $\delta(S') \in \mathbb{Z}_{\geq 0}$ associated to each connected component S' of $\overset{\circ}{S}$; to each circle $L \in \mathcal{L}$ which is in the closure of the connected components S' and S'' of $\overset{\circ}{S}$ (note that we may have $S' = S''$), we associate the number $\gamma(L) = |\delta(S') - \delta(S'')|$;
- a partition $\mu(p)$ of $\delta(S')$ associated to each point $p \in \mathcal{P}$, where S' is the connected component of $\overset{\circ}{S}$ containing p;
- a partition $\mu(L)$ of $\gamma(L)$ associated to each circle $L \in \mathcal{L}$.

We identify two continuous maps $f : S_1 \to S$ and $f' : S_1' \to S$ if there exists a homeomorphism $\Phi : S_1 \to S_1'$ such that $f' \circ \Phi = f$.

Now let us denote by $\mathcal{S}$ the set of all (equivalence class of) ramified coverings $f : S_1 \to S$ where

- S_1 is a connected compact oriented surface with boundary;
- $f(\partial S_1) \subset \cup_{L\in\mathcal{L}} L$;
- f is unramified over $S \setminus \mathcal{P}$;
- $f_{|f^{-1}(S')}$ has degree $\delta(S')$ for each connected component S' of $\overset{\circ}{S}$;
- for each point $p \in \mathcal{P}$, if $\mu(p) = (\lambda_1, \ldots, \lambda_k)$, then $f^{-1}(p)$ contains exactly k points, denoted by $q_1, \ldots, q_k$, and f has ramification index λ_i at q_i;
- for each circle $L \in \mathcal{L}$, if $\mu(L) = (\lambda_1, \ldots, \lambda_k)$, then $f^{-1}(L)$ contains exactly k boundary components of S_1, denoted by $c_1, \ldots, c_k$, and $f_{|c_i} : c_i \to L$ is an unramified covering of degree λ_i.

Note that the Riemann-Hurwitz formula gives us

$$\chi(S_1) = \sum_{S'} \delta(S')\,(\chi(S') - |\mathcal{P} \cap S'|) + \sum_{p \in \mathcal{P}} l(\mu(p))$$

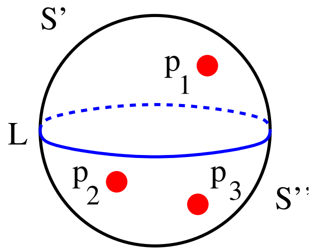
where $l(\mu(p))$ is the length of the partition $\mu(p)$ (i.e. its cardinality as a multi-set of natural numbers).

Definition 5.1 ([BBM11, Definition 1.1]) *The open Hurwitz number $H_S^\delta(\mathcal{L}, \mathcal{P}, \mu)$ is defined as*

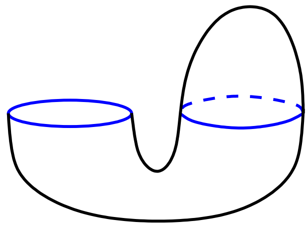
$$H_S^\delta(\mathcal{L}, \mathcal{P}, \mu) = \sum_{f \in \mathcal{S}} \frac{1}{|Aut(f)|}$$

where $Aut(f)$ is the set of automorphisms of f.

The open Hurwitz number $H_S^\delta(\mathcal{L}, \mathcal{P}, \mu)$ is a topological invariant that depends only on the topological type of the triple $(S, \overset{\circ}{S}, \mathcal{P})$, and the functions δ and μ.



a)



b)

Figure 5.1:

Example 5.2 *Let S be the sphere, L be a circle in S, and p_1, p_2, and p_3 three points distributed in S as depicted in figure 5.1a. Let us also denote by S' and S'' the two connected components of $S \setminus L$ according to Figure 5.1a. We define $\mu(p_1) = \mu(p_2) = \mu(p_3) = (2)$. The table below lists some values of $H_S^\delta(\mathcal{L}, \mathcal{P}, \mu)$ easily computable by hand. Figure 5.1b depicts the only map to be taken into account in the second row of the table.*

$\delta(S)$	$\delta(S')$	$\delta(S'')$	$\mathcal{L}$	$\mu(\mathcal{L})$	$\mathcal{P}$	$H_S^\delta(\mathcal{L}, \mathcal{P}, \mu)$
2			$\emptyset$		$\{p_2, p_3\}$	$\frac{1}{2}$
	1	2	$\{L\}$	(1)	$\{p_2, p_3\}$	1
	1	2	$\{L\}$	(1)	$\{p_1, p_3\}$	0
	0	2	$\{L\}$	$(1,1)$	$\{p_2, p_3\}$	$\frac{1}{2}$

In the special case where $\mathcal{L}$ is empty, we recover the usual Hurwitz numbers. In particular δ is just a positive integer number, the degree of the maps we are counting and that we denote by d. We simply denote Hurwitz numbers by $H_S^d(\mathcal{P}, \mu)$.

The problem of computing $H_S^d(\mathcal{P}, \mu)$ is equivalent to counting the number of some group morphisms from the fundamental group of a punctured surface to the symmetric group $\mathfrak{S}_d$. Hence, Hurwitz numbers are theoretically computed by Frobenius's Formula (see for example [LZ04, Appendix, Theorems A.1.9 and A.1.10]). As an example of computation, let us mention the following nice closed formula due to Hurwitz.

Proposition 5.3 (Hurwitz) *If $\mu(p) = (2, 1, \dots, 1)$ for all p in $\mathcal{P}$ except for one point p_0 for which we have $\mu(p_0) = (\lambda_1, \dots, \lambda_k)$, then*

$$H^d_{S^2}(\mathcal{P}, \mu) = \frac{d^{k-3}(d+k-2)!}{|Aut(\mu(p_0))|} \prod_{i=1}^{k} \frac{\lambda_i^{\lambda_i}}{\lambda_i!}$$

5.1.2 Tropical morphisms between tropical curves

Here we extend Definition 2.4 to tropical morphisms between two tropical curves. Given a tropical curve C, we denote by $\mathrm{Vert}^0(C)$ the set of its vertices which are not 1-valent, and by $\mathrm{Edge}^0(C)$ the set of its edges which are not adjacent to a 1-valent vertex. In order to avoid unnecessary formal complications, we treat points on edges of a tropical curve as 2-valent vertices of genus 0 in the next definition.

Definition 5.4 ([BBM11, Definition 2.2]) *A continuous map $f : C_1 \to C$ between two tropical curves C_1 and C is a tropical morphism if*

- *$f^{-1}(\partial C) \subset \partial C_1$;*
- *for any edge e of C_1, the set $f(e)$ is contained in an edge of C, and the restriction $f_{|e}$ is a dilatation by some integer $w_{f,e} > 0$;*
- *for any vertex v in $Vert^0(C_1)$, if we denote by $e_1, \dots, e_k$ the edges of C adjacent to $f(v)$, and by $e'_{i,1}, \dots, e'_{i,l_i}$ the edges of C_1 adjacent to v such that $f(e'_{i,j}) \subset e_i$, then one has the balancing condition*

$$\forall i, j, \quad \sum_{l=1}^{l_i} w_{f,e'_{i,l}} = \sum_{l=1}^{l_j} w_{f,e'_{j,l}} \tag{5.1}$$

  This number is called the local degree of f at v, and is denoted by $d_{f,v}$;
- *for any vertex v in $Vert^0(C_1)$, if l (resp. k) denotes the number of edges e of C (resp. of C_1) adjacent to $f(v)$ (resp. to v) and $k > 0$ then one has the Riemann-Hurwitz condition*

$$k - d_{f,v}(2g_{f(v)} + l - 2) + 2g_v - 2 \geq 0 \tag{5.2}$$

  This number is denoted by $r_{f,v}$.

The Riemann-Hurwitz condition in the previous definition is of course analogous to the one from Definition 4.37, and comes from the classical Riemann-Hurwitz Theorem: if S_1 is a genus g_v oriented surface with k punctures, S is a genus $g_{f(v)}$ oriented surface with l punctures, and $h : S_1 \to S$ is a ramified covering of degree $d_{f,v}$, then the left hand side of inequality (5.2) is the sum of the ramification index of all points of S_1. In particular, it is non-negative.

The integer $w_{f,e}$ is called the *weight of the edge e with respect to f*. When no confusion is possible, we will speak about the weight of an edge, without referring to the morphism f. Note that the length of edges of C and the weights of edges of C_1 determine the length of edges of C_1.

Remark 5.5 *This is again a simplified definition of tropical morphisms, in principle one should allow C to have edges of weight 0. We refer to [BBM11] for more details.*

Example 5.6 *We depicted in Figure 5.2a (resp. 5.2b) a tropical morphism from a rational tropical curve with one boundary component (resp. a rational closed tropical curve) to a rational curve with four ends. Three edges have weight 2 with respect to f.*

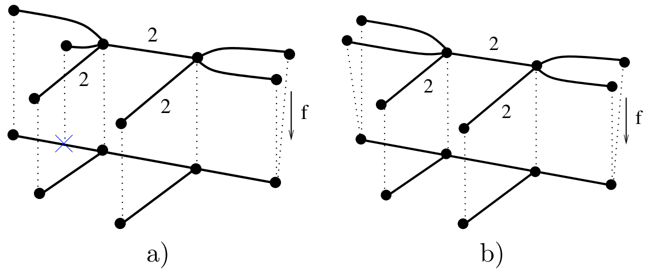


Figure 5.2: Two tropical morphisms.

The sum of all local degrees of elements in $f^{-1}(v)$ is a locally constant function on $C \setminus f(\partial C_1)$; if C' is a connected component of $C \setminus f(\partial C_1)$ then this sum over a point of C' is called the *degree* of f over C'. In the example of the morphism from Figure 5.2a, the set $C \setminus f(\partial C_1)$ has two connected components and f has degree 1 and 2 over them.

Definition 5.7 *Let $f : C_1 \to C$ be a tropical morphism. A vertex v of C_1 is called a ramification point of f if it is either a vertex $v \in Vert^0(C_1)$ with $r_{f,v} > 0$, or a vertex $v \in Vert^\infty(C_1)$ adjacent to an edge e with $w_{f,e} > 1$. If $p \in C$ is such that $f^{-1}(p)$ does not contain any ramification component of f, we say that f is unramified over p.*

Let $\nu = (\lambda_1, \dots, \lambda_l)$ be an unordered l-tuple of positive integer numbers. We say that the map f has ramification profile ν over $v \in Vert^\infty(C)$ if $f^{-1}(v) = \{v_1, \dots, v_l\}$ where $v_i \in Vert^\infty(C_1)$ is adjacent to an edge of weight λ_i.

As usual, we identify two tropical morphisms $f : C_1 \to C$ and $f' : C_1' :\to C$ if there exists a tropical isomorphisms $\Phi : C_1 \to C_1'$ such that $f' \circ \Phi = f$.

5.1.3 Tropical open Hurwitz numbers

Similarly to section 5.1.1, we start with the following data

- C a closed explicit tropical curve with $\text{Vert}^0(C) \neq \emptyset$;
- $\mathcal{R}$ a finite collection of points in $C \setminus \text{Vert}(C)$ such that any connected component of the set $C \setminus \mathcal{R}$, denoted by $\overset{\circ}{C}$, contains a vertex of C;
- $\mathcal{Q}$ be a finite collection of points in $\text{Vert}^\infty(C)$;
- a number $\delta(C') \in \mathbb{Z}_{\geq 0}$ associated to each connected component C' of $\overset{\circ}{C}$; to each point $q \in \mathcal{R}$ which is in the closure of the connected components C' and C'' of $\overset{\circ}{C}$, we associate the number $\gamma(q) = |\delta(C') - \delta(C'')|$;
- a partition $\nu(q)$ of $\delta(C')$ associated to each point $q \in \mathcal{Q}$, where C' is the connected component of $\overset{\circ}{C}$ containing q;
- a partition $\nu(q)$ of $\gamma(q)$ associated to each point $q \in \mathcal{R}$.

We denote by $\mathcal{C}^{\mathbb{T}}$ the set of all tropical morphisms $f : C_1 \to C$ such that

- C_1 is an irreducible tropical curve with boundary;

- $f(\partial C_1) \subset \mathcal{R}$;
- f is unramified over $C \setminus \mathcal{Q}$;
- $f_{|f^{-1}(C')}$ has degree $\delta(C')$ for each connected component C' of $\overset{\circ}{C}$;
- for each point $q \in \mathcal{Q}$, the map f has ramification profile $\nu(q)$ over q;
- for each point $q \in \mathcal{R}$, if $\nu(q) = (\lambda_1, \dots, \lambda_k)$, the set $f^{-1}(q)$ contains exactly k boundary components of C_1, denoted by $c_1, \dots, c_k$, and c_i is adjacent to an edge of C_1 of weight λ_i.

Note that the set $\mathcal{C}^{\mathbb{T}}$ is finite. As usual in tropical geometry, a tropical morphism $f : C_1 \to C$ in $\mathcal{C}^{\mathbb{T}}$ should be counted with some multiplicity. Given $v \in \mathrm{Vert}^0(C)$ a such that $f(v)$ is adjacent to the edges $e_1, \dots, e_{k_v}$ of C, we choose a configuration $\mathcal{P}' = \{p'_1, \dots, p'_{k_v}\}$ of k_v points on the sphere S^2, and we define $\mu'(p'_i)$ as the partition of $d_{f,v}$ defined by f at v above the edge e_i (cf the balancing condition (5.1)).

Definition 5.8 ([BBM11, Definition 2.6]) *The multiplicity of $f : C_1 \to C$ is defined as*

$$\mu(f) = \frac{1}{|Aut(f)|} \prod_{e \in Edge^0(C_1)} w_{f,e} \prod_{v \in Vert^0(C_1)} \left(\prod_{i=1}^{k_v} |Aut(\mu'(p'_i))| \right) H_{S^2}^{d_{f,v}}(\mathcal{P}', \mu')$$

The tropical open Hurwitz number $\mathbb{T}H_C^{\delta}(\mathcal{R}, \mathcal{Q}, \nu)$ is defined as

$$\mathbb{T}H_C^{\delta}(\mathcal{R}, \mathcal{Q}, \nu) = \sum_{f \in \mathcal{C}^{\mathbb{T}}} \mu(f)$$

As in section 5.1.1, if $\mathcal{R} = \emptyset$ then δ is a number denoted by d, and we denote by $\mathbb{T}H_C^{d}(\mathcal{Q}, \nu)$ the corresponding tropical (closed) Hurwitz number.

Example 5.9 *Let $h : C_1 \to C$ be the tropical morphism depicted in figure 5.2a. It is the tropical analog of the map considered in figure 5.1. Let q_1 be the image of the boundary component of C_1, and q_2 and q_3 be the leaves of C which are image of a leaf of C_1 adjacent to an edge of weight 2. We denote by C' (resp. C'') the connected component of $C \setminus \{q_1\}$ which does not contain (resp. contains) q_2 and q_3, and we define $\delta(C') = 1$, $\delta(C'') = 2$, $\nu(q_1) = (1)$, and $\nu(q_2) = \nu(q_3) = (2)$. To compute $\mathbb{T}H_C^{\delta}(\mathcal{R}, \mathcal{Q}, \nu)$, the morphism of figure 5.2a is the only one to consider and it has multiplicity 1 so $\mathbb{T}H_C^{\delta}(\mathcal{R}, \mathcal{Q}, \nu) = 1$ (see the second row of the table in example 5.2).*

Example 5.10 *Let C be the genus 2 closed explicit tropical curve depicted in figure 5.3. We set $\mathcal{Q} = Vert^{\infty}(C)$ and $\nu(q) = (2)$ for $q \in \mathcal{Q}$. Then according to Figure 5.3, we have $\mathbb{T}H_C^{2}(\mathcal{Q}, \nu) = 8$.*

Let us relate these tropical open Hurwitz numbers to the open Hurwitz numbers we defined in section 5.1.1. Let C be a tropical curve as in definition 5.8 with the data introduced at the beginning of this subsection. Let S be an oriented connected compact closed surface whose genus is the genus of C. We choose a collection $\mathcal{L} = \{L_q\}_{q \in \mathcal{R}}$ of disjoint smoothly embedded circles in S such that there is a natural correspondence $C' \to S'_{C'}$ between the connected components of $\overset{\circ}{C}$ and $\overset{\circ}{S}$ which preserves incidence relations and such that $b_1(C') = g(C'_{S'})$. For each point $q \in \mathcal{Q}$, we choose a point $p_q \in S'_{C'}$ where C' is the connected component of $\overset{\circ}{C}$ containing q, such that $p_q \neq p_{q'}$ for $q \neq q'$ (see figure 5.4 for an example). Finally we define $\mathcal{P} = \cup_{q \in \mathcal{Q}} \{p_q\}$, $\delta(S'_{C'}) = \delta(C')$, $\mu(L_q) = \nu(q)$, and $\mu(p_q) = \nu(q)$.

Theorem 5.11 ([BBM11, Theorem 2.11]) *For any $\delta, \mathcal{R}, \mathcal{Q}$, and ν, one has*

$$\mathbb{T}H_C^{\delta}(\mathcal{R}, \mathcal{Q}, \nu) = H_S^{\delta}(\mathcal{L}, \mathcal{P}, \mu)$$

Remark 5.12 *We may allow points with ramification profile $\nu(q) = (2, 1, \dots, 1)$ in $C \setminus Vert^{\infty}(C)$, and recover in this way results from [CJM10].*

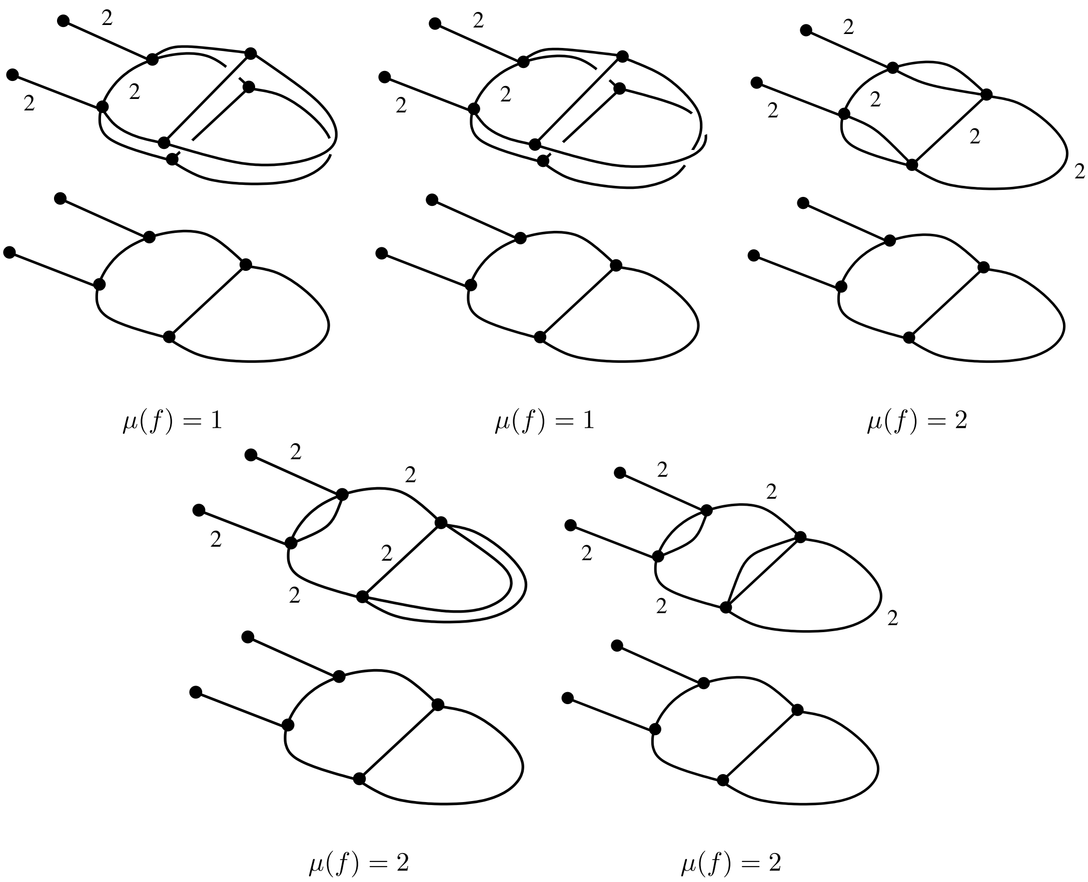


Figure 5.3:

5.2 Genus 0 characteristic numbers of the tropical projective plane

Finding the so-called characteristic numbers of the complex projective plane $\mathbb{C}P^2$ is a classical problem of enumerative geometry posed by Zeuthen more than a century ago. For a given d and g one has to find the number of degree d genus g curves that pass through a certain generic configuration of points and at the same time are tangent to a certain generic configuration of lines. The total number of points and lines in these two configurations is $3d - 1 + g$ so that the answer is a finite integer number.

In this section we translate this classical problem to the corresponding tropical enumerative problem in the case when $g = 0$, relate these two problems thanks to the Correspondence Theorem 5.22, and apply the floor decomposition technique to provide a new insight on these characteristic numbers and their relation to Hurwitz numbers in Theorem 5.38.

Correspondence Theorem 5.22, together with its generalization [BBM14, Theorem 3.12] to immersed constraints, are the first ones concerning plane curves satisfying tangency conditions to a given set of curves. In the enumeration of curves satisfying simple incidence conditions, the (finitely many) tropical curves arising as the limit of amoebas of the enumerated complex curves could be identified considering the tropical limit of embedded complex curves. This is no longer enough to identify the tropical limit of tangent curves, and one has to consider *phase-tropical curves and morphisms.* For the sake of shortness,

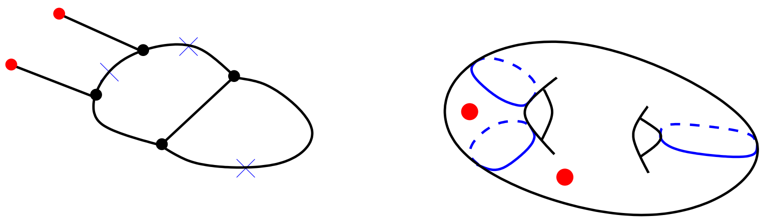


Figure 5.4: A tropical curve C is depicted on the left and the corresponding surface S on the right. The two leaves of C are elements of $\mathcal{Q}$ and correspond to elements of $\mathcal{P}$ (depicted by dots) on S while crosses on C represent points of $\mathcal{R}$ and correspond to the circles of $\mathcal{L}$ pictured on S.

we will no enter into the details of phase-tropical geometry in this typescript, we refer instead to [BBM14, Sections 6 and 7] (see also [Mik06, Section 6] or [BM13, Section 5]). We also provide few examples of applications to real enumerative geometry in Section 5.2.2.3. There is no doubt that Theorems 5.22 and [BBM14, Theorem 3.12] should lead to further results in real enumerative geometry.

Once the computation of characteristic numbers has been reduced to a tropical enumerative problem, we apply the floor decomposition technique of tropical curves to reduce again the computations to pure combinatoric. Since the computation of characteristic numbers of $\mathbb{C}P^2$ generalizes the computation of Gromov-Witten invariants of $\mathbb{C}P^2$, the floor diagrams defined in Section 5.2.3 are a generalization of those introduced in Section 3.2.1 in the case of the projective plane. Recall that the floor decomposition technique allows one to solve an enumerative problem by induction on the dimension of the ambient space, i.e. to reduce enumerative problems in $\mathbb{C}P^n$ to enumerative problems in $\mathbb{C}P^{n-1}$. In the case under consideration, the enumerative problem we are concerned with is to count curves which interpolate a given configuration of points and are tangent to a given set of curves. On the level of maps, tangency conditions are naturally interpreted as ramification conditions. In particular, the 1-dimensional analogues of characteristic numbers are Hurwitz numbers, Hence floor diagrams express characteristic numbers of $\mathbb{C}P^2$ in terms of Hurwitz numbers. Surprisingly, not only closed Hurwitz numbers appear in this expression, but also open ones. Computations of characteristic numbers of $\mathbb{C}P^2$ performed in [Pan99], [Vak01], and [GKP02], were done by induction on the degree of the enumerated curves. To our knowledge, this is the first time that characteristic numbers are expressed in terms of their analogue in dimension 1, i.e. in terms of (open) Hurwitz numbers.

We define classical characteristic numbers of $\mathbb{C}P^2$, and provide some examples, in Section 5.2.1. We settle the corresponding tropical problem and state our correspondence theorem in Section 5.2.2. A floor diagrammatic computation of characteristic numbers is performed in Section 5.2.3.

5.2.1 Classical characteristic numbers

Let d, g and k be non negative integer numbers such that $g \leq \frac{(d-1)(d-2)}{2}$ and $k \leq 3d+g-1$ and $d_1, \ldots, d_{3d+g-1-k}$ be positive integer numbers. For any configurations $\mathcal{P} = \{p_1, \ldots, p_k\}$ of k points in $\mathbb{C}P^2$, and $\mathcal{L} = \{L_1, \ldots, L_{3d+g-1-k}\}$ of $3d+g-1-k$ complex non-singular algebraic curves in $\mathbb{C}P^2$ such that L_i has degree d_i, we consider the set $\mathcal{C}(d, g, \mathcal{P}, \mathcal{L})$ of holomorphic maps $f : C \to \mathbb{C}P^2$ from an irreducible non-singular complex algebraic curve of genus g, passing through all points $p_i \in \mathcal{P}$, tangent to all curves $L_i \in \mathcal{L}$, and such that $f(C)$ has degree d in $\mathbb{C}P^2$.

If the constraints $\mathcal{P}$ and $\mathcal{L}$ are chosen generically, then the set $\mathcal{C}(d, g, \mathcal{P}, \mathcal{L})$ is finite, and the characteristic number $N_{d,g}(k; d_1, \ldots, d_{3d+g-1-k})$ is defined as

$$N_{d,g}(k; d_1, \ldots, d_{3d+g-1-k}) = \sum_{f \in \mathcal{C}(d,g,\mathcal{P},\mathcal{L})} \frac{1}{|Aut(f)|}$$

where $Aut(f)$ is the group of automorphisms of the map $f : C \to \mathbb{C}P^2$, i.e. isomorphisms $\Phi : C \to C$ such that $f \circ \Phi = f$. It depends only on d, g, k and $d_1, \ldots, d_{3d+g-1-k}$ (see for example [Vak01]). In this text, we will use the shorter notation $N_{d,g}(k; d_1^{i_1}, \ldots, d_l^{i_l})$ which indicates that the integer d_j is chosen i_j times.

Characteristic numbers were considered by nineteenth century geometers among which S. Maillard ([Mai71]) who computed them in degree 3, H. Zeuthen ([Zeu73]) who did third and fourth degree cases, and H. Schubert ([Sch79]). Modern mathematicians confirmed and extended their predecessor's results thanks in particular to intersection theory. P. Aluffi computed, for instance, all characteristic numbers for plane cubics and some of them for plane quartics (see [Alu88], [Alu90], [Alu91] and [Alu92]), and R. Vakil completed to confirm Zeuthen's computation of all characteristic numbers of plane quartics in [Vak99]. R. Pandharipande computed characteristic numbers in the rational case in [Pan99], Vakil achieved the genus 1 case in [Vak01], and T. Graber, J. Kock and Pandharipande computed genus 2 characteristic numbers of plane curves in [GKP02]. Let us describe the characteristic numbers in some special instances.

Example 5.13 *The number $N_{d,g}(3d-1+g)$ is the usual Gromov-Witten invariant of degree d and genus g of $\mathbb{C}P^2$.*

Example 5.14 *The numbers $N_{2,0}(5)$, $N_{2,0}(4;1)$, and $N_{2,0}(3;1^2)$ are easy to compute by hand, and thanks to projective duality we have*

$$N_{2,0}(k; 1^{5-k}) = N_{2,0}(5-k; 1^k) = 2^k \quad for\ 0 \leq k \leq 2$$

Example 5.15 *All characteristic numbers $N_{3,0}(k; 1^{8-k})$ for rational cubic curves have been computed by Zeuthen ([Zeu72]) and confirmed by Aluffi ([Alu91]). We sum up part of their results in the following table.*

k	8	7	6	5	4	3	2	1	0
$N_{3,0}(k;1^{8-k})$	12	36	100	240	480	712	756	600	400

Example 5.16 *The number $N_{2,0}(0; 2^5)$ has been computed independently by Chasles ([Cha64]) and De Jonquiere. More than one century later, Ronga, Tognoli, and Vust showed in [RTV97] that it is possible to choose 5 real conics in such a way that all conics tangent to these 5 conics are real. See also [Sot] and [Ghy08] for a historical account and digression on this subject. See also Example 5.26 for a tropical version of the arguments from [RTV97]. We list below the numbers $N_{2,0}(k; 2^{5-k})$.*

$$N_{2,0}(4;2) = 6 \qquad N_{2,0}(3;2^2) = 36 \qquad N_{2,0}(2;2^3) = 184$$

$$N_{2,0}(1;2^4) = 816 \qquad N_{2,0}(0;2^5) = 3264$$

More generally, the characteristic numbers $N_{d,g}(k; 1^{3d-1+g-k})$ of $\mathbb{C}P^2$ determine all the numbers $N_{d,g}(k; d_1, \ldots, d_{3d-1+g-k})$. Indeed, by degenerating the non-singular curve $L_{d_{3d-1+g-k}}$ to the union of two non-singular curves of lower degrees intersecting transversely, we obtain the following formula (see for example [RTV97, Theorem 8])

$$\begin{aligned} N_{d,g}(k; d_1, \ldots, d_{3d-1+g-k}) &= 2d'_{3d-1+g-k} d''_{3d-1+g-k} N_{d,g}(k+1; d_1, \ldots, d_{3d-2+g-k}) \\ &\quad + N_{d,g}(k; d_1, \ldots, d'_{3d-1+g-k}) + N_{d,g}(k; d_1, \ldots, d''_{3d-1+g-k}) \end{aligned} \tag{5.3}$$

where $d_{3d-1+g-k} = d'_{3d-1+g-k} + d''_{3d-1+g-k}$.

5.2.2 Rational characteristic numbers of the tropical projective plane

5.2.2.1 Tropical pretangencies in $\mathbb{R}^2$

Here we define tangency for tropical morphisms from a tropical curve. Note that A. Dickenstein and L. Tabera also studied in [DT12] tropical tangencies but in a slightly different context, i.e. tangencies between tropical hypersurfaces instead of tropical morphisms.

Let $f : C \to \mathbb{R}^2$ be a tropical morphism, and L be a smooth tropical curve in $\mathbb{R}^2$.

Definition 5.17 *The tropical morphism f is said to be* pretangent *to L if there exists a connected component E of the set theoretic intersection of $f(C)$ and L which contains either a vertex of L or the image of a vertex of C.*

The set $E \subset \mathbb{R}^2$ is called a pretangency set *of f and L. A connected component of $f^{-1}(E) \subset C$ is called a* pretangency component *of f with L if E contains either a vertex of C or a point p such that $f(p)$ is a vertex of L.*

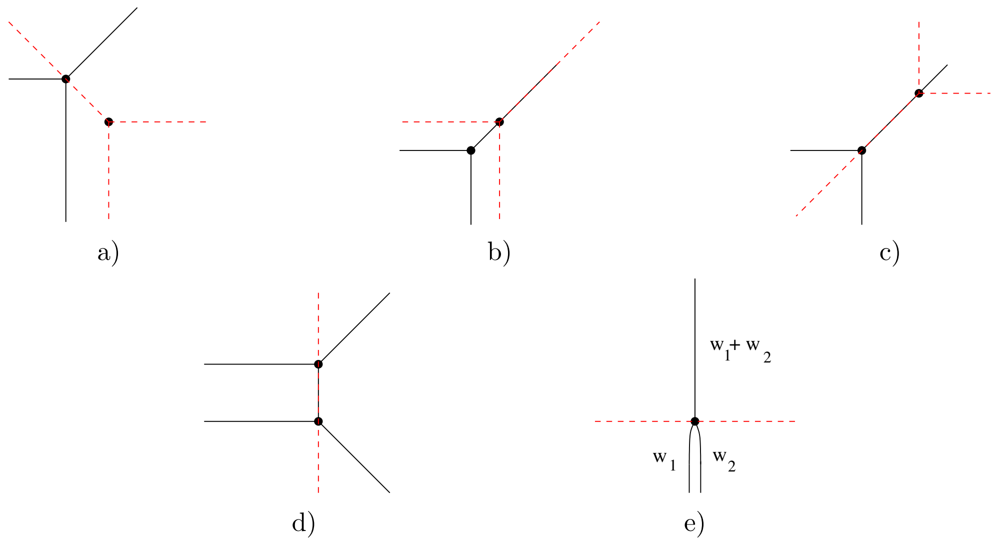


Figure 5.5: Pretangent morphisms

Example 5.18 *In Figure 5.5 we depicted several examples of the image of a morphism pretangent to a plane tropical curve represented by doted lines.*

It is clear that not any pretangency set corresponds to some classical tangency point. For example, the two tropical lines in Figure 5.5b are pretangent, but this pretangency set doesn't correspond to any tangency point between two complex algebraic lines in $\mathbb{C}P^2$. However, given any approximation of f (if one exists) and any approximation of L by algebraic curves, the accumulation set of tangency points of these approximations must lie inside the pretangency sets of f and L (see [BBM14, Section 7]).

5.2.2.2 Correspondence

As in Section 5.2.1 let $d \geq 1$, $k \geq 0$, and $d_1, \ldots, d_{3d-1-k} > 0$ be some integer numbers, and choose $\mathcal{P} = \{p_1, \ldots, p_k\}$ a set of k points in $\mathbb{R}^2$, and $\mathcal{L} = \{L_1, \ldots, L_{3d-1-k}\}$ a set of $3d - 1 - k$ non-singular tropical curves in $\mathbb{R}^2$ such that L_i has degree d_i. We denote by $\mathbb{T}\mathcal{C}(d, \mathcal{P}, \mathcal{L})$ the set of tropical morphisms

$f : C \to \mathbb{R}^2$ of degree d, where C is an irreducible closed rational tropical curve passing through all points p_i and pretangent to all curves L_i.

We suppose now that the configuration $(\mathcal{P}, \mathcal{L})$ is *generic*, in the sense of [BBM14, Definition 4.7]). Recall that as usual, the set of generic configurations is a dense open subset of the set of all configurations with a given number of points and tropical curves. Genericity of the configuration implies the following nice behavior of the curves we are counting.

Proposition 5.19 ([BBM14, Proposition 3.3]) *The set $\mathbb{T}\mathcal{C}(d, \mathcal{P}, \mathcal{L})$ is finite, and any of its element $f : C \to \mathbb{R}^2$ satisfies*

- *C is a 3-valent curve and generic at infinity;*
- *$f(\mathit{Vert}(C)) \bigcap \left(\bigcup_{L\in\mathcal{L}} \mathit{Vert}(L) \cup \mathcal{P} \right) = \emptyset$, i.e. no vertex of C is mapped to a vertex of a curve in $\mathcal{L}$ nor to a point in $\mathcal{P}$;*
- *given $p \in \mathcal{P}$, if x and x' are in $f^{-1}(p)$, then the (unique) path in C from x to x' is mapped to segment in $\mathbb{R}^2$ by f;*
- *given $L \in \mathcal{L}$, there exists a connected subgraph $\Gamma \subset C$ which contains all pretangency components of f with L, and such that $f(\Gamma)$ is a segment in $\mathbb{R}^2$.*

As usual, an element $f : C \to \mathbb{R}^2$ of $\mathbb{T}\mathcal{C}(d, \mathcal{P}, \mathcal{L})$ has to be counted with some multiplicity that we define now. Choose $3d-1$ points $x_1, \ldots, x_{3d-1}$ on C such that $f(x_i) = p_i$ if $i \leq k$, and for $i \geq k+1$, either x_i is a vertex of C mapped to L_{i-k}, or x_i is mapped to a vertex of L_{i-k}. We define $\overset{\circ}{C} = C \setminus \{x_1, \ldots, x_{3d-1}\}$.

First, we define an orientation on $\overset{\circ}{C}$. Let x be a point on an edge of $\overset{\circ}{C}$. Since C is rational, $C \setminus \{x\}$ has 2 connected components C_1 and C_2 containing respectively s_1 and s_2 ends, and $s_1 + s_2 = 3d + 2$. Moreover, since $(\mathcal{P}, \mathcal{L})$ is generic, C_1 (resp. C_2) contains $k_1 \leq s_1 - 1$ (resp. $k_2 \leq s_2 - 1$) marked points. Since $k_1 + k_2 = 3d - 1 = s_1 + s_2 - 3$, up to exchanging C_1 and C_2, we have $k_1 = s_1 - 1$ and $k_1 = s_2 - 2$. We orient $\overset{\circ}{C}$ at x from C_1 to C_2. Note that $\overset{\circ}{C}$ and its orientation depends on the choice of the points x_i, but this won't play a role in what follows.

Next, we define a multiplicity $\mu_{(\mathcal{P},\mathcal{L})}(v)$ for each vertex v in $\mathrm{Vert}(C)$. If $f(v) \notin \bigcup_{L\in\mathcal{L}} L$, then the genericity of $(\mathcal{P}, \mathcal{L})$ implies that there exist two edges $e_1, e_2 \in \mathrm{Edge}(C)$ adjacent to v and oriented toward v. We define

$$\mu_{(\mathcal{P},\mathcal{L})}(v) = |\det(u_{f,e_1}, u_{f,e_2})|.$$

If $f(v) \in L_i$, we denote by u_{L_i} the primitive integer direction of the edge of L_i containing $f(v)$. If $f(v) \in L_i \setminus \bigcup_{L\neq L_i} L$, then the genericity of $(\mathcal{P}, \mathcal{L})$ implies that there exists exactly one edge $e \in \mathrm{Edge}(C)$ oriented toward v with $u_{f,e} \neq u_{L_i}$, and we define

$$\mu_{(\mathcal{P},\mathcal{L})}(v) = |\det(u_{f,e_1}, u_{L_i})|.$$

If $f(v) \in L_i \cap L_j$, we define

$$\mu_{(\mathcal{P},\mathcal{L})}(v) = |\det(u_{L_i}, u_{L_j})|.$$

Finally, we associate a weight to all constraints in $\mathcal{P} \cup \mathcal{L}$. Given $p \in \mathcal{P}$, we denote by $\mathcal{E}(p)$ the set of edges of C which contain a point of $f^{-1}(p)$ and we define

$$w_p = \sum_{e\in\mathcal{E}(p)} w_{f,e}.$$

Given $L \in \mathcal{L}$, we denote by E_L the union of all pretangency components of f with L, by μ the cardinal of $E_L \cap \mathrm{Vert}(C)$, and by λ the number of ends of C contained in E_L. If $v \in \mathrm{Vert}(L)$, we denote by $\mathcal{E}(v)$ the set of edges of C which contain a point of $f^{-1}(v)$, and we define

$$w_L = \left(\sum_{v \in \mathrm{Vert}(L)} \sum_{e \in \mathcal{E}(v)} w_{f,e} \right) + \mu - \lambda. \tag{5.4}$$

Equivalently, we can define w_L as follows

$$w_L = \sum_{e \in \mathcal{E}(v)} w_{f,e} \qquad \text{if } f(E_L) = v \in \mathrm{Vert}(L),$$

$$w_L = \left(\sum_{v \in \mathrm{Vert}(L)} \sum_{e \in \mathcal{E}(v)} (w_{f,e} + 1) \right) + \kappa - 2b_0(E_L) \quad \text{otherwise}$$

where κ is the number of edges of $C \setminus E_L$ adjacent to a vertex of C in E_L.

Definition 5.20 *The* $(\mathcal{P}, \mathcal{L})$-multiplicity of $f : C \to \mathbb{R}^2$*, denoted by* $\mu_{(\mathcal{P},\mathcal{L})}(f)$*, is defined as*

$$\mu_{(\mathcal{P},\mathcal{L})}(f) = \frac{1}{|Aut(f)|} \prod_{q \in \mathcal{P} \cup \mathcal{L}} w_q \prod_{e \in Edge(C)} w_{f,e} \prod_{v \in Vert(C)} \mu_{(\mathcal{P},\mathcal{L})}(v).$$

Remark 5.21 *The* $(\mathcal{P}, \mathcal{L})$*-multiplicity of* f *has a more conceptual definition as an intersection number in the deformation space of* f*, see [BBM14, Section 4]. The above definition is more practical for computation purposes.*

Theorem 5.22 (Correspondence Theorem, [BBM14, Theorem 3.8 and Proposition 5.1]) *With the hypothesis above, we have*

$$N_{d,0}(k; d_1, \dots, d_{3d-1-k}) = \sum_{f \in \mathbb{T}\mathcal{C}(d,\mathcal{P},\mathcal{L})} \mu_{(\mathcal{P},\mathcal{L})}(f).$$

Remark 5.23 *Theorem 5.22 can actually be generalized to the case when the constraints are not necessarily non-singular tropical or complex curves but any immersed curves. We refer to [BBM14, Theorem 3.12] for more details.*

Example 5.24 *Figure 5.6 depicts a configuration of two lines and three points together with the only morphism in* $\mathbb{T}\mathcal{C}(2, \mathcal{P}, \mathcal{L})$*. The multiplicity of this morphism is thus* $\mu_{(\mathcal{P},\mathcal{L})}(f) = \frac{1}{2} \times 2^2 \times 2 \times 1 = 4$ *which is indeed the number of conics tangent to two lines and passing through three points provided that the configuration is generic.*

5.2.2.3 Enumeration of real curves

As already mentioned, the proof of Theorem 5.22 establishes a correspondence between phase-tropical curves and complex curves close to the tropical limit. In particular, if we choose real phases for all constraints in $(\mathcal{P}, \mathcal{L})$, it is possible to recover all real algebraic curves passing through a configuration of real points and tangent to a configuration of real lines when these points and lines are close to the tropical limit. For the sake of shortness, we refer to [BBM14, Sections 6 and 7] for basic definitions in phase-tropical geometry.

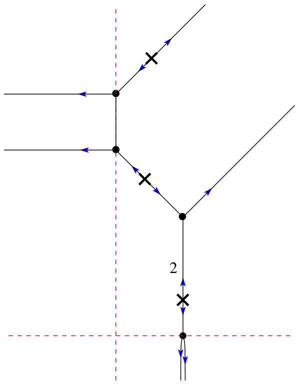


Figure 5.6: A conic tangent to two lines and passing through three points

Example 5.25 *Let us revisit Example 5.24 from a real point of view. For example, if all the three points in Example 5.24 have phase* $(1,1)$*, then the tropical curve in Figure 5.6 ensures that there exists a configuration of three points and two lines in* $\mathbb{R}P^2$ *such that all four conics passing through these points and tangent to these lines are real. On the opposite, if the middle point has phase* $(-1,1)$ *and the two other points have phase* $(1,1)$*, then there exists a corresponding configuration of three points and two lines in* $\mathbb{R}P^2$ *such that none of the four conics passing through these points and tangent to these lines are real.*

Example 5.26 *One can interpret in tropical terms the method used in [RTV97] to construct a configuration of 5 real conics such that all 3264 conics tangent to these 5 conics are real. The main step in this construction is to find 5 real lines* $L_1,\dots,L_5$ *in* $\mathbb{R}P^2$ *and 5 points* $p_1\in L_1,\dots,p_5\in L_5$ *such that for any set* $I\subset\{1,\dots,5\}$*, all the conics passing through the points* p_i*,* $i\in I$*, and tangent to the lines* L_j*,* $j\in\{1,\dots,5\}\setminus I$ *are real. As in [RTV97], let us start with the configuration depicted in Figure 5.7a, whose tropical analog is depicted in Figure 5.7b (without phase) and 5.7c (equipped with the appropriate real phases). Next, we perturb the double lines* L_i^2 *as depicted in Figure 5.7d (without phase, the cycle defined by the image is a twice a line) and 5.7e (equipped with the appropriate real phases). Then there exist 5 families of real conics converging to our 5 phase conics and producing 3264 real conics as in [RTV97].*

It would be interesting to explore the possible numbers of real conics tangent to 5 real conics, in connection to [Wel06] and [Ber08]. In particular, does there exist a configuration of 5 real conics, any one of which lying outside the others, such that exactly 32 real conics are tangent to them?

Note that once the lines and points L_i and p_i are chosen as above, arguments used in Example 5.26 also prove Proposition 3.62.

5.2.3 Floor diagrams

Now we reformulate the computation of characteristic numbers in a purely combinatorial way using floor diagrams. The strategy is the same as in Sections 3.2.1 and 3.2.3: we stretch our configuration of constraints in the vertical direction, i.e. we only consider configurations $(\mathcal{P},\mathcal{L})$ for which the difference of the y-coordinates of any two elements of the set $\mathcal{P}\cup_{L\in\mathcal{L}}\mathrm{Vert}(L)$ is very big compared to the difference of their x-coordinates. For a sufficiently stretched configuration $(\mathcal{P},\mathcal{L})$, tropical morphisms $f:C\to\mathbb{R}^2$ in $\mathbb{T}\mathcal{C}(d,\mathcal{P},\mathcal{L})$ will have a very simple decomposition into floors linked together by *shafts*, which are generalized elevators as defined in Definition 3.2.1. Marked floor diagrams and their multiplicities will encode the combinatoric of these decompositions together with the distribution of $f^{-1}(\mathcal{P})$ and the tangency components of f with elements of $\mathcal{L}$. In the case where no tangency condition is imposed, these new floor diagrams get simplified to those introduced Section 3.2.1. As mentioned in the beginning of Section 5.2,

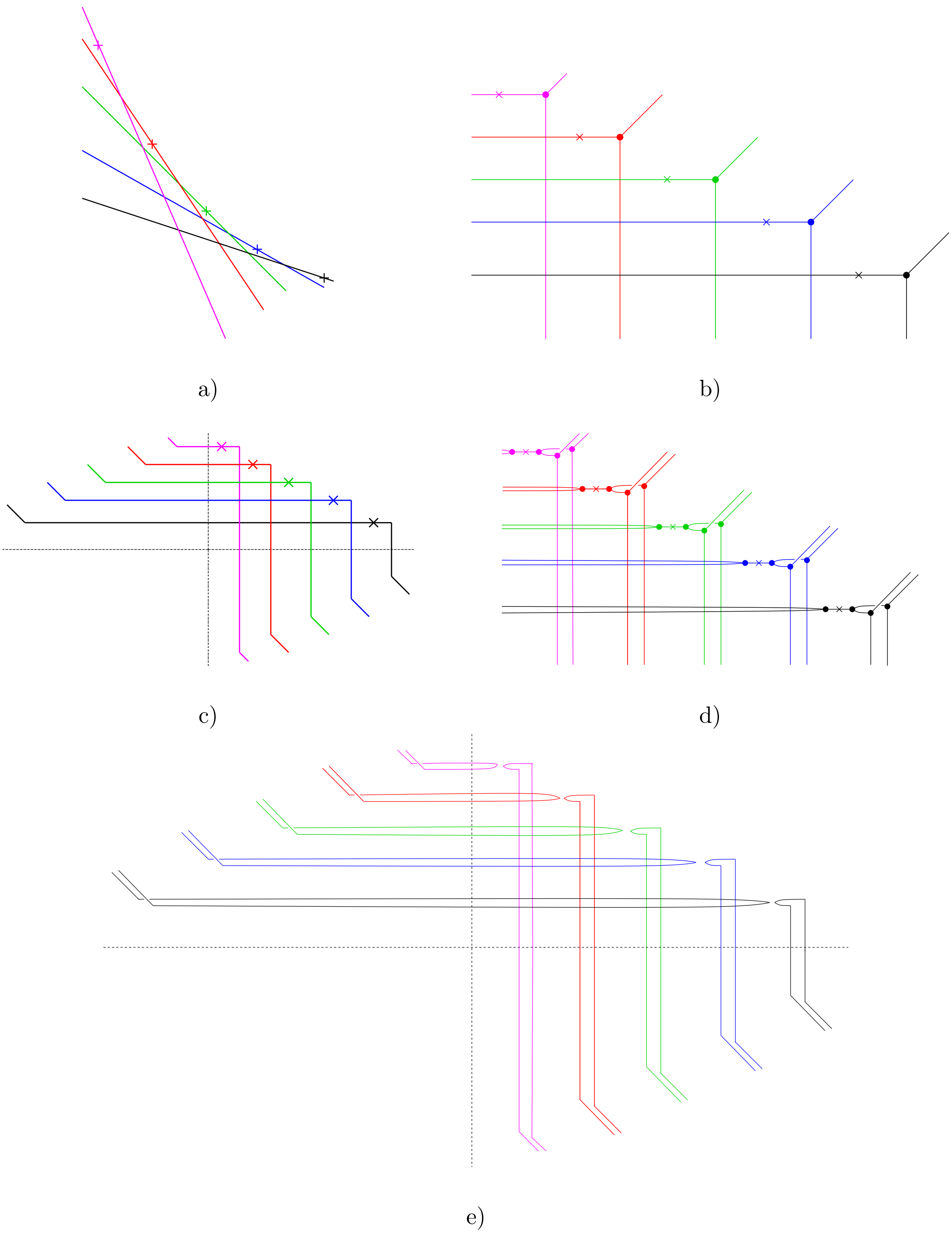


Figure 5.7: 3264 real conics tangent to 5 real conics

these floor diagrams compute characteristic numbers of the plane in terms of open Hurwitz numbers, which appear in two distinct ways in the count of $N_{d,0}(k; d_1, \ldots, d_{3d-1-k})$.

Definition 5.27 *An* elevator *of a tropical morphism $f : C \to \mathbb{R}^2$ is an edge e of C with $u_{f,e} = \pm(0,1)$. A* shaft *of f is a connected component of the topological closure of the union of all elevators of f. The set of shafts of $f : C \to \mathbb{R}^2$ is denoted by $Sh(f)$.*

A floor *of a tropical morphism $f : C \to \mathbb{R}^2$ is a connected component of the topological closure of $C \setminus Sh(f)$. The* degree *of a floor $\mathcal{F}$ of f, denoted by* $\deg(\mathcal{F})$*, is the tropical intersection number of $f(\mathcal{F})$ with a generic vertical line of $\mathbb{R}^2$.*

Theorem 5.38 involves some special open Hurwitz numbers that we describe now. Let $s \geq 0$ be an integer number, and $\delta,\ n : \{0, \ldots, s\} \to \mathbb{Z}_{\geq 0}$ be two functions. Choose a collection of s embedded circles $c_1, \ldots, c_s$ in the sphere S^2 such that c_1 (resp. c_s) bounds a disk D_0 (resp. D_s), and c_i and c_{i+1} bound an annulus D_i for $1 \leq i \leq s-1$. Choose also a collection $\mathcal{Q}$ of points in $S^2 \setminus \cup_{i=1}^s c_i$, such that each D_i contains exactly $n(i)$ points of $\mathcal{Q}$. Let us consider the set $\mathcal{H}(\delta, n)$ of all equivalence class of ramified coverings $f : \Sigma \to S^2$ where

- Σ is a connected compact oriented surface of genus 0 with s boundary components;
- $f(\partial\Sigma) \subset \cup_{i=1}^s c_i$;
- f is unramified over $S^2 \setminus \mathcal{Q}$;
- $f_{|f^{-1}(D_i)}$ has degree $\delta(i)$ for each i;
- each point in $\mathcal{Q}$ is a simple critical value of f;
- for each circle c_i, the set $f^{-1}(c_i)$ contains exactly one connected component c of $\partial\Sigma$, and $f_{|c} : c \to c_i$ is an unramified covering of degree $|\delta(i) - \delta(i-1)|$.

Definition 5.28 *The number $H(\delta, n)$ is defined as*

$$H(\delta, n) = \sum_{f \in \mathcal{H}(\delta,n)} \frac{1}{|Aut(f)|}.$$

Note that we can naturally extend the definition of the numbers $H(\delta, n)$ to the case where $\delta : \{0, \ldots, s\} \to \mathbb{Z}$ is any function by setting

$$H(\delta, n) = 0 \quad \text{if} \quad Im(\delta) \not\subseteq \mathbb{Z}_{\geq 0}.$$

In the special case where $s = 0$, we recover usual Hurwitz numbers. In particular δ is just a positive integer number, the degree of the maps we are counting and that we just denote by d. We simply denote this Hurwitz number by $H(d)$.

5.2.3.1 A simple example

Let us start by illustrating our approach on a simple case. Let us consider $\mathcal{L}$ the set composed of the five tropical lines depicted in Figure 5.8. The set $\mathbb{T}\mathcal{C}(2, \emptyset, \mathcal{L})$ is then reduced to the tropical morphism $f : C \to \mathbb{R}^2$ depicted in Figure 5.8b, which is of multiplicity 1. This morphism has one floor of degree 2, and one shaft made of three elevators. Let us represent the morphism f by the graph depicted in Figure 5.8c, where the black vertex represents the shaft of f, the white vertex represents the floor of f, and the edge represents the weight 2 elevator of f which join the shaft and the floor of f. By remembering on this graph how are distributed the tangency components of f with the lines L_i, we obtain the labeled graph depicted in Figure 5.8d.

Let us define the two projections π_x and π_y as follows

$$\begin{array}{cccc} \pi_x : & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{cccc} \pi_y : & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & y \end{array}.$$

Our main observation is the following:

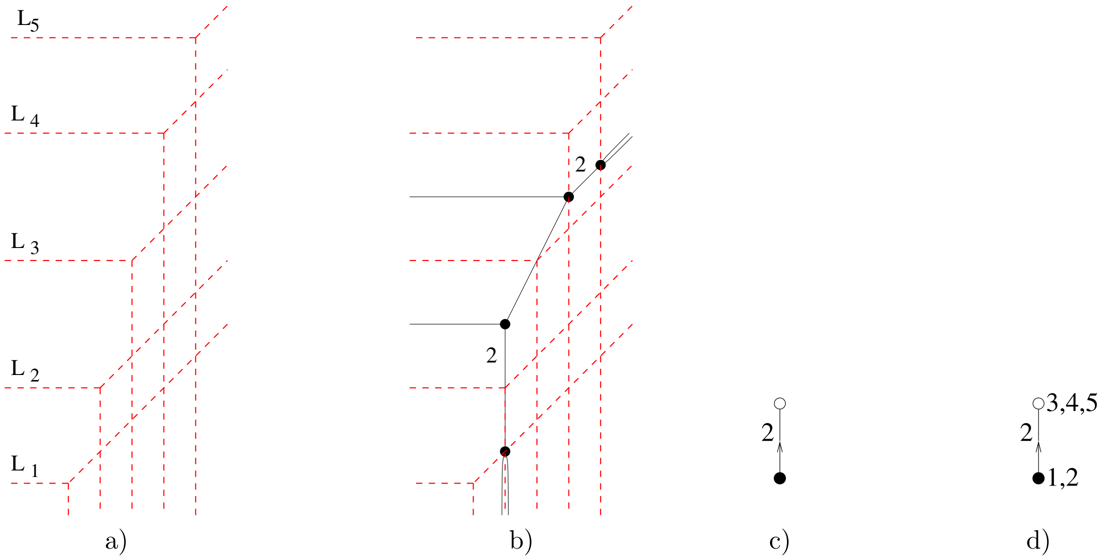


Figure 5.8: A tropical conic tangent to five lines, and its associated marked floor diagram

- the map $\pi_x \circ f$ restricted to the floor of f is a tropical ramified covering of $\mathbb{R}$ of degree 2; its critical values correspond to the vertical edge of the lines L_4 and L_5 (see Figure 5.9);
- the map $\pi_y \circ f$ restricted to the shaft of f is a tropical morphism with source a tropical curve with one boundary component; its critical value correspond approximately to the horizontal edge of L_1; the image of its boundary component corresponds approximately to the horizontal edge of L_3.

Vice versa, the morphism $f : C \to \mathbb{R}^2$ can be reconstructed out of the labeled diagram of Figure 5.8d in the following way: we first find the tropical solutions of two tropical open Hurwitz problems, one for the floor of f and one for its shaft; next we glue them according to the elevator joining this floor and this shaft, and the lines L_2 and L_3.

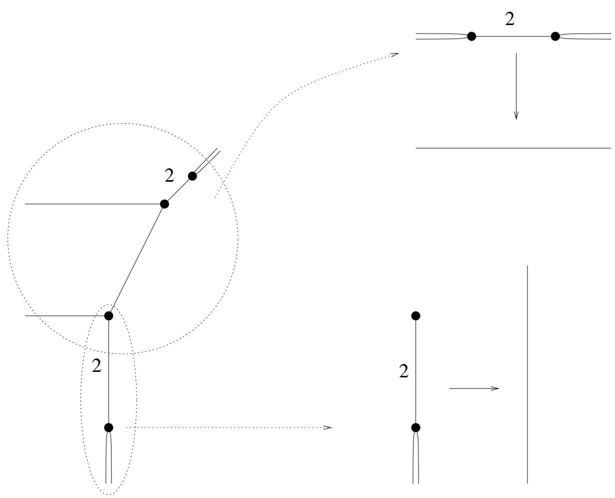


Figure 5.9: From characteristic numbers to open Hurwitz numbers

The floor of f leads to the Hurwitz number $H(2) = \frac{1}{2}$; we have two possibilities to attach the weight 2 elevator; making the floor tangent to L_3 gives a factor 1; the Hurwitz number we have to compute to reconstruct the shaft of f is $H(\delta, n) = \frac{1}{2}$ where $\delta(0) = 2, \delta(1) = n(1) = 0$, and $n(0) = 1$ making the shaft tangent to L_2 gives us a factor 1; gluing the floor and the shaft along the weight 2 elevator gives an extra factor 2. Hence, the total multiplicity of f is

$$\frac{1}{2} \times 2 \times 1 \times \frac{1}{2} \times 1 \times 2 = 1$$

as expected.

5.2.3.2 The general case

Let us define now floor diagrams to arbitrary degree and set of constrains. For simplicity, we only explain in detail how to turn the problem of computing the numbers $N_{d,0}(k; 1^{3d-1-k})$ into the enumeration of marked floor diagrams. The general computation of the numbers $N_{d,0}(k; d_1, \ldots, d_{3d-1-k})$ in terms of floor diagrams require no more substantial efforts, but makes the exposition heavier. Hence we restrict ourselves to the case of tangency with lines, which by Equation (5.3) is enough to recover all genus 0 characteristic numbers of $\mathbb{C}P^2$.

The floor diagrams we deal with in this paper underlie bipartite trees, whose vertices are divided between *white* and *black* vertices. In this section we use the following definition of a floor diagram, which is slightly different from Definition 3.1.

Definition 5.29 *A* floor diagram *(of genus 0) is an oriented bipartite tree $\mathcal{D}$ equipped with a weight function w : Edge($\mathcal{D}$) $\to \mathbb{Z}_{>0}$ such that white vertices have positive divergence, and black vertices have non-positive divergence.*

The sum of the divergence of all white vertices is called the *degree* of $\mathcal{D}$. We denote by $\mathrm{Vert}^\circ(\mathcal{D})$ the set of white vertices of $\mathcal{D}$, and by $\mathrm{Vert}^\bullet(\mathcal{D})$ the set of its black vertices. As explained in Section 5.2.3.1, a white vertex represents a *floor* of a tropical morphism, whereas a black vertex represents one of its shafts.

Example 5.30 *All floor diagrams of degree 2 are depicted in Figure 5.10. We precise the weight of an edge of $\mathcal{D}$ only if this latter is not 1.*

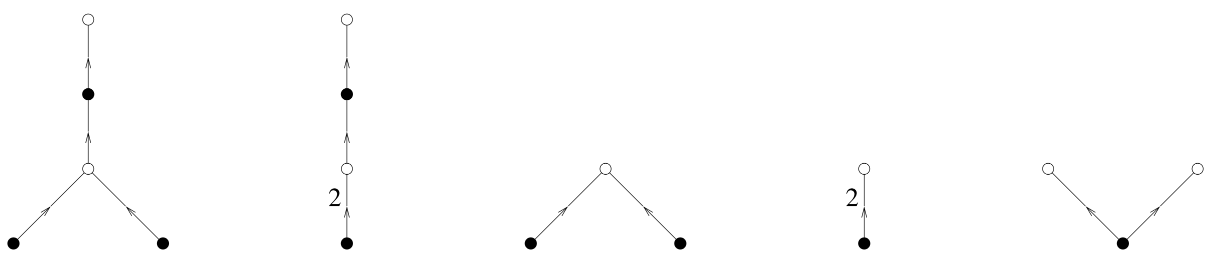


Figure 5.10: Floor diagrams of degree 2

Given a vertex v of $\mathcal{D}$, we denote by $\mathrm{Vert}(v)$ the set of vertices of $\mathcal{D}$ adjacent to v.

Definition 5.31 *Let $\mathcal{L}^{comb} \sqcup \mathcal{P}^{comb}$ be a partition of the set $\{1, \ldots, 3d-1\}$. A $\mathcal{L}^{comb}$-*marking *of a floor diagram $\mathcal{D}$ of degree d is a surjective map $m : \{1, \ldots, 3d-1\} \to$ Vert($\mathcal{D}$) such that*

- *for any $v \in Vert(\mathcal{D})$, the set $m^{-1}(v)$ contains at most one point in $\mathcal{P}^{comb}$; moreover, if $v \in Vert^{\circ}(\mathcal{D})$ and $m^{-1}(v) \cap \mathcal{P}^{comb} = \{i\}$, then $i = \min(m^{-1}(v))$;*
- *for any $v \in Vert^{\circ}(\mathcal{D})$, $|m^{-1}(v)| = 2div(v) - 1$;*
- *for any $v \in Vert^{\bullet}(\mathcal{D})$, $|m^{-1}(v)| = val(v) - div(v) - 1$; moreover there exists at most one element i in $m^{-1}(v)$ such that $i > \max_{v' \in Vert(v)} \min(m^{-1}(v'))$, and if such an element exists then we have $m^{-1}(v) \cap \mathcal{P}^{comb} = \emptyset$.*

Two $\mathcal{L}^{comb}$-markings $m : \{1, \ldots, 3d-1\} \to \mathrm{Vert}(\mathcal{D})$ and $m' : \{1, \ldots, 3d-1\} \to \mathrm{Vert}(\mathcal{D}')$ are *isomorphic* if there exists an isomorphism of bipartite graphs $\phi : \mathcal{D} \to \mathcal{D}'$ such that $m = m' \circ \phi$. In this text, $\mathcal{L}^{comb}$-marked floor diagrams are considered up to isomorphism.

The set $\{1, \ldots, 3d-1\}$ represents the configuration of constraints in the increasing height order (see Section 5.2.3.1), the set $\mathcal{L}^{comb}$ represents the lines in the configuration, and the set $\mathcal{P}^{comb}$ represents the points. Note that unlike in Section 3.2.1, we do not consider the partial order on $\mathcal{D}$ defined by its orientation. In particular, it makes no sense here to require the marking m to be an increasing map.

In order to define the multiplicity of an $\mathcal{L}^{comb}$-marked floor diagram, we first define the multiplicity of a vertex of $\mathcal{D}$.

Definition 5.32 *The* multiplicity of a vertex v in $\mathrm{Vert}^{\circ}(\mathcal{D})$ *is defined as*

- *if* $\min(m^{-1}(v)) \in \mathcal{P}^{comb}$*, then*
$$\mu_{\mathcal{L}^{comb}}(v) = div(v)^{val(v)+1} H(div(v))$$
- *otherwise*
$$\mu_{\mathcal{L}^{comb}}(v) = (div(v) - 2 + val(v)) div(v)^{val(v)} H(div(v)).$$

Example 5.33 *We give in Figure 5.11 some examples of multiplicities of white vertices of a marked floor diagram. The corresponding Hurwitz numbers are given in Proposition 5.3. We write the elements of $m^{-1}(v)$ close to the vertex v.*

$a \in \mathcal{P}^{comb}$, $\mu = 1$	$a \in \mathcal{P}^{comb}$, $\mu = 324$	$a \in \mathcal{P}^{comb}$, $\mu = 1$	$a \in \mathcal{P}^{comb}$, $\mu = \frac{d^{d-1}(2d-2)!}{d!}$
$a \in \mathcal{L}^{comb}$, $\mu = 2$	$a \in \mathcal{L}^{comb}$, $\mu = 432$	$a \in \mathcal{L}^{comb}$, $\mu = 0$	$a \in \mathcal{L}^{comb}$, $\mu = \frac{d^{d-3}(2d-2)!}{(d-2)!}$

Figure 5.11: Example of multiplicities of white vertices of $\mathcal{D}$

The definition of the multiplicity of a black vertex v of $\mathcal{D}$ requires a preliminary construction. The order on $\{1, \ldots, 3d-1\}$ induces an order on $\mathrm{Vert}(v)$ via the map $v' \mapsto \min(m^{-1}(v'))$. Note that this order doesn't have to be compatible with the orientation of $\mathcal{D}$. Let us denote by $v'_1 < \ldots < v'_s$ the elements of $\mathrm{Vert}(v)$ according to this order. We denote by e_i the edge of $\mathcal{D}$ joining the vertices v and v'_i, and define $\varepsilon_i = 1$ if e_i is oriented toward v, and $\varepsilon_i = -1$ otherwise. Given $j \in m^{-1}(v)$ we define the integer i_j by

$i_j = 0$ if $j < \min(m^{-1}(v'_1))$;
$i_j = i$ if $\min(m^{-1}(v'_i)) < j < \min(m^{-1}(v'_{i+1}))$;
$i_j = s$ if $j > \min(m^{-1}(v'_s))$.

We define two functions $\delta, \tilde{n} : \{0, \ldots, s\} \to \mathbb{Z}$ by

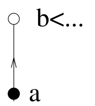
- $\delta(0) = -div(v)$,
  $\delta(i+1) = \delta(i) + \varepsilon_{i+1} w(e_{i+1})$;
- $\tilde{n}(i) = |\{j \in m^{-1}(v) \mid i_j = i\}|$.

Given $i_0 \in \{0, \dots, s\}$, we define the function $n_{i_0} : \{0, \dots, s\} \to \mathbb{Z}_{\geq 0}$ by $n_{i_0}(i_0) = \tilde{n}(i_0) - 1$ and $n_{i_0}(i) = \tilde{n}(i)$ if $i \neq i_0$. Finally, we define $\tilde{N}(i) = \sum_{l=0}^{i} \tilde{n}(l)$ and $\tilde{N}(-1) = 0$.

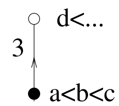
Definition 5.34 *The* multiplicity of a vertex v in $\mathrm{Vert}^\bullet(\mathcal{D})$ *is defined by the following rules*

- *if* $m^{-1}(v) \cap \mathcal{P}^{comb} = \{j\}$*, then*
$$\mu_{\mathcal{L}^{comb}}(v) = \delta(i_j) H(\delta, n_{i_j})$$
- *if* $m^{-1}(v) \cap \mathcal{P}^{comb} = \emptyset$ *and* $m^{-1}(v)$ *contains an element* j *such that* $j > \max_{v' \in Vert(v)} \min(m^{-1}(v'))$*, then*
$$\mu_{\mathcal{L}^{comb}}(v) = (2val(v) - 2) H(\delta, n_s)$$
- *otherwise,*
$$\mu_{\mathcal{L}^{comb}}(v) = \frac{1}{2} \sum_{i=0}^{s} \left(\tilde{n}(i) \left(2\delta(i) + 2i + \tilde{N}(i) + \tilde{N}(i-1) - 1 + 2div(v) \right) H(\delta, n_i) \right).$$

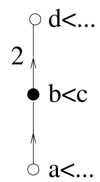
Example 5.35 *We give in Figure 5.12 some examples of multiplicities of black vertices of a marked floor diagram.*



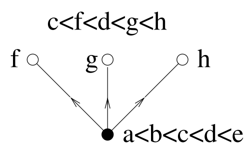
$a > b$, $\mu = 0$
$a < b$, $a \in \mathcal{P}^{comb}$, $\mu = 1$
$a < b$, $a \in \mathcal{L}^{comb}$, $\mu = 0$



$c > d$, $\mu = 0$
$c < d$, $\{a, b, c\} \cap \mathcal{P}^{comb} \neq \emptyset$, $\mu = 3$
$c < d$, $\{a, b, c\} \subset \mathcal{L}^{comb}$, $\mu = 3$



$b < a$, $b \in \mathcal{P}^{comb}$, $\mu = 1$
$b < a$, $b \in \mathcal{L}^{comb}$, $\mu = 0$
$a < b < c < d$, $\{b, c\} \cap \mathcal{P}^{comb} \neq \emptyset$, $\mu = 2$
$a < b < c < d$, $\{b, c\} \subset \mathcal{L}^{comb}$, $\mu = 5$
$a < b < d < c$, $\{b, c\} \subset \mathcal{L}^{comb}$, $\mu = 2$



$e > h$, $e \in \mathcal{L}^{comb}$, $\mu = 32$
$e < g$, $\{d, e\} \cap \mathcal{P}^{comb} \neq \emptyset$, $\mu = 16$
$e < g$, $c \in \mathcal{P}^{comb}$, $\mu = 6$
$e < g$, $\{a, b, c, d, e\} \in \mathcal{L}^{comb}$, $\mu = 62$

Figure 5.12: Example of multiplicities of black vertices of $\mathcal{D}$

Definition 5.36 *The* multiplicity of an $\mathcal{L}^{comb}$-marked floor diagram *is defined as*

$$\mu_{\mathcal{L}^{comb}}(\mathcal{D}, m) = \prod_{e \in Edge(\mathcal{D})} w(e) \prod_{v \in Vert(\mathcal{D})} \mu_{\mathcal{L}^{comb}}(v).$$

Note that $\mu_{\mathcal{L}^{comb}}(\mathcal{D}, m)$ can be equal to 0.

Example 5.37 *We give in Figure 5.13 a few examples of multiplicities of $\mathcal{L}^{comb}$-marked floor diagrams.*

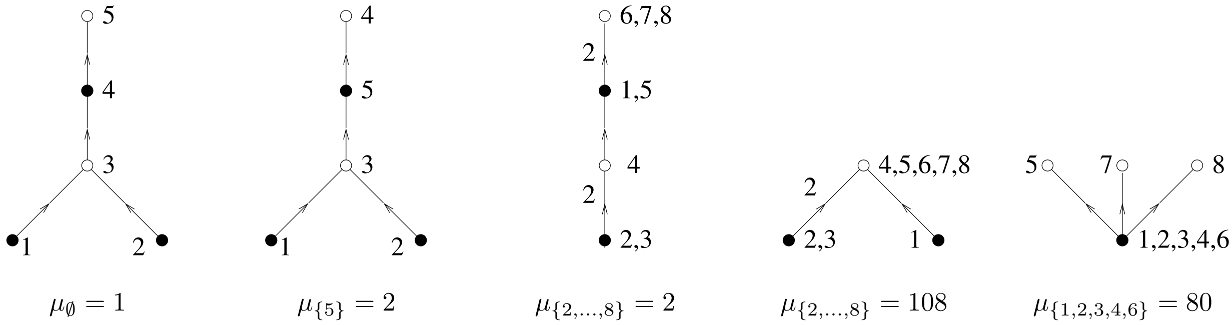


Figure 5.13: Example of multiplicities of $\mathcal{L}^{comb}$-marked floor diagrams

Next theorem reduces the computation of characteristic numbers in genus 0 of $\mathbb{C}P^2$ to the combinatorial enumeration of marked floor diagrams.

Theorem 5.38 ([BBM14, Theorem 8.11]) *For any $d \geq 1$, $k \geq 0$, and $\mathcal{L}^{comb} \subset \{1, \dots, 3d-1\}$ of cardinal $3d-1-k$, we have*

$$N_{d,0}(k; 1^{3d-1-k}) = \sum \mu_{\mathcal{L}^{comb}}(\mathcal{D}, m)$$

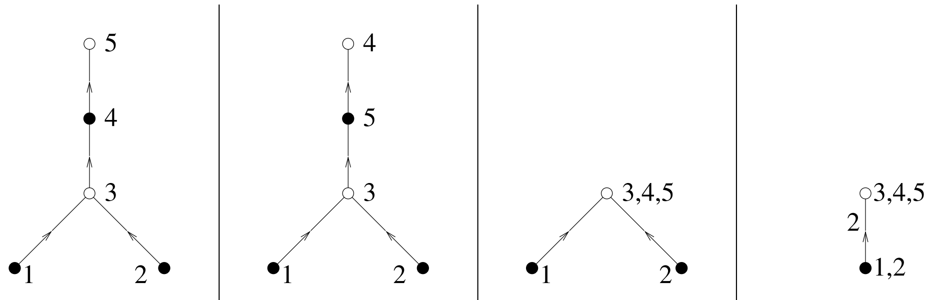
where the sum ranges over all $\mathcal{L}^{comb}$-marked floor diagrams of degree d.

Note that in the case $k = 3d-1$, Theorem 5.38 agrees with Theorem 3.11. Indeed, a $\emptyset$-marked floor diagram has non-null multiplicity if and only if the marking is increasing with respect to the partial order on $\mathcal{D}$ defined by its orientation; in this case, the different definitions of multiplicity of a marked floor diagram coincide.

Example 5.39 *We compute the numbers $N_{2,0}(k; 1^{5-k})$, with $\mathcal{L}^{comb} = \{k+1, \dots, 5\}$. In each case, there is exactly one marked floor diagram of positive multiplicity, depicted in Figure 5.14.*

Example 5.40 *Figure 5.15 represents all marked floor diagrams of degree 3 with positive multiplicity when $\mathcal{L}^{comb} = \{2, \dots, 8\}$. Hence there are exactly 600 rational cubics passing through 1 point and tangent to 7 lines.*

We refer to [BBM14, Section 8] for more examples of computations using marked floor diagrams.



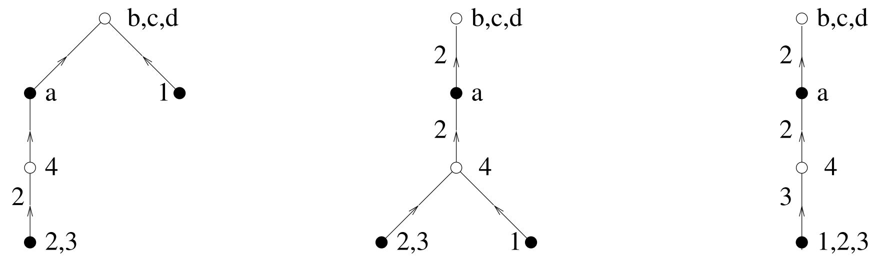
$\mu_\emptyset$	1	0	0	0
$\mu_{\{5\}}$	0	2	0	0
$\mu_{\{4,5\}}$	0	0	4	0
$\mu_{\{3,4,5\}}$	0	0	4	0
$\mu_{\{2,3,4,5\}}$	0	0	0	2
$\mu_{\{1,2,3,4,5\}}$	0	0	0	1

Figure 5.14: Computation of $N_{2,0}(k; 1^{5-k})$ with $\mathcal{L}^{comb} = \{k+1, \dots, 5\}$

5.2.4 Further comments

All statements, and proofs given in [BBM14] should generalize with no difficulty to the case of rational curves in $\mathbb{C}P^n$ intersecting cycles and tangent to non-singular hypersurfaces. The resulting floor diagrams would then be a generalization of those defined in Section 3.2.3. The enumeration of plane curves with higher order tangency conditions to other curves should also be doable in principle using our methods. This would first necessitate to identify tropicalizations of higher order tangencies between curves, generalizing the simple tangency case treated in Section 5.2.2.1. However this identification might be intricate, and will certainly lead to much more different cases than for simple tangencies (third order tangencies to a line are dealt with in [BLdM12]). In turn, the use of tropical techniques in the computation of higher genus characteristic numbers requires some substantial additional work. The main difficulty is that superabundancy appears for positive genus: some combinatorial types appearing as solution of the enumerative problem might be of actual dimension strictly bigger than the expected one (see Remark 2.5). Hence before enumerating tropical curves, in addition to the balancing condition one has first to understand extra necessary conditions for a tropical morphism to be the tropical limit of a family of algebraic maps. Using techniques based on a combination of tropical modifications and local obstructions (see Section 6.2), we succeeded with Bertrand and Mikhalkin to compute genus 1 characteristic numbers of $\mathbb{C}P^2$. Also for a small number of tangency constraints, it is possible to find a configuration of constraints for which no superabundant curve shows up. In this case Theorem 5.22 applies, the proof only requiring minor adjustments.

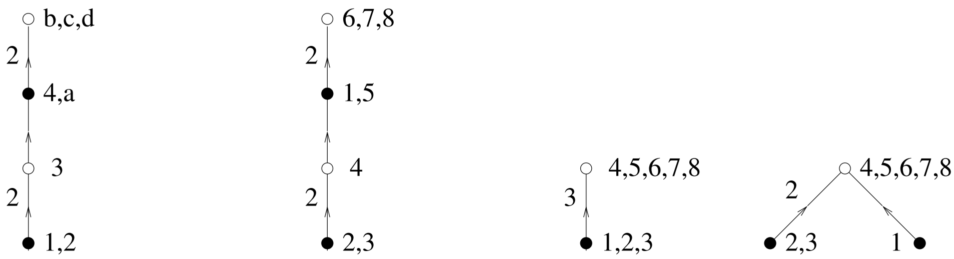
Also, as mentioned above, it would be interesting to explore applications of Theorem 5.22 to enumeration of real curves.



$a = 5, 6, 7$, or 8 and $\mu = 8$

$a = 5$ and $\mu = 12$
$a = 6, 7$, or 8 and $\mu = 8$

$a = 5$ and $\mu = 54$
$a = 6, 7$, or 8 and $\mu = 36$



$a = 5$ and $\mu = 20$
$a = 6, 7$, or 8 and $\mu = 8$

$\mu = 2$

$\mu = 216$

$\mu = 108$

Figure 5.15: Computation of $N_{3,0}(1; 1^7) = 600$ with $\mathcal{L}^{comb} = \{2, \ldots, 8\}$

Chapter 6

A brief presentation of other related works

In this chapter we give a very brief overview of some other of our works, which are less directly related to the central theme of this memoir. However they were partly motivated by the development of tropical geometry with a view towards applications to enumerative geometry, which justify their presence here.

6.1 Real inflection points of real algebraic curves

The fact that a non-singular real algebraic curve in $\mathbb{R}P^2$ of degree d has at most $d(d-2)$ real inflection points was proved by Klein in 1876, see [Kle76]; see also [Ron98, Sch04, Vir88]. A non-singular real plane algebraic curve having this maximal number of real inflection points is called *maximally inflected*. The existence of maximally inflected curves of any degree was also proved by Klein. However, many questions about real inflection points of (maximally inflected) real algebraic curves remain widely open. For example, which rigid-isotopy classes of real algebraic curves contain a maximally inflected curve? How real inflection points can be distributed among the connected component of a maximally inflected curve? These possible distributions of real inflection points turn out to be subject to non-trivial obstructions that mainly remain mysterious (see for example [KS03, BLdM12, ABdLdM14]).

We studied these questions in the two papers [BLdM12, ABdLdM14]. The former is a joint work with López de Medrano, and the latter with Arroyo and López de Medrano. Using patchworking of real algebraic curves, we provided in [BLdM12] a systematic method to construct maximally inflected real algebraic curves with a controlled position of their real inflection points.

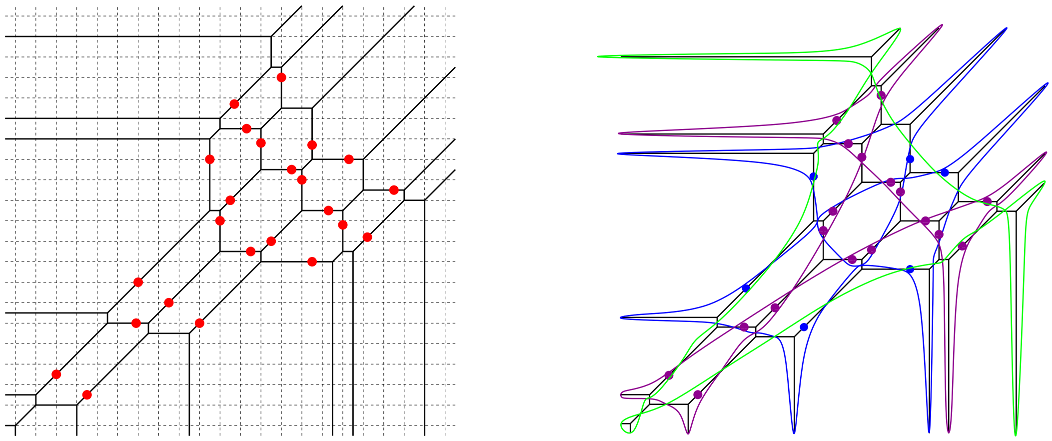
Theorem 6.1 ([BLdM12, Theorem 5.7]) *Let C be a non-singular tropical curve in $\mathbb{R}^2$ such that if v is a vertex of C dual to Δ_1, then its 3 adjacent edges have 3 different length. Then any real algebraic curve constructed by a patchworking of C is maximally inflected. Moreover, the position of its real inflection points can be read out of this patchworking.*

For an account of Viro's patchworking, we refer for example to the tropical presentation made in [Vir01, Mik04, Bru13].

Thanks to Theorem 6.1 combined with other tools from real algebraic geometry (rigid isotopy classification results, braid theoretical methods), we obtain some classifications results. For example, we obtained the complete classification of possible distributions of real inflection points among the components of a hyperbolic curve of degree 6.

Theorem 6.2 ([ABdLdM14, Theorem 1.2]) *Let C be a non-singular maximally inflected real algebraic curve curve of degree 6 in $\mathbb{R}P^2$ whose real part consists of three nested ovals. Then, the outer oval of C contains at least 6 real inflection points. Moreover, for any $0 \leq k \leq 9$, there exists such a real algebraic curve with exactly $6 + 2k$ real inflection points on the outer oval.*

As an example of application of Theorem 6.1, the patchworking of a maximally inflected hyperbolic curve of degree 6 with only 6 real inflection points on the outer oval is depicted in Figure 6.1 (we refer for example to [Bru13, Section 3] for the ribbon interpretation of Patchworking).



a) b)

Figure 6.1: Patchworking of a maximally inflected hyperbolic curve of degree 6 with only 6 real inflection points on the outer oval (each tropical inflection point is colored according to the arc it comes from)

As mentioned above, obstructions concerning possible distributions of real inflection points on non-singular real algebraic curves mainly remain mysterious, and the proof of non-trivial obstructions known to us are rather ad hoc. It would be interesting to find more general proofs. Note also that real inflection points are related to real Weierstrass points on real algebraic curves, and that the study of these latter is also widely open.

6.2 On the approximation of tropical curves in tropical surfaces

One of the challenging problems in tropical geometry is to understand which tropical varieties are *approximable*, i.e. arise as the tropicalization of classical algebraic varieties. Not all tropical varieties are approximable, the first example was given by Mikhalkin who constructed in [Mik05] a spatial elliptic tropical cubic C which is not tropically planar: by the Riemann-Roch Theorem any classical spatial elliptic cubic is planar, therefore the tropical curve C cannot be approximable. It follows from the works of Kapranov, Viro, Mikhalkin, and Rullgård (see [Kap00, Vir01, Mik04, Rul01]) that any tropical hypersurface in $\mathbb{R}^N$ is approximable. In addition, many nice partial results about approximation of tropical curves in $\mathbb{R}^N$ have been proved by different authors (e.g. [Mik05, Mik06, Spe07, NS06, Mik, Nis09, Tyo12, Kat12a, BBM14]).

Tropical varieties in $\mathbb{R}^N$ are related to classical subvarieties of toric varieties. When considering non-toric varieties, or when working in tropical models of the torus different from $\mathbb{R}^N$ (for example as in Section 4.3), one is naturally led to the approximation problem for pairs, that is to say the simultaneous approximation of a tropical variety together with one if its subvarieties. Non-approximable pairs of tropical objects show up even in very simple situations. Some well known pathological examples of such pairs were given by Vigeland, who constructed in [Vig09] examples of generic non-singular tropical surfaces in $\mathbb{R}^3$ of any degree $d \geq 3$ containing infinitely many tropical lines. Moreover, the surfaces constructed by Vigeland form an *open* subset of the space of all tropical surfaces of the given degree d, which means that these families of lines survive when perturbing the coefficients of a tropical equation of the surface. Vigeland's construction dramatically contrasts with Segre's Theorem [Seg43] asserting that any non-singular complex surface of degree $d \geq 3$ in $\mathbb{C}P^3$ can contain only finitely many lines.

In a joint work with Shaw [BS14], we provided combinatorial local obstructions in the case of tropical curves in non-singular tropical surfaces. More precisely, we studied the following problem.

Question 6.3 *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a plane, and $C \subset Trop(\mathcal{P}) \subset \mathbb{R}^N$ be a fan tropical curve. Does there exist a complex algebraic curve $\mathcal{C} \subset \mathcal{P} \subset (\mathbb{C}^*)^N$ such that $Trop(\mathcal{C}) = C$?*

We refer to [BS14, Section 2] for precise definitions of fan tropical curves and of tropicalizations. As a first rough approximation, given an algebraic subvariety X of $(\mathbb{C}^*)^N$, one can think of $\text{Trop}(X)$ as $\lim_{t\to\infty} \text{Log}_t(X)$. Partial answers to Question 6.3 and its generalizations were previously obtained by the author and Mikhalkin in an unpublished work, by Bogart and Katz [BK12], and subsequently by Gathman, Schmitz and Winstel [GSW13].

The strategy used in [BS14] is to use the relation between tropical and complex intersection theories in order to translate classical results (e.g. adjunction formula, intersection with Hessian) into combinatorial formulas involving only tropical data ([BS14, Theorem 3.8]). In the case of stable intersections, such a relation has been previously obtained in [Rab12, BLdM12, OR13, Kat12b]. As an example of general obstructions proved in [BS14], we sate a weak but easy-to-state version of [BS14, Theorem 4.1], which is a consequence of complex adjunction formula. A plane $\mathcal{P}$ in $(\mathbb{C}^*)^N$ is called *uniform* if its compactification $\overline{\mathcal{P}} \subset \mathbb{C}P^N$ as a projective linear subspace does not meet any $N-k$-coordinate linear space with $k \geq 3$.

Theorem 6.4 ([BS14, Theorem 1.3]) *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a uniform plane, and $C \subset Trop(\mathcal{P}) \subset \mathbb{R}^N$ be a fan tropical curve of degree d. If there exists an irreducible and reduced complex algebraic curve $C' \subset \mathcal{P}$ such that $Trop(C') = C$, then*

$$C^2 + (N-2)d - \sum_{e \subset Edge(C)} w_e + 2 \geq 2g(C').$$

In particular, if the left hand side is negative then C is not approximable in $\mathcal{P}$ by an irreducible and reduced complex algebraic curve.

Note that even in the case of *rational* tropical curves in surfaces, there exist non-local obstructions to the approximation in pairs, see [BS14, Remark 1.9]. This contrast with the fact that any tropical rational curve in $\mathbb{R}^n$ is approximable (see for example [Spe07]).

The last two sections of [BS14] are devoted to applications of the general obstructions proved in [BS14, Theorems 4.1 and 5.3].

We first classify in [BS14, Theorem 6.9] all 2 or 3-valent approximable fan tropical curves in a plane $\mathcal{P}$. Next theorem provides two simple instances of this classification.

Theorem 6.5 ([BS14, Theorem 1.3]) *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane, and let $C \subset Trop(\mathcal{P})$ be a reduced 2 or 3-valent fan tropical curve.*

1. *If $N = 3$, then C is approximable in $\mathcal{P}$ if and only if $C^2 = 0$ or $C^2 = -1$;*

2. *If $N \geq 6$ and C is of degree at least 2, then C is not approximable in $\mathcal{P}$.*

Secondly, we apply our methods to the study of tropical lines in tropical surfaces. Next theorem shows that when we restrict our attention to those which are approximable in the tropical surface, the situation turns out to be analogous to the case of complex algebraic surfaces. This solves the problem raised in [Vig09] of generic tropical surfaces of degree $d \geq 4$ containing tropical lines, and tropical surfaces of degree $d = 3$ containing infinitely many tropical lines.

Theorem 6.6 ([BS14, Theorem 1.8]) *Let S be a generic non-singular tropical surface in $\mathbb{T}P^3$ of degree d. If $d = 3$, then there exist finitely many tropical lines $L \subset S$ such that the pair (S, L) is approximable.*
If $d \geq 4$, then there exist no tropical lines $L \subset S$ such that the pair (S, L) is approximable.

We end this section with a few comments. First, it is interesting to note that all fan tropical curves $C \subset \mathbb{R}^3$ known to us to be approximable by an irreducible and reduced complex algebraic curve in a plane $\mathcal{P} \subset (\mathbb{C}^*)^3$ satisfy $C^2 \geq -1$ in $\mathrm{Trop}(\mathcal{P})$. This leads us to the following open question: does there exist a fan tropical curve $C \subset \mathbb{R}^3$ which is approximable by an irreducible and reduced complex algebraic curve in a plane $\mathcal{P} \subset (\mathbb{C}^*)^3$ and satisfies $C^2 \leq -2$ in $\mathrm{Trop}(\mathcal{P})$?

Next, one possible way of getting non-local obstructions to the approximation problem is to combine local obstructions with tropical modifications. This strategy is the subject of a joint work in progress with G. Mikhalkin that has ramifications to many directions.

6.3 Approximation of tropical morphisms between tropical curves

In a joint work with Amini, Baker, and Rabinoff [ABBR13a, ABBR13b], we carefully study which morphisms between tropical curves, in the sense of Definition 5.4, arise as tropicalizations of morphisms of algebraic curves. In these two papers, tropicalization is defined via Berkovich's theory of analytic spaces (see also [Pay09, BRP11, CLD12]). This setting is related but somewhat different from the frameworks proposed by Kontsevich and Soibelman [KS01] on one hand, and by Mikhalkin [Mik06] on the other hand, where the link between tropical geometry and complex algebraic geometry is provided by real one-parameter families of complex varieties.

In [ABBR13a] we prove many approximability results concerning several variations on the above question, and extend earlier works by Saïdi [Saï97] and Wewers [Wew99]. In particular, one of our main results is that in characteristic 0, the only obstructions are local for tropical morphisms in the sense of Definition 5.4. We refer to [ABBR13a] for a more rigorous statement.

Theorem 6.7 ([ABBR13a, Theorem B, Corollary D]) *A tropical morphism between two tropical curves is globally approximable in characteristic 0 if and only it is locally approximable everywhere.*

In Mikhalkin's framework of phase-tropical geometry (see for example [Mik06, Section 6], [BMa, Section 6]), Theorem 6.7 is a consequence of Riemann's Existence Theorem. Note that the local approximation problem of a tropical morphism between two tropical curves (in characteristic 0) is equivalent to the study of the vanishing of certain Hurwitz numbers, which is known to be a subtle problem.

We prove a number of additional results which supplement and provide applications of Theorem 6.7 and its variations: approximation of a tropical curve equipped with a *tame action* of a finite group H, classify tropical curves approximable by hyperelliptic curves, how gonality and rank of a divisor change under tropicalization, ... In particular, we prove that neither the tropical gonality nor the tropical rank of a divisor are sharp.

Theorem 6.8 ([ABBR13b, Theorem 5.4]) *There exists a choice of edges length such that the explicit tropical curve depicted in Figure 6.2 has gonality* 4*. However, for any choice of edges length, any approximation of this tropical curve has gonality at least* 5*.*

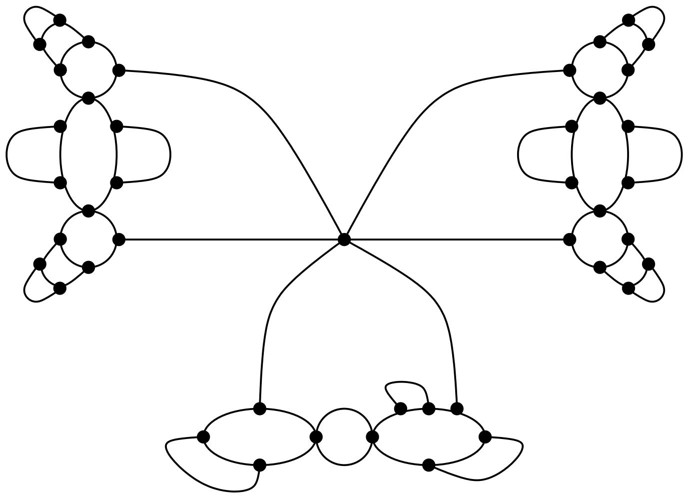


Figure 6.2: An explicit tropical curve of gonality 4 with no approximation of gonality 4

Proposition 6.9 ([ABBR13b, Proposition 5.14]) *There exists an effective divisor D on a tropical curve C such that D has tropical rank equal to* 1*, but any effective lifting of D has rank* 0*.*

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