DEFORMATION OF TROPICAL HIRZEBRUCH SURFACES AND ENUMERATIVE GEOMETRY

ERWAN BRUGALLÉ AND HANNAH MARKWIG

Abstract. We illustrate the use of tropical methods by generalizing a formula due to Abramovich and Bertram, extended later by Vakil. Namely, we exhibit relations between enumerative invariants of the Hirzebruch surfaces Σ_n and Σ_{n+2}, obtained by deforming the first surface to the latter.

Our strategy involves a tropical counterpart of deformations of Hirzebruch surfaces, and tropical enumerative geometry on a tropical surface in three-space.

Contents

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Key words and phrases. Enumerative geometry, Hirzebruch surfaces, deformation of complex surfaces, tropical geometry, tropical modifications, degeneration formula.

1. Introduction

1.1. **Results.** In [AB01], Abramovich and Bertram related genus 0 enumerative invariants of $\Sigma_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ and the second Hirzebruch surface Σ_2 (i.e. the quadratic cone $x^2 + y^2 + z^2 = 0$ in $\mathbb{C}P^3$ blown up at the node). The strategy of their proof is to understand how algebraic curves on Σ_0 behave when this latter surface deforms to Σ_2. Later on this method was extended by Vakil in [Vak00] to non-rational enumerative invariants of Σ_0 and Σ_2, and more generally to relate enumerative invariants of an almost Fano complex surface and any of its deformations.

The goal of this paper is to illustrate the use of tropical deformation techniques by generalizing this formula to the case of Σ_n and Σ_{n+2}. Here, $n \geq 0$ and $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \mathbb{C})$ is the so called *n-th Hirzebruch surface*. Hirzebruch surfaces are toric.

We consider enumerative invariants $N_\chi(\delta)$ counting curves in Σ_n with prescribed Euler characteristic χ, data of intersections with the toric boundary, and enough point conditions (see Definition 2.2). We encode the intersections with the toric boundary in terms of a *Newton fan* δ (see Definition 2.1). A Newton fan is a multiset of vectors $\delta = \{v_1^{m_1}, \ldots, v_k^{m_k}\}$ (where the notation used here indicates that the vector v_i appears m_i times) that refines the notion of the fan dual to a Newton polygon. In particular, to any Newton fan $\delta \subset \mathbb{Z}^2$ we can naturally associate a dual polygon Π_δ. The toric surface Σ_n in which we consider the curves to be counted is determined once we fix a Newton fan (see Notation 1.1).

In our formula, we also consider enumerative invariants that we call the (1-1)-relative invariants of Σ_n (see Definition 2.4). They count curves passing through prescribed points and having prescribed tangent germs at a fixed point. We refer to these enumerative invariants as $\mathcal{N}(u, n, d, \alpha)$, they depend on the integer u prescribing the number of irreducible components, the integer n defining the Hirzebruch surface Σ_n in question, and two vectors d and α encoding the intersection multiplicities with boundary divisors of the tangent germs. For fixed n and u, there exist only finitely many possibilities for d and α, as explained (along with further details on the computation of these invariants) in Remark 2.5.

Before stating our formula relating these enumerative invariants, we need to introduce some notation.

Notation 1.1
Let δ_0 be the Newton fan

$$\delta_0 = \{(1,n)^a, (0,-1)^{an+b}, (-1,0)^a, (0,1)^b\}.$$

We write $\delta \vdash \delta_0$ if δ is a Newton fan satisfying

$$\begin{aligned}\delta = \quad & \{(1,n+2)^m, (0,-1)^{a(n+1)+b}, (-1,0)^A, \\ & (-\alpha_1,\beta_1),\dots,(-\alpha_r,\beta_r),(0,\beta_{r+1}),\dots,(0,\beta_{r+s}),(0,1)^U\}\end{aligned}$$

with

$$0 < m \leq a, \quad 0 \leq A \leq \min\{m,b\},$$
$$\alpha_i, \beta_i > 0 \quad \text{for} \quad i = 1\dots,r, \quad \text{and} \quad \beta_{r+1},\dots,\beta_{r+s} > 1.$$

Note that for every $\delta \vdash \delta_0$,

$$\Pi_\delta \subset Conv\{(0,0),(0,a),(b-a,a),(a(n+1)+b,0)\},$$

where the latter is the polygon dual to

$$\{(1,n+2)^a, (0,-1)^{a(n+1)+b}, (-1,0)^a, (0,1)^{b-a}\}.$$

In particular once δ_0 is fixed, the choices of such δ are limited (see Figure 1).

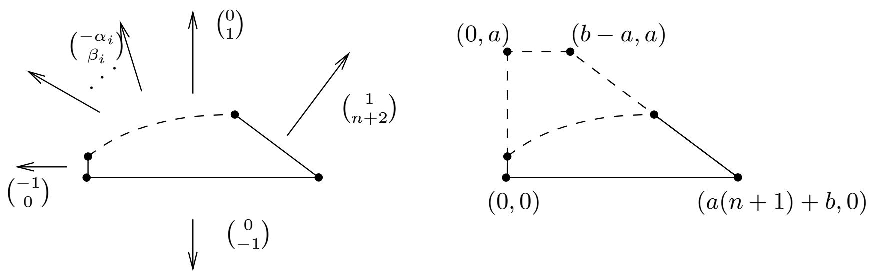

FIGURE 1. Finitely many fans $\delta \vdash \delta_0$.

For a Newton fan $\delta \vdash \delta_0$, we define the following quantities:

- $\chi' = \chi - 2(a+b-m-r-s-A-U)$;
- $d = (\beta_1+\alpha_1,\dots,\beta_r+\alpha_r,\beta_{r+1},\dots,\beta_{r+s},1^{U+A-b})$;
- $\alpha = (\alpha_1,\dots,\alpha_r)$.

Our main result is a formula relating enumerative invariants of Σ_n and Σ_{n+2} which we prove at the end of Section 4.3.

Theorem 1.2
Let $n \geq 0$ and $\chi \in \mathbb{Z}$ be two integers, and let δ_0 as in Notation 1.1 be the Newton fan

$$\delta_0 = \{(1,n)^a, (0,-1)^{an+b}, (-1,0)^a, (0,1)^b\}.$$

Then we have the following equation:

$$N_\chi(\delta_0) = \sum_{\delta \vdash \delta_0} \left(\binom{U}{b-A} \cdot \prod_{i=1}^{r} \gcd(\alpha_i,\beta_i) \cdot \prod_{i=1}^{s} \beta_{r+i} \cdot \mathcal{N}(a-m,n,d,\alpha)\ N_{\chi'}(\delta) \right).$$

Note that the factor $N_{\chi'}(\delta)$ indeed counts curves on Σ_{n+2} by Notation 1.1 and Example 2.3. Note also that as pointed out in Remark 2.5, for a fixed n the computations of all numbers $\mathcal{N}(a-m,n,d,\alpha)$ reduces to the computation of finitely many cases.

We use methods from tropical geometry to prove Theorem 1.2. In the tropical world, algebraic curves are replaced by certain balanced piece-wise linear graphs called tropical curves. Tropical geometry has gained lots of attention recently. One of the interesting results is that we can determine enumerative invariants of toric surfaces by counting the corresponding tropical curves instead (Mikhalkin's Correspondence Theorem, see [Mik05]). This fact is at the base of our tropical approach to enumerative invariants of Hirzebruch surfaces.

The proof of Theorem 1.2 basically consists of the suitable Correspondence Theorems (see Theorem 4.13 and Theorem 3.17) relating the above enumerative invariants to their tropical counterparts, and a proof of the tropical version of the above formula (see Section 4.3). The main idea underlying our tropical strategy is a deformation of tropical Hirzebruch surfaces, which is a tropical analogue of Kodaira's deformation of complex Hirzebruch surfaces. We provide a more detailed description of our method use in Section 1.2.

The two Correspondence Theorems 4.13 and 3.17 and the formula of Theorem 1.2 together with its tropical method of proof should be viewed as the main contributions of our paper.

As examples of applications, we specialize Theorem 1.2 to the cases $n = 0, 1$ and 2.

Example 1.3 (The case $n = 0$)
If $n = 0$ our formula reduces to the Abramovich-Bertram Formula [AB01] relating enumerative invariants of Σ_0 and Σ_2. According to Example 2.7, $\mathcal{N}(u,0,d,\alpha) \neq 0$ if and only if $d = (1^u)$ and $\alpha = 0$, so the only Newton fans δ which contribute to $N_\chi(\delta_0)$ are of the form

$$\delta = \delta_m = \{(1,2)^m, (0,-1)^{a+b}, (-1,0)^m, (0,1)^{a+b-2m}\},$$

for which we have $r = s = 0$, $A = m$, $U = a+b-2m$, and $\chi' = \chi$. Eventually, Theorem 1.2 reduces to Abramovich-Bertram Formula

$$N_\chi(\delta_0) = \sum_{m=0}^{a} \binom{a+b-2m}{a-m} \cdot N_\chi(\delta_m).$$

We point out that the above sum starts at $m = 0$ although the original Abramovich-Bertram formula starts at $m = 1$. This difference comes from the fact our formula involves *reducible* enumerative invariants, whereas the original involves *irreducible* invariants. In this latter situation one has $N_\chi^{irr}(\delta_0) = 0$ which explains the difference in the initial value in both summations.

In the special case $n = 0$, it is immediate to adapt the proof of Theorem 1.2 to get a formula only involving irreducible curves, and of course one recovers the original Abramovich-Bertram formula

$$N_\chi^{irr}(\delta_0) = \sum_{m=1}^{a} \binom{a+b-2m}{a-m} \cdot N_\chi^{irr}(\delta_m).$$

However in the general case, the most natural way of phrasing Theorem 1.2 is via reducible enumerative invariants. See also Proposition 4.17 for another application of our methods to irreducible enumerative invariants.

Example 1.4 (The case $n = 1$)
We now set up a formula relating enumerative invariants of Σ_1 and Σ_3. This formula can also be deduced from Ionel's and Parker's symplectic sum formula [IP04] (note however that this latter provides another, though equivalent, formula than ours). According to Example 2.7 we have a nonzero contribution for a fan δ only if $d = (2^{u-t_a}, 1^{2t_a})$ and $\alpha = (1^{u-t_a-t_b})$.

By notation 1.1, we thus have directions $(-\alpha_i, \beta_i) = (-1, 1)$ and $(0, \beta_{r+i}) = (0, 2)$ only. Hence the only Newton fans δ which contribute to $N_\chi(\delta_0)$ are of the form

$$\delta = \delta_{m,t_a,t_b} = \{(1,3)^m, (0,-1)^{2a+b}, (-1,0)^A, (0,2)^{t_b}, (-1,1)^R, (0,1)^{b-A+2t_a}\}$$

with $R = a - t_a - t_b - m$ and $A = 2m + t_a + t_b - a$. Theorem 1.2 reduces to

$$N_\chi(\delta_0) = \sum_{m=0}^{a} \sum_{t_a+t_b=0}^{a-m} 2^{t_b} \cdot (2t_a - 1)!! \cdot \binom{a + b + t_a - t_b - 2m}{2t_a} \cdot N_{\chi+2t_a}(\delta_{m,t_a,t_b}).$$

Example 1.5 (The case $n = 2$)
According to Example 2.7 we have a nonzero contribution only for fans of the form

$$\begin{aligned}\delta = \delta_{m,\underline{t}} = \ & \{(1,4)^m, (0,-1)^{3a+b}, (-1,0)^A, (-2,1)^{t_a}, (-1,2)^{t_b}, \\ & (-1,1)^{t_c}, (0,3)^{t_f}, (0,2)^{t_e}, (0,1)^{b-A+3t_d+t_c+t_e}\}\end{aligned}$$

with

$$\underline{t} = (t_a, t_b, t_c, t_d, t_e, t_f),$$

$t_a = a - t_c - t_b - t_f - t_e - t_d - m$, and $A = 3m + 2(t_f + t_e + t_d - a) + t_b + t_c$.
Theorem 1.2 reduces to

$$N_\chi(\delta_0) = \sum_{m=0}^{a} \sum_{t_b+t_c+t_d+t_e+t_f=0}^{a-m} \mathcal{D}_{m,\underline{t}} \cdot N_{\chi+2t_c+2t_e+4t_d}(\delta_{m,r,s})$$

with

$$\mathcal{D}_{m,\underline{t}} = 3^{t_b+t_f} \cdot 2^{t_e} \cdot \binom{3t_d + t_c + t_e}{t_c, t_e, 3, \dots, 3} \cdot \frac{t_c! t_e!}{t_d!} \cdot \binom{2a + b + t_d - t_e - 2t_f - 3m - t_b}{3t_d + t_c + t_e}.$$

There are t_d copies of 3 in the above multinomial coefficient.

1.2. **Our method.** Inspired by Kodaira's deformation of Hirzebruch surfaces, we consider an analogous deformation in the tropical world (see Appendix A). Since the tropical world has a *discrete* nature, quite often one does not need to go to the limit to actually *see* the limit. This is also the case here: for our proof, it is not really necessary to fully formalize the point of view of a tropical Kodaira deformation, and for that reason we only present it in the Appendix A, as motivation and background for our tropical deformation technique (see, in particular, Remark A.10). For our technique, it is enough to work on a tropical model of $\mathbb{T}\Sigma_n$ close enough to the limit of the deformation.

We model this closeness to the limit as follows: the tropical model of (an open part of) $\mathbb{T}\Sigma_n$ we use is the tropical surface X in $\mathbb{R}^3$ defined by the tropical polynomial "$x + y + z$". It consists of three half-planes $\sigma_1 = \{x = y \geq z\}$, $\sigma_2 = \{x = z \geq y\}$

and $\sigma_3 = \{y = z \geq x\}$ meeting along the line $L = \{x = y = z\} = \mathbb{R}(1,1,1)$, see Figure 2. The configuration of points through which we require curves to pass through are chosen on the face σ_1 very far down from the line L. In this setting, the

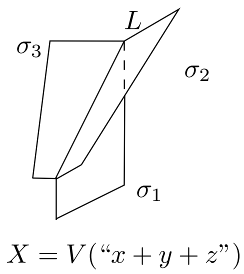

FIGURE 2. The tropical surface X.

tropical curves we are enumerating naturally contain two parts: one corresponding to the face σ_1, and one corresponding to the two upper faces σ_2 and σ_3. The former correspond to curves in $\mathbb{T}\Sigma_{n+2}$, and the latter to curves in $\mathbb{T}\Sigma_n$. This behaviour is parallel to Kodaira's deformation in the complex world as we discuss in more detail in Section 1.3.

Note that this ambient tropical surface X, different from the "usual" $\mathbb{R}^2$, is imposed by our strategy based on deformation of tropical Hirzebruch surfaces. Indeed the presence of a unique exceptional curve on Σ_n, with different self-intersection for different values of n, is an obstruction for any deformation of Σ_n to Σ_{n+2} to be toric. As a consequence, to model this deformation tropically one needs to make use of *tropical modifications* (see [Mik06]) of tropical Hirzebruch surfaces. In particular the tropical model of $\mathbb{T}\Sigma_n$ involved in this deformation is no longer a tropical toric compactification of $\mathbb{R}^2$.

Our two main ingredients for the proof of Theorem 1.2 are the enumeration of tropical curves in the tropical surface X and its relation, via Correspondence Theorems 4.13 and 3.17, to the enumeration of algebraic curves in Hirzebruch surfaces. Both aspects have not been much explored in the literature yet. Most papers about tropical enumerative geometry deal with the case of tropical curves in $\mathbb{R}^n$. In the case of curves in X, the tropical inclusion is more subtle than the set theoretic one, as it has already been noticed by several people (see for example [Vig09, BMb, BS]). In other words a tropical curve C might be set-theoretically contained in X without being tropically contained in X. One has to require extra conditions for this latter inclusion to hold. This phenomenon quite complicates the enumerative geometry of general tropical varieties. However, the surface X we are interested in here is simple enough so that only one extra condition, the so-called *Riemann-Hurwitz condition*, suffices to rule out parasitic tropical curves. The necessity of this condition has been observed earlier by the first author and Mikhalkin ([BMb], see also [BBM11], [ABBRa], and [ABBRb]). In the other direction, we prove Correspondence Theorems for tropical curves in X by reducing to Correspondence Theorems for tropical curves in $\mathbb{R}^2$ with a multiple point at some fixed point on the toric boundary. Thanks to this reduction, our proofs goes by a mild adaptation of the proofs of Correspondence Theorems from [Mik05] and [Shu12], in the framework of phase-tropical morphisms given in [BBM].

1.3. **Background, context, and difficulties.** Kodaira proved that the surface Σ_n can be deformed to Σ_{n+2k} with $k \geq 0$. As in [AB01], one may try to relate enumerative invariants of Σ_n and Σ_{n+2k} by studying the limit of curves when Σ_n deforms to Σ_{n+2k}. However this analysis gets much more complicated when $n > 0$. The main reason for that is that as soon as $n > 0$ both surfaces Σ_n and Σ_{n+2k} contain an exceptional curve, say E_n and E_{n+2k}, and that curves in Σ_n degenerate to curves in Σ_{n+2k} with singularities at k fixed points on E_{n+2k}. Those latter points may be thought as the "virtual intersection points of E_n and E_{n+2k}" defined by the chosen deformation from Σ_n to Σ_{n+2k}.

Let us explain in details the origin of the complications arising when $n > 0$.

First let us decompose the Kodaira deformation of Σ_n to Σ_{n+2k} into two steps: a deformation of Σ_n to the normal cone of a curve of bidegree $(1, k)$, followed by the blow-down of the Σ_n copy in the special fiber. More precisely, let V be a non-singular curve of bidegree $(1, k)$ in Σ_n[1], and let Σ be the trivial family $\Sigma_n \times \mathbb{C}$ blown-up along the curve $V \times \{0\}$. The natural projection $\mathrm{pr} : \Sigma \to \mathbb{C}$ defines a flat degeneration of Σ_n into the reducible surface $\mathrm{pr}^{-1}(0) = \Sigma_n \cup \mathbb{P}(\mathcal{N}_{V/\Sigma_n} \oplus \mathbb{C})$ intersecting transversally along $V \subset \Sigma_n$ and $V_\infty \subset \mathbb{P}(\mathcal{N}_{V/\Sigma_n} \oplus \mathbb{C})$. Since V has self-intersection number $n + 2k$ in Σ_n, we get that $\mathbb{P}(\mathcal{N}_{V/\Sigma_n} \oplus \mathbb{C}) = \Sigma_{n+2k}$ and $V_\infty = E_{n+2k}$ is the exceptional curve in Σ_{n+2k}.

It turns out that the Σ_n copy in $\mathrm{pr}^{-1}(0)$ can be contracted to V by a blow down $\mathrm{bl} : \Sigma \to \Sigma'$, and that the induced projection $\mathrm{pr}' : \Sigma' \to \mathbb{C}$ is precisely the Kodaira deformation of Σ_n to Σ_{n+2k} (see Figure 3). With this picture in mind, the "virtual

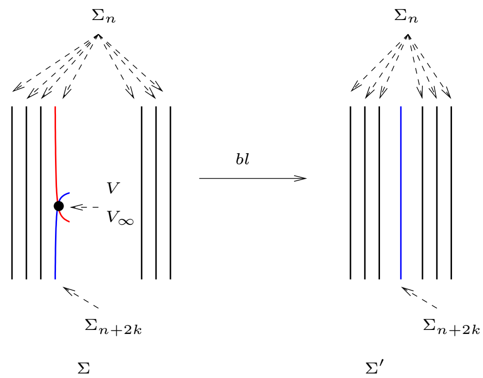

FIGURE 3. The 3-folds Σ and Σ'.

intersection points of E_n and E_{n+2k}" we mentioned above are now simply the intersection points of V and E_n in the Σ_n copy of the central fiber $\mathrm{pr}^{-1}(0) \subset \Sigma$. Relations between enumerative invariants of Σ_n and Σ_{n+2k} should now be derived from a careful analysis of how curves in Σ_n degenerate when this latter surface degenerates to $\Sigma_n \cup \Sigma_{n+2k}$.

At this point, the origin of the complications might appear more clearly with a symplectic point of view on the problem and the methods. On the level of the underlying

[1]The presentation $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \mathbb{C})$ induces a projection $\pi : \Sigma_n \to \mathbb{C}P^1$, and a curve of bidegree $(1, k)$ is characterized by the fact that it intersects in exactly k points the exceptional section and in one point any fiber of the map π; see Section 2.1 for details.

symplectic manifolds (recall that Σ_n is a Kähler manifold), the above deformation Σ of the reducible surface $\Sigma_n \cup \Sigma_{n+2k}$ to Σ_n can be seen as a symplectic sum of Σ_n and Σ_{n+2k} glued respectively along V and E_{n+2k}. There exist powerful symplectic sum formulas (see [IP04, LR01, EGH00], or [Li02] for a degeneration formula in the algebraic setting) relating Gromov-Witten invariants of a symplectic sum with those of the symplectic summands, but these Gromov-Witten invariants are *relative* to some *smooth* symplectic hypersurface. Symplectic sum formulas provide an alternative proof of the Abramovich-Bertram-Vakil results stated above, and more generally express Gromov-Witten invariants of a symplectic 4-manifold in term of its Gromov-Witten invariants relative to an embedded symplectic sphere E with self-intersection $-l$. Note however that as soon as $l \geq 2$, these formulas involve enumeration of ramified coverings, which singularly complicates actual computations when $l \geq 3$. Still, one can deduce in this way a relation among enumerative invariants of Σ_1 and Σ_3 equivalent to our Theorem 1.2 specialized to the case $n = 1$ (see Example 1.4). However when $n > 1$, relating enumerative invariants of Σ_n and Σ_{n+2k} requires to consider Gromov-Witten invariants relative to a *singular* symplectic curve. Indeed, the complex structure on the algebraic surface Σ_n is not generic as soon as $n > 1$ and enumerating algebraic curves on Σ_n is the same than computing Gromov-Witten invariants of the underlying symplectic manifold relative to the symplectic divisor E_n. In particular, relating enumerative invariants of Σ_n and Σ_{n+2k} using the deformation Σ can be seen as expressing Gromov-Witten invariants of Σ_n relative to E_n in terms of Gromov-Witten invariants of Σ_n relative to $E_n \cup V$ and of Σ_{n+2k} relative to E_{n+2k}. Since the curves E_n and V intersect in k points, Gromov-Witten invariants relative to a singular divisor show up naturally with the method we intend to apply. Up to our knowledge, Gromov-Witten invariants relative to a singular divisor have been defined only recently (see [Ion, Par], or [GS13, AC] in the algebraic setting) and there does not exist a general symplectic sum/degeneration formula for those invariants yet.

We view our tropical approach as a powerful tool to overcome these problems.

If one translates the above strategy in the tropical setting, then the family Σ is replaced by a single tropical surface X, and the study of degenerations of holomorphic curves is replaced by the enumeration of tropical curves in X. We then perform this enumeration in the special case of the degeneration of Σ_n to $\Sigma_n \cup \Sigma_{n+2}$. We obtain in this way Theorem 1.2, which may be seen as such a symplectic sum/degeneration formula in some particular instance of normal crossing divisor. The case of the degeneration of Σ_n to $\Sigma_n \cup \Sigma_{n+2k}$ should also be doable tropically, but requires some additional efforts (see Section 6).

The aim of the above discussion is to replace our work in the context of current mathematical developments and to explain where the difficulties we have to deal with come from. Having said that, we formulate our main result, Theorem 1.2, in the algebraic language without referring explicitly to relative Gromov-Witten invariants. Hence our formula expresses (some) enumerative invariants of Σ_n in terms of (some) enumerative invariants of Σ_{n+2} and some finitely many simple relative enumerative invariants of Σ_n, both of these latter invariants involving curves with a prescribed very singular point.

Organization of the paper. Our paper is organized as follows. In Section 2 we define Hirzebruch surfaces, and the enumerative invariants we are interested in. We recall basic tropical definitions in Section 3, and state a Correspondence Theorem (Theorem 3.17) to compute $(1,1)$-relative invariants of Σ_n in Section 3.2. Section 4 is devoted to the study of basic tropical enumerative geometry in X. In particular we state a second Correspondence Theorem (Theorem 4.13) in this setting, and prove Theorem 1.2. Both Correspondence Theorems are proved in Section 5. Section 6 contains some remarks about possible extensions of our present work. We end this paper by giving in Appendix A the tropical counterpart of Kodaira deformation of Hirzebruch surfaces.

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2. Complex Hirzebruch surfaces

2.1. **Definition.** A *Hirzebruch surface*, also called a rational geometrically ruled surface, is a compact complex surface which admits a holomorphic fibration to $\mathbb{C}P^1$ with fiber $\mathbb{C}P^1$. The classification of Hirzebruch surfaces is well known (see for example [Bea83]): they are all isomorphic to exactly one of the surfaces $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \mathbb{C})$ with $n \geq 0$. The surface Σ_n is called the *nth Hirzebruch surface*. For example $\Sigma_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$. Let us denote by B_n (resp. E_n, and F_n) the section $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \{0\})$ (resp. the section $\mathbb{P}(\{0\} \oplus \mathbb{C})$, and a fiber). The curves B_n, E_n, and F_n have self-intersections $B_n^2 = n$, $E_n^2 = -n$, and $F_n^2 = 0$. When $n \geq 1$, the curve E_n itself determines uniquely the Hirzebruch surface since it is the only reduced and irreducible algebraic curve in Σ_n with negative self-intersection. For example Σ_1 is the projective plane blown up at a point, and Σ_2 is the quadratic cone with equation $x^2 + y^2 + z^2$ is $\mathbb{C}P^3$ blown up at the node. In both cases the fibration is given by the extension of the projection from the blown-up point to a line (if $n = 1$) or a hyperplane section (if $n = 2$) which does not pass through the blown-up point.

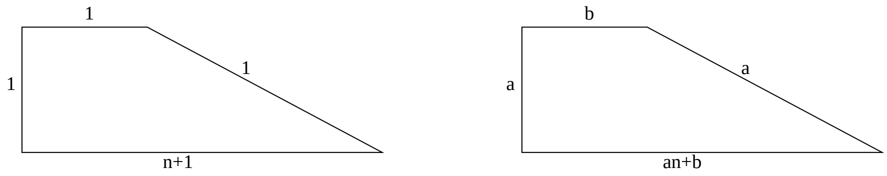
The group $\mathrm{Pic}(\Sigma_n) = \mathrm{H}_2(\Sigma_n, \mathbb{Z})$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and is generated by the classes of B_n and F_n. Note that we have $E_n = B_n - nF_n$ in $\mathrm{H}_2(\Sigma_n, \mathbb{Z})$. An algebraic curve S in Σ_n is said to be of *bidegree* (a, b) if it realizes the homology class $aB_n + bF_n$ in $\mathrm{H}_2(\Sigma_n, \mathbb{Z})$.

The surface Σ_n is a projective toric surface which can be obtained by taking two copies of $\mathbb{C} \times \mathbb{C}P^1$ glued by the biholomorphism

$$\begin{array}{ccc} \mathbb{C}^* \times \mathbb{C}P^1 & \longrightarrow & \mathbb{C}^* \times \mathbb{C}P^1 \\ (x_1, y_1) & \longmapsto & (\frac{1}{x_1}, \frac{y_1}{z_1^n}) \end{array}$$

The coordinate system (x_1, y_1) in the first chart is called *standard*. The surface Σ_n is the toric surface defined by the polygon depicted in Figure 4a (the number labeling an edge corresponds to its integer length). If S is a curve of bidegree

(a, b) in Σ_n then its Newton polygon in a standard coordinate system lies inside the trapeze with vertices $(0, 0)$, $(0, a)$, (b, a), and $(an + b, 0)$ (see Figure 4b), with equality if S is generic and $a, b \geq 0$. A curve of class B_n, E_n, or F_n is defined by the equation $y_1 = P(x_1)$, $y_1 = +\infty$, or $x_1 = c$, respectively, where $P(x)$ is a complex polynomial of degree at most n and $c \in \mathbb{C}P^1$.



a) A polygon defining Σ_n.

b) The Newton polygon of a curve of bidegree (a, b) in Σ_n.

FIGURE 4. Polygons of Hirzebruch surfaces.

2.2. **Toric relative invariants of Hirzebruch surfaces.** Since we work with curves in toric surfaces having possibly non-transversal intersection with toric divisors, it is more suitable to deal with Newton fans rather than Newton polygons. We will also use Newton fans in $\mathbb{Z}^3$ in Sections 4 and 5.

Definition 2.1 (Newton fans)
A *Newton fan* is a multiset $\delta = \{v_1, \ldots, v_k\}$ of vectors $v_i \in \mathbb{Z}^r$ satisfying

$$\sum_{i=1}^{k} v_i = 0.$$

The positive integer $w_i = \gcd(v_{i1}, \ldots, v_{ir})$ (resp. the vector $\frac{1}{w_i} v_i$) is called the *weight* (resp. the *primitive direction*) of v_i. We will use the notation

$$\delta = \{v_1^{m_1}, \ldots, v_k^{m_k}\}$$

to indicate that the vector v_i appears m_i times in δ. If $\delta = \{(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}$ is a Newton fan of vectors in $\mathbb{Z}^2$, one can construct the *dual polygon* Π_δ in $\mathbb{R}^2$ in the following way: for each primitive integer direction (α, β) in δ, we consider the vector $w(-\beta, \alpha)$, where w is the sum of the weights of all vectors in δ with primitive integer direction (α, β); Π_δ is the unique (up to translation) polygon whose oriented edges (the orientation is induced by the usual orientation of $\mathbb{R}^2$) are exactly the vectors $w(-\beta, \alpha)$.

To any complex algebraic curve S in $(\mathbb{C}^*)^n$, we may associate its Newton fan δ_S as follows: consider the toric compactification $\mathrm{Tor}(\Pi_S)$ of $(\mathbb{C}^*)^n$ given by a polytope Π_S such that S does not intersect boundary components of $\mathrm{Tor}(\Pi_S)$ of codimension two or more. Then each puncture p of S corresponds to a facet γ of Π_S. We associate to p the element $w_p v_p$ where v_p is the primitive normal vector to γ oriented outward Π_S, and w_p is the order of contact at p of S with the toric divisor corresponding to γ in the toric variety $\mathrm{Tor}(\Pi_S)$. The choice of Π_S is clearly not unique however δ_S does not depend on this choice. Note that if $n = 2$, one may choose Π_S equal to Π_{δ_S}.

The enumerative invariants of Σ_n we are mainly interested in here are defined by counting curves with prescribed intersection profile with the toric divisors of $\mathrm{Tor}(\Sigma_n)$, and hence correspond to *toric relative invariants* of the surface Σ_n.

Definition 2.2 (Toric relative invariants of Σ_n)
Given a Newton fan δ in $\mathbb{Z}^2$ and an integer $\chi \in \mathbb{Z}$, the number of algebraic curves (resp. irreducible algebraic curves) in $\mathrm{Tor}(\Pi_\delta)$ with Newton fan δ, whose normalization has Euler characteristic χ, and passing through a generic configuration ω of $\#\delta - \frac{\chi}{2}$ points does not depend on ω; we denote this number by $N_\chi(\delta)$ (resp. $N_\chi^{irr}(\delta)$).

Alternatively $N_\chi(\delta)$ (resp. $N_\chi^{irr}(\delta)$) is the number of algebraic curves (resp. irreducible algebraic curves) in $(\mathbb{C}^*)^2$ with Newton fan δ, whose normalization has Euler characteristic $\chi - \#\delta$, and passing through a generic configuration of $\#\delta - \frac{\chi}{2}$ points in $(\mathbb{C}^*)^2$.

Remember that if one defines the genus of a compact non-singular reducible complex algebraic curve $S = S_1 \cup \ldots \cup S_k$, where S_i is irreducible, as $g(S) = g(S_1) + \ldots g(S_k) + 1 - k$, then one has $\chi(S) = 2 - 2g(S)$.

Example 2.3
In this text we consider complex curves in $(\mathbb{C}^*)^2$ with Newton fans of the form

$$\delta = \{(1,n)^a, (0,-1)^{an+b}, (-\alpha_1, \beta_1), \ldots, (-\alpha_k, \beta_k)\}$$

with $\alpha_i, \beta_i \geq 0$ for all i. A complex curve with such a Newton fan can naturally be seen as a curve of bidegree (a,b) in Σ_n with a singularity at the point $(0,\infty)$ in the standard coordinates corresponding to Π_δ. In particular, a generic algebraic curve in Σ_n of bidegree (a,b) will have the Newton fan

$$\delta(a,b,n) = \{(1,n)^a, (0,-1)^{an+b}, (-1,0)^a, (0,1)^b\}$$

in standard coordinates.

We give below some well known enumerative invariants of Hirzebruch surfaces:

$$N_2(\delta(1,0,n)) = N_2(\delta(0,1,n)) = 1 \quad \text{and} \quad N_2(\delta(2,2,0)) = 12.$$

2.3. $(1,1)$**-relative invariants of Hirzebruch surfaces.** Theorem 1.2 involves some additional relative enumerative invariants of Hirzebruch surfaces Σ_n. We give their definition below, and refer to Sections 3.2 and 5.3 for more details about those invariants and their tropical computation. Given two algebraic curves S and S' in an algebraic surface intersecting in finitely many points, we denote by $S \circ S'$ the intersection number of S and S', and by $(S \circ S')_p$ the intersection multiplicity of S and S' at a point p.

Let S_0 be a curve of bidegree $(1,1)$ in Σ_n and let $p_0 \in S_0 \setminus E_n$ be a point. We denote by F_0 the unique curve of bidegree $(0,1)$ passing through p_0 and we choose a (non-standard) local system of coordinates (x_0, y_0) on Σ_n at p_0 such that S_0 has local equation $y_0 = 0$ and F_0 has local equation $x_0 = 0$. Given two integer numbers d_1 and d_2, a (d_1, d_2)-*germ* at p_0 is a curve in Σ_n with local equation $x_0^{d_2'} + c y_0^{d_1'} = 0$ with $c \in \mathbb{C}^*$, $d_1' = \frac{d_1}{\gcd(d_1,d_2)}$, and $d_2' = \frac{d_2}{\gcd(d_1,d_2)}$. Let D be a local branch at p_0 of a reduced algebraic curve in Σ_n containing neither S_0 nor F_0 as an irreducible component. Denote by d_{S_0} (resp. d_{F_0}) the local intersection multiplicity of D with S_0 (resp. F_0) at p_0. Then there exists a unique (d_{S_0}, d_{F_0})-germ at p_0 whose intersection multiplicity with D at p_0 is maximal. We call this curve the *tangent germ* of D at p_0.

Definition 2.4 ((1-1)-relative enumerative invariants of Σ_n)
Let $n \geq 0$, and $u, \alpha_1, \ldots \alpha_r, d_1, \ldots d_{r+s} > 0$ be integer numbers such that $\alpha_i \leq d_i - 1$ and $\sum_{i=1}^{r+s} d_i = u(n+1)$, and let S_0, F_0, and p_0 as above. Choose a configuration $\omega = \{G_1, \ldots, G_r, p_1, \ldots, p_s\}$ of s distinct point $p_1, \ldots, p_s$ in $S_0 \setminus (E_n \cup \{p_0\})$, and r distinct germs at p_0 such that G_i is a $(d_i + \alpha_i(n+1), \alpha_i)$-germ at p_0. We denote by $\mathcal{S}_{S_0,p_0}(\omega)$ the set of all algebraic curves S in Σ_n of bidegree $(a, 0)$ such that

- S has a smooth branch tangent to S_0 at p_i with intersection multiplicity d_{r+i} for $i = 1, \ldots, s$;
- S has exactly r branches $D_1, \ldots, D_r$ at p_0, $(D_i \circ S_0)_{p_0} = d_i + \alpha_i(n+1)$ and $(D_i \circ F_0)_{p_0} = \alpha_i$;
- G_i is the tangent germ of D_i at p_0;
- S has u connected components, whose normalization are all rational;
- each connected component of S intersect $F_0 \setminus \{p_0\}$ in a single point, with intersection multiplicity 1.

For ω generic, the set $\mathcal{S}_{S_0,p_0}(\omega)$ is finite and its cardinal is independent of S_0, p_0, and ω. We denote it by $\mathcal{N}(u, n, d, \alpha)$. Note that it is independent of the ordering of the pairs $(d_1, \alpha_1), \ldots, (d_r, \alpha_r)$ and of $d_{r+1}, \ldots, d_{r+s}$. If $r = 0$, we write $\alpha = 0$.

Remark 2.5
Given fixed n and u, there clearly exist finitely many choices for d and α. Furthermore the computation of $\mathcal{N}(u, n, d, \alpha)$ reduces to the computation of all finitely many possible $\mathcal{N}(1, n, d', \alpha')$, and to the combinatorial enumeration of how elements of ω can be distributed among the u irreducible components of S.

Lemma 2.6
We use the notation of Definition 2.4. If $\widetilde{S}$ is an irreducible component of an element S of $\mathcal{S}_{S_0,p_0}(\omega)$, then

$$\sum_{D_i \text{ branch of } \widetilde{S}} d_i = n + 1$$

and all intersection points of S with S_0 are exactly the points $p_0, p_1, \ldots, p_s$.

Proof:
The curve $\widetilde{S}$ has bidegree $(\widetilde{a}, 0)$ in Σ_n, so $\widetilde{S} \circ S_0 = \widetilde{a}(n+1)$ and $\widetilde{S} \circ F_0 = \widetilde{a}$. By the hypothesis we have $\widetilde{S} \circ S_0 \geq \sum d_i + (n+1)\sum \alpha_i$ and $\widetilde{S} \circ F_0 = \sum \alpha_i + 1$, where the sums are taken over integers i such that D_i is a branch of $\widetilde{S}$. So we deduce that $\sum d_i \leq n+1$. But since $\sum_{i=1}^{r+s} d_i = u(n+1)$, the latter inequality is in fact an equality. □

We show in Corollary 5.19 in Section 5.3 that $\mathcal{N}(u, n, d, \alpha)$ has an equivalent definition in terms of the enumeration of algebraic curves in $(\mathbb{C}^*)^2$ with a given Newton fan δ and a fixed multiple point on the toric boundary of $\mathrm{Tor}(\Pi_\delta)$. We give below the values of $\mathcal{N}(u, n, d, \alpha)$ in the cases $n = 0, 1, 2$, for which we will specialize

Theorem 1.2. Recall that $k!! = k \cdot (k-2) \cdot (k-4) \cdot \ldots 3 \cdot 1$ if k is odd, and that multinomial coefficients are defined by

$$\binom{t}{t_1, \ldots, t_k} = \binom{t}{t_k}\binom{t-t_k}{t_1, \ldots, t_{k-1}}.$$

Example 2.7
We suppose that $d_1 \geq \ldots \geq d_r$ and $d_{r+1} \geq \ldots \geq d_{r+s}$. The following values for $n = 0, 1$ and 2 are computed in Lemmas 3.19, 3.20, and 3.21:

$$\mathcal{N}(u,0,d,\alpha) = \begin{cases} 1 & \text{if } d = (1^u) \text{ and } \alpha = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{N}(u,1,d,\alpha) = \begin{cases} (2t_a-1)!! & \text{if } d = (2^{u-t_a}, 1^{2t_a}) \text{ and } \alpha = (1^{u-t_a-t_b}) \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{N}(u,2,d,\alpha) = \begin{cases} 3^{t_b} \cdot \binom{3t_d+t_c+t_e}{t_c, t_e, 3, \ldots, 3} \frac{t_c! t_e!}{t_d!} \\ \quad \text{if } d = (3^{t_a+t_b}, 2^{t_c}, 3^{t_f}, 2^{t_e}, 1^{3t_d+t_c+t_e}) \text{ and } \alpha = (2^{t_a}, 1^{t_c+t_b}) \\ \quad \text{with } t_a+t_b+t_c+t_f+t_e+t_d = u \\ 0 \quad \text{otherwise} \end{cases}$$

where there are t_d copies of 3 in the above multinomial coefficient.

3. Tropical enumerative geometry of Hirzebruch surfaces

3.1. **Basics in tropical geometry.** Here we briefly recall standard definitions in tropical geometry in order to fix notations we use in the rest of paper. For a more comprehensive introduction to the subject the reader may refer for example to [BIT08], [Gat06], [Mar08], [Mik05], [Mik06], [RGST05] and references therein.

3.1.1. *Tropical curves and morphisms.* We start by defining tropical curves and their morphisms to $\mathbb{R}^n$.

Given a (non-necessarily compact) topologically complete metric graph C, we denote by $\text{Edge}(C)$ the set of edges of C, by $\text{Vert}(C)$ the set of vertices of C, and by $\text{Edge}^\infty(C)$ the set of non-compact edges of C. Elements of $\text{Edge}^\infty(C)$ are called *ends* of C, and elements of $\text{Edge}^0(C) = \text{Edge}(C) \setminus \text{Edge}^\infty(C)$ are called *bounded edges*. The valency of a vertex V is the number of edges of C adjacent to V, and is denoted by $\text{val}(V)$.

A *degenerate metric graph* is the data of a graph C equipped with a complete metric on

$$C \setminus \left(\bigcup_{e \in E} e \right),$$

called the non-degenerate part of C, where E is some subset of $\text{Edge}^0(C)$. We write $l(e) \in \mathbb{R}_{>0}$ for the length of the bounded non-degenerate edge e. Edges in E are called *degenerate*, and their length $l(e)$ is defined to be 0. A connected component of the union of degenerate edges of C is called a *degenerate component* of C. The metric graph C is said to be *tamely degenerate* if C does not have a non-degenerate edge whose adjacent vertices are both contained in the same degenerate component.

Definition 3.1 (Abstract tropical curves)
An *abstract (punctured) tropical curve* is a tamely degenerate metric graph without degenerate loop-edges, equipped with a function

$$\begin{array}{ccc} \mathrm{Vert}(C) & \longrightarrow & \mathbb{Z}_{\geq 0} \\ V & \longmapsto & g_V \end{array}$$

satisfying the stability condition. The integer $g_V \in \mathbb{Z}_{\geq 0}$ is called the *genus* of the vertex V. The stability condition states that vertices of genus 0 are at least 3-valent, and vertices of genus 1 are at least 1-valent. Two abstract tropical curves are isomorphic (and will from now on be identified) if there exists an homeomorphism between them, restricting on an isometry on the non-degenerate parts, that respects the genus function on the vertices.

The tropical curve C is said to be *irreducible* if it is connected.

The *topological Euler characteristic* of C is $\chi(C) = b_0(C) - b_1(C)$ where $b_i(C)$ is the ith Betti number of C.

The *tropical Euler characteristic* of C is $\chi_{\text{trop}}(C) = \chi(C) - \sum_{V\in\mathrm{Vert}(C)} g_V$.

The *genus* of an abstract tropical curve C is defined to be $g(C) = 1 - \chi_{\text{trop}}(C)$.

We say that an abstract tropical curve is *explicit* if $g_V = 0$ for all vertices V.

The *combinatorial type* of an abstract tropical curve is the homeomorphism class of C together with the genus function and the degenerate edges, i.e. we drop the information about the positive lengths.

Remark 3.2
The consideration of tropical curves with degenerate edges has already been suggested in the literature (see for example [BBM, Definition 6.2]). However to our knowledge they were not rigorously introduced and used up to now since tropical curves without degenerate edges suffice for many applications of tropical geometry, for example for the enumeration of curves in toric varieties. Degenerate edges turn out to be necessary when considering tropical morphisms to tropical varieties different for $\mathbb{R}^n$, where it is essential to distinguish the set-theoretic inclusion from the tropical-theoretic inclusion. We refer to Section 5.1 for more details.

Remark 3.3
We recover the classical formula $\chi(C) = \#\mathrm{Vert}(C) - \#\mathrm{Edge}(C)$ only in the case when C is compact. When C is not compact, we have

$$\chi(C) = \#\mathrm{Vert}(C) - \#\mathrm{Edge}^0(C) = \#\mathrm{Vert}(\overline{C}) - \#\mathrm{Edge}(\overline{C}),$$

where $\overline{C}$ is the compact tropical curve obtained from C by adding one vertex to each of its ends.

Note that $\chi(C) = \chi_{\text{trop}}(C)$ if and only if C is explicit, and that $g(C) = b_1(C) + \sum_{V\in\mathrm{Vert}(C)} g_V + 1 - b_0(C)$.

Definition 3.4 (Tropical morphisms of curves to $\mathbb{R}^n$)
A tropical morphism is a continuous map $h : C \to \mathbb{R}^n$ from an abstract tropical curve C satisfying the following conditions:

- On each non-degenerate edge e the map h is integer affine linear, i.e. of the form $h_{|e}(t) = a + t \cdot v$ with $a \in \mathbb{R}^n$ and $v \in \mathbb{Z}^n$ once we have identify e with an interval of $\mathbb{R}$ by an isometry. If $V \in \partial e$ and we parameterize the edge e starting at V, the vector v in the above equation will be denoted $v(V, e)$ and called the *direction vector* of e starting at V. If V is understood from the context (e.g. in case e is an end, having only one adjacent vertex) we will also write $v(e)$ instead of $v(V, e)$. The greatest common divisor of the entries of $v(e)$ is called the *weight* of e for h.
- each degenerate edge e adjacent to the vertices V and V' is equipped with two vectors $v(V, e) = -v(V', e)$ in $\mathbb{Z}^n$; moreover e is mapped to a point by h (i.e. the restriction of h on e can be thought as an *infinitesimal* integer affine linear map).
- At each vertex V the *balancing condition* is satisfied, i.e.
$$\sum_{e\in \mathrm{Edge}(C);\ V\in\partial e} v(V, e) = 0.$$

Note that degenerate edges of C contribute to the balancing condition. Two tropical morphisms $h : C \to \mathbb{R}^n$ and $\tilde{h} : \tilde{C} \to \mathbb{R}^n$ are called *isomorphic* (and will from now on be identified) if there is an isomorphism of tropical curves $\phi : C \to \tilde{C}$ such that $\tilde{h} \circ \phi = h$.

The Newton fan δ of a tropical morphism $h : C \to \mathbb{R}^n$ is the multiset $\delta = \{v(e)\}_{e\in \mathrm{Edge}^\infty(C)}$.

The *combinatorial type* of $h : C \to \mathbb{R}^n$ is the combinatorial type of the underlying abstract curve C together with the direction vectors of all edges.

3.1.2. *Tropical morphisms of curves to a tropical surface.* The proof of Theorem 1.2 works by enumerating of tropical curves in a tropical surface $X \subset \mathbb{R}^3$ (see Remark A.10 for a justification). We first recall the definition of a tropical hypersurface of $\mathbb{R}^n$, and then define tropical morphisms $h : C \to X$ for a particular surface X.

Recall that the tropical operations on $\mathbb{R}$, denoted by "$+$" and "$\times$", are defined by
$$\text{“}a + b\text{”} = \max(a, b) \quad \text{and} \quad \text{“}a \times b\text{”} = a + b.$$

A tropical polynomial is then a piecewise-affine function
$$\begin{array}{cccll} P: & \mathbb{R}^n & \longrightarrow & \mathbb{R} & \\ & x & \longmapsto & \text{“}\sum_i a_i x^i\text{”} & = \max_i(a_i + \langle x, i\rangle), \end{array}$$
where $\langle , \rangle$ denotes the standard Euclidean product on $\mathbb{R}^n$.

Definition 3.5 (Tropical hypersurfaces)
The tropical hypersurface $Z(P)$ defined by a tropical polynomial $P(x) = \text{“}\sum_i a_i x^i\text{”}$ in n variables is the subset of $\mathbb{R}^n$ where the value of $P(x)$ is given by (at least) two distinct monomials of P.

In other words, a points x_0 is in $Z(P)$ if and only if there exist two distinct i and j in $\mathbb{Z}^n$ such that $P(x_0) = a_i + \langle x, i\rangle = a_j + \langle x, j\rangle$.

Remark 3.6
We work with a simplified definition of a tropical hypersurface: we neglect the weights that one usually assigns to the facets of the polyhedral complex $Z(P)$. We do not discuss these weights here, since for the particular tropical surface X considered in the rest of this paper they do not play any role.

Example 3.7
In the whole paper, we denote by P_X the tropical polynomial $P_X(x, y, z) =$ “$x + y + z$” and by X the tropical surface $Z(P_X)$ in $\mathbb{R}^3$. The surface X consists of three 2-dimensional cells, $\sigma_1 = \{x = y \geq z\}$, $\sigma_2 = \{x = z \geq y\}$ and $\sigma_3 = \{y = z \geq x\}$, that meet in the line $L = \{x = y = z\} = \mathbb{R}(1, 1, 1)$ (see Figure 2).

Now we define tropical morphisms $h : C \to X$. As it has already been mentioned in the introduction, the tropical inclusion is more subtle than the set theoretic inclusion. In the simple situation we deal with in this paper, i.e. X is just made of three faces meeting along the line L, there are only two extra conditions we have to impose on h to be tropically contained in X: the direction vectors of degenerate edges should be contained in the union of the linear spans of the faces of X, and h should satisfy the so-called Riemann-Hurwitz condition. The necessity of this latter condition has first been observed by the first author and Mikhalkin ([BMb], see also [BBM11], [ABBRa], and [ABBRb]). More conditions that have to be required for a tropical morphism $h : C \to \widetilde{X}$ to a more general tropical surface $\widetilde{X}$ can be found in [BS] and [GSW].

The above-mentioned Riemann-Hurwitz condition is based on tropical intersection theory. We first recall basic facts and definitions in tropical intersection theory restricting to the information we need for our purposes. We use notations from Example 3.7 in next definitions.

We define $\tilde{\sigma}_1 = \{x = y\}$, $\tilde{\sigma}_2 = \{x = z\}$, $\tilde{\sigma}_3 = \{y = z\}$, and $T_L X = \tilde{\sigma}_1 \cup \tilde{\sigma}_2 \cup \tilde{\sigma}_3$.

Definition 3.8 (Tropical premorphisms to X)
A *tropical premorphism* $h : C \to X$ is a tropical morphism $h : C \to \mathbb{R}^3$ such that

- $h(C) \subset X$;
- for any edge e adjacent to the vertex V, one has $v(V, e) \in T_L X$; moreover if $v(V, e) \in \tilde{\sigma}_i \setminus X$ one has $v(V, e') \in \tilde{\sigma}_i$ for all edges e' adjacent to V.

An edge e of C is said to be *tropically mapped to* L if $h(e) \subset L$ and $v(V, e)$ is parallel to $(1, 1, 1)$.

Note that the second condition above is non-empty only for degenerate edges. Obviously an edge e is tropically mapped to L if e is non-degenerate and $h(e) \subset L$. On the other hand, if e is degenerate then the fact that $f(e) \subset L$ does not imply that $v(V, e)$ is parallel to $(1, 1, 1)$.

Definition 3.9 (Intersection multiplicities)
Let $h : C \to X$ be a tropical premorphism, and let V be a vertex of C. If h does not map a neighborhood of V entirely in some plane $\tilde{\sigma}_i$, the *intersection multiplicity* of

C with L at V, denoted by d_V, is defined as

$$d_V = \sum_{V \in \partial e \text{ and } h(e) \subset \sigma_i} |P_X(v(V,e)) + P_X(-v(V,e))|$$

for any choice of face σ_i of X. Otherwise we set $d_V = 0$.

A vertex V of C is said to be *tropically mapped to* L if either $d_V > 0$ or each edge adjacent to V is tropically mapped to L.

The *overvalency* of V is defined by

$$\mathrm{ov}_V := k_V - d_V - 2 + 2g_V,$$

where k_V is the number of edges of C adjacent to V and not tropically mapped to L by h.

The *total intersection number* of C with L, denoted by $C \circ L$, is defined as

$$C \circ L = \sum_{V \in \mathrm{Vert}(C)} d_V.$$

Remark 3.10
The intersection multiplicity d_V does not depend on the choice of the face σ_i (see [AR10], [Sha13]). More precisely, if we choose $\sigma_i = \sigma_1$, then

$$v(V,e) = (x,x,z) \quad \text{and} \quad |P_X(v(V,e)) + P_X(-v(V,e))| = |x - z|.$$

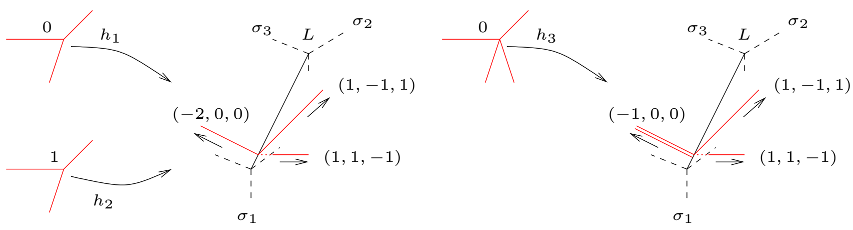
The total intersection number of C and L depends only on the Newton fan of $h : C \to X$: the Newton fan equals the recession fan of the (embedded) tropical curve $h(C)$, i.e. the fan obtained by shrinking all bounded edges of $h(C)$ to length 0; the total intersection number of C with L equals the tropical intersection of $h(C)$ with X in $\mathbb{R}^3$; since $h(C)$ is rationally equivalent to its Newton fan, this intersection depends only on the Newton fan.

Definition 3.11 (Tropical morphisms to X)
A tropical premorphism $h : C \to X$ is a tropical morphism to X if $\mathrm{ov}_V \geq 0$ for any vertex V of C with $d_V > 0$.

This extra condition for a premorphism to be a morphism is a consequence of the Riemann-Hurwitz formula in complex geometry (see Section 5.1 for more details) and is usually referred to as the *Riemann-Hurwitz condition*.

Example 3.12
Two examples of tropical premorphisms $h_1 : C_1 \to X$ and $h_2 : C_2 \to X$, where C_1 and C_2 are two tropical curves made of one vertex and 3 ends, are depicted in Figure 5a. The integer close the the vertex of C_i denotes its genus, and the vector close to the image of an edge e is the vector $v(e)$. In each case, $C_i \circ L = 2$. The tropical curve C_1 has genus 0, and C_1 has genus 1, so h_1 is not a tropical morphism to X, while h_2 is.

Figure 5b shows an example of a tropical morphism $h_3 : C_3 \to X$, where C_3 is a tropical curve of genus 0 made of one vertex and 4 ends. In the whole paper we use the same convention to encode both an abstract tropical curve and its image by a tropical premorphism in the same picture: the two edges we draw close to each other in the $(-1,0,0)$-direction are distinct edges in C_3 which have the same image by h_3.



a) b)

FIGURE 5. Examples of tropical (pre-)morphisms.

The *combinatorial type* of a tropical morphism $h : C \to X$ is the combinatorial type of the underlying abstract tropical curve together with the direction vectors for all edges, and together with the set of vertices and edges that are tropically mapped into L.

Notation 3.13
Given $h : C \to X$ a tropical morphism, we define $C_i = h^{-1}(\sigma_i)$ and $h_i = h_{|C_i}$ for $i = 1, 2, 3$. Note that the projection $(x, y, z) \mapsto (x, y)$ identifies each face σ_2 and σ_3 of X with a half-space in $\mathbb{R}^2$, and maps the lattice $\sigma_i \cap \mathbb{Z}^3$ isomorphically to the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. Hence using this projection and extending edges of C that meet L to ends, the map $h_i : C_i \to \mathbb{R}^2$ can be understood as a tropical morphism to the plane for $i = 1, 2$. In the same way, the projection $(x, y, z) \mapsto (y, z)$ identifies the face σ_1 of X with a half-space in $\mathbb{R}^2$, and using this projection the map $h_1 : C_1 \to \mathbb{R}^2$ can be understood as a tropical morphism to the plane.

3.2. **Tropical analogues of $(1,1)$-Relative invariants of Σ_n.** We now define tropical analogues of $(1,1)$-relative invariants and state their correspondence to the numbers $\mathcal{N}(u, n, d, \alpha)$ of Definition 2.4 in Theorem 3.17. We end this section by computing explicitly those invariants in the cases $n = 0, 1$, and 2. The proof of Theorem 3.17. is postponed to Section 5.3.

3.2.1. *Tropical analogues of $\mathcal{N}(u, n, d, \alpha)$.* If C_1 and C_2 are two tropical curves in the plane $\mathbb{R}^2$, we denote by $C_1 \circ C_2$ their tropical intersection number, and by $(C_1 \circ C_2)_p$ the stable intersection multiplicity of C_1 and C_2 at the point p (see for example [RGST05], Section 4).

Definition 3.14
Fix the following data:

- two integers $u \geq 1$ and $n \geq 0$,
- a tuple $d = (d_1, \ldots, d_{r+s})$ of positive integers satisfying $d_i \leq n + 1$ and $\sum_{i=1}^{r+s} d_i = u(n + 1)$,
- a tuple $\alpha = (\alpha_1, \ldots, \alpha_r)$ of positive integers satisfying $\alpha_i \leq d_i - 1$ (we write $\alpha = 0$ if $r = 0$).

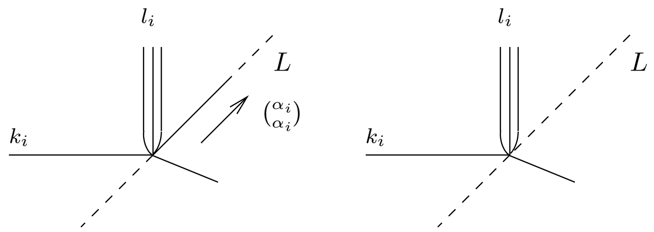
Suppose in addition that

$$\frac{d_i}{\alpha_i} \leq \frac{d_{i+1}}{\alpha_{i+1}} \quad \forall i = 1, \dots r-1.$$

Furthermore, choose a collection of $r+s$ points $\omega = (p_1, \dots, p_{r+s})$ on the line $L = \{x = y\}$ in $\mathbb{R}^2$, in such a way that the x-coordinates of the points p_i increase as i increases.

Let $\mathbb{T}\mathcal{S}'(u, n, d, \alpha, \omega)$ be the set of tropical morphisms $h : C \to \mathbb{R}^2$ passing through ω (i.e. satisfying $\omega \subset h(C)$) such that

- C is the disjoint union of of u rational tropical curves;
- for each $i = 1, \dots, r+s$, the curve C has exactly one vertex V_i with $h(V_i) = p_i$ such that the intersection multiplicity of C with L at V_i equals d_i. Furthermore, each vertex V_i is adjacent to k_i ends of direction $(-1, 0)$ and to $l_i := d_i - k_i$ ends of direction $(0, 1)$. In addition, these vertices satisfy:
  - *for* $i = 1, \dots, r$*:* V_i is $d_i + 2$-valent, in addition to the ends described above it is adjacent to one end of direction (α_i, α_i). In the following, we sum up all these requirements by saying that V_i is a vertex *of type 1* (see Figure 6a).
  - *for* $i = r+1, \dots, r+s$*:* V_i is $d_i + 1$-valent. In the following, we sum up all these requirements by saying that V_i is a vertex *of type 2* (see Figure 6b).
- each connected component of C contains exactly one end with direction vector $(1, -n)$;
- C has exactly $u(n+2) + r$ ends (in particular they all have been described above).



a) Vertex of type 1 b) Vertex of type 2

FIGURE 6. Vertex types on L of tropical morphisms in $\mathbb{T}\mathcal{S}'(u, n, d, \alpha, \omega)$.

Let $h : C \to \mathbb{R}^2$ be an element of $\mathbb{T}\mathcal{S}'(u, n, d, \alpha, \omega)$. For each vertex V of C not mapped to L, we choose two adjacent vectors $e_{V,1}$ and $e_{V,2}$, and we define the multiplicity of $h : C \to \mathbb{R}^2$ as

$$\mu_h = \prod_{i=1}^{r+s} \binom{k_i + l_i}{k_i} \prod_{V \in \text{Vert}(C),\ V \neq V_i} |\det(v(e_{V,1}), v(e_{V,2}))|.$$

Finally we define the number

$$\mathbb{T}\mathcal{N}(u,n,d,\alpha) = \left(\prod_{i=1}^{r} \frac{1}{\gcd(d_i,\alpha_i)}\right) \cdot \left(\prod_{i=1}^{s} \frac{1}{d_{r+i}}\right) \cdot \sum_{h \in \mathbb{T}\mathcal{S}'(u,n,d,\alpha,\omega)} \mu_h.$$

We also set

$$\mathbb{T}\mathcal{N}(0,n,d,\alpha) = 1 \text{ if } d = \alpha = 0, \quad \text{and } 0 \text{ otherwise.}$$

Remark 3.15
It follows from the definition that the total intersection number of a component $\widetilde{C}$ of C with L equals $\sum_{V_i \in \widetilde{C}} d_i = n+1$.

Note also that since $0 < \alpha_i \leq d_i - 1$, we have $d_i \geq 2$ for $i \leq r$.

The next lemma states in particular that $\mathbb{T}\mathcal{N}(u,n,d,\alpha)$ is indeed finite. The proof follows exactly along the lines of [GM07b], so we do not include it here. Denote by $\mathcal{U}$ the subset of L^{r+s} consisting of tuple $(p_1,\ldots,p_{r+s})$ such that p_i has a strictly smaller x-coordinate than p_j if $i < j$, for all $i,j = 1\ldots r+s$.

Lemma 3.16
There exists a dense open subset U of $\mathcal{U}$ such that for any choice of collection ω in U satisfying the requirement of Definition 3.14 one has

- *the set $\mathbb{T}\mathcal{S}'(u,n,d,\alpha,\omega)$ is finite;*
- *for an element $h : C \to \mathbb{R}^2$ in $\mathbb{T}\mathcal{S}'(u,n,d,\alpha,\omega)$, any vertex V of C not mapped to L is 3-valent; furthermore h is an embedding in a neighborhood of V;*
- *the number $\mathbb{T}\mathcal{N}(u,n,d,\alpha)$ does not depend on the choice of ω in U.*

The independence of the number $\mathbb{T}\mathcal{N}(u,n,d,\alpha)$ of ω is also an immediate corollary of the next theorem, whose proof is postponed to Section 5.3.

Theorem 3.17
For any $u,n,d,$ and α satisfying the hypothesis of Definition 3.14 one has

$$\mathbb{T}\mathcal{N}(u,n,d,\alpha) = \mathcal{N}(u,n,d,\alpha).$$

3.2.2. *Examples.* We end this section by computing the numbers $\mathcal{N}(u,n,d,\alpha)$ for $n = 0,1$ and 2. The cases $n = 0$ and 1 could easily by done without the whole machinery of tropical geometry. However we perform their computation tropically as a warm-up for less straightforward computations. We first show that there are only two kinds of vertices of type 2, and that μ_h only gets non-trivial contribution from connected components of C containing a vertex of type 1. We now fix a tuple $\omega = (p_1,\ldots,p_{r+s})$ in U, and an element $h : C \to \mathbb{R}^2$ of $\mathbb{T}\mathcal{S}'(u,n,d,\alpha,\omega)$.

Lemma 3.18
Let $\widetilde{C}$ be a connected component of C containing an end e with direction vector $(-1,0)$ adjacent to a vertex V_{r+j} of type 2. Then $\widetilde{C}$ does not contain any vertex of type 1, and e is the only end of $\widetilde{C}$ with direction vector $(-1,0)$. In particular one

has

$$\prod_{i\ |\ V_i\in\widetilde{C}}\frac{1}{d_i}\cdot\prod_{i\ |\ V_i\in\widetilde{C}}\binom{k_i+l_i}{k_i}\cdot\prod_{V\in Vert(\widetilde{C}),\ V\neq V_i}|\det(v(e_1),v(e_2))|=1.$$

If $\alpha = 0$ (i.e. no vertices of type 1) and $u = 1$ (i.e. C is irreducible) the curve C is mapped to $\mathbb{R}^2$ as depicted in Figure 7.

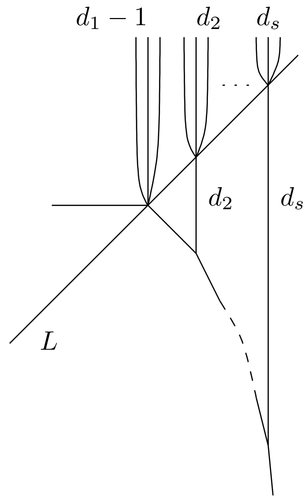


FIGURE 7. The element of $\mathbb{T}\mathcal{S}'(1, n, d, 0, \omega)$.

Proof:
Let T be the smallest subtree of $\widetilde{C}$ containing the end e' of $\widetilde{C}$ with direction vector $(1, -n)$, as well as all the vertices of type 2 contained in $\widetilde{C}$ (see Figure 8 where we depict only what happens in the half-plane $x > y$). Orient the edges in $\widetilde{C}$ such that they point towards the end e'. Assume that $T \neq \widetilde{C}$. For each vertex V_i of $\widetilde{C}$ we denote by e_i the unique edge of $\widetilde{C}$ adjacent to V_i and mapped to the half-plane $x > y$. By definition of the set $\mathbb{T}\mathcal{S}'(u, n, d, \alpha, \omega)$, the direction vector of the edge e_{r+i} has x-coordinate k_{r+i}. Any edge of $\widetilde{C}$ adjacent to T but not in T connects T to some of the points $p_1, \ldots, p_r$, which have a strictly less x-coordinate than the points $p_{r+1}, \ldots, p_{r+s}$. Thus any such edge has a positive x-coordinate. Since the sum of the x-coordinates of all such edges and of all edges e_{r+i} in $\widetilde{C}$ equals 1 we can conclude that there is in fact only one nonzero summand which equals one. By assumption, we have $k_{r+j} > 1$, so we must have $k_{r+j} = 1$ and $T = \widetilde{C}$. The claim follows. The claim about the contribution of such a component to $\mathbb{T}\mathcal{N}(u, n, d, \alpha)$ follows since the vertices below L contribute $\prod_{i=2}^{s} d_i$, the binomial factors are all one except for V_1 where we get d_1, and we divide by $\prod_{i=1}^{s} d_i$. □

Lemma 3.19
We have

$$\mathcal{N}(u,0,d,\alpha)=\begin{cases}1 & \textit{if } d=(1^u) \textit{ and } \alpha=0\\ 0 & \textit{otherwise.}\end{cases}$$

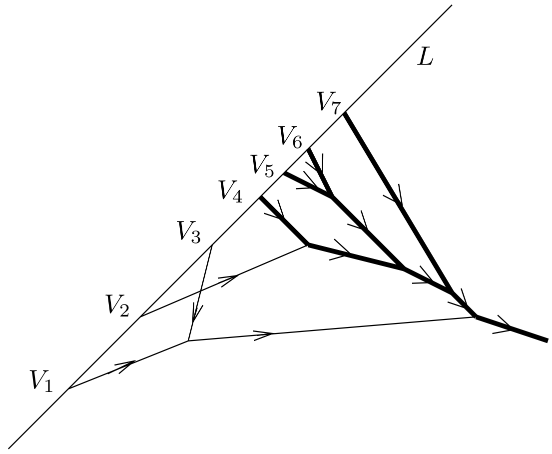


FIGURE 8. A component $\tilde{C}$, together with the tree T in thick.

In particular, for any morphism in $\mathbb{T}\mathcal{S}'(u, 0, (1^u), 0, \omega)$ *every connected component is mapped to a horizontal line as in Figure 9a.*

Proof:
It is enough to prove the lemma in the case $u = 1$. Let $h : C \to \mathbb{R}^2$ be in $\mathbb{T}\mathcal{S}'(1, 0, d, \alpha, \omega)$. Remark 3.15 implies that $\sum d_i = 1$ and $r = 0$. So $s = 1$, which completes the proof by Theorem 3.17. □

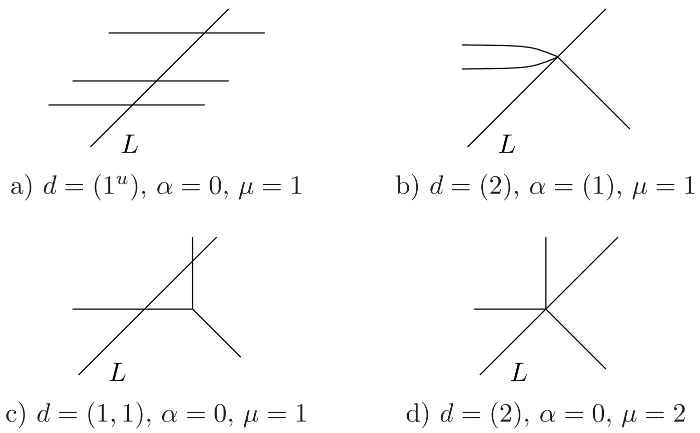


FIGURE 9. $\mathcal{N}(u, n, d, \alpha)$ for $n = 0$ and 1.

Lemma 3.20
Suppose that $d_{r+1} \geq \ldots \geq d_{r+s}$. *We have*

$$\mathcal{N}(u, 1, d, \alpha) = \begin{cases} (2t_a - 1)!! & \text{if } d = (2^{u-t_a}, 1^{2t_a}) \text{ and } \alpha = (1^{u-t_a-t_b}) \\ 0 & \text{otherwise.} \end{cases}$$

Proof:
Let $h : C \to \mathbb{R}^2$ be an element of $\mathbb{T}\mathcal{S}'(u,1,d,\alpha,\omega)$. We study the connected components of C, i.e. we assume first that $u=1$, which implies $\sum d_i = n+1 = 2$.

If $r > 0$, then by Remark 3.15 we have $d = (2)$ and $\alpha = (1)$. This implies immediately that C has only one vertex, and that the only possibility for the Newton fan of h is $\{(1,-1),(1,1),(-1,0)^2\}$. In this case C is mapped to $\mathbb{R}^2$ as depicted in Figure 9b and $\mu_h = 1$.

If $r = 0$ according to Lemma 3.18 the Newton fan of h is $\{(-1,0),(0,1),(1,-1)\}$, the curve C is mapped to $\mathbb{R}^2$ as depicted in Figure 9c or d, and μ_h equals respectively 1 and 2.

In the general case, every connected component of C is mapped to $\mathbb{R}^2$ as depicted in Figures 9b, c, and d. Thus we must have $\alpha = (1^r)$, and r equals the number of components as in Figure 9b. Denote by t_b the number of vertices of type 2 with $d_i = 2$ (i.e. the number of components as in Figure 9d) and by $2t_a$ the number of vertices of type 2 with $d_i = 1$ (i.e. t_a is the number of components as in Figure 9c). Since u is the total number of components we have $r + t_a + t_b = u$. We can thus express $d = (2^r, 2^{t_b}, 1^{2t_a})$ as $d = (2^{u-t_a}, 1^{2t_a})$ and $\alpha = (1^r) = (1^{u-t_a-t_b})$.

There is a one to one correspondence between $\mathbb{T}\mathcal{S}'(u,1,d,\alpha,\omega)$ and the partitions into pairs of the set of p_i such that $d_i = 1$. Hence we have $\#\mathbb{T}\mathcal{S}'(u,1,d,\alpha,\omega) = (2t_a - 1)!!$. The multiplicities of the components as in Figure 9d exactly cancels with the global factor $\prod \frac{1}{d_{r+i}}$ in the definition of $\mathbb{T}\mathcal{N}(u,1,d,\alpha)$, thus we have $\mathbb{T}\mathcal{N}(u,1,d,\alpha) = \mathcal{N}(u,1,d,\alpha) = (2t_a-1)!!$ by Theorem 3.17. □

Lemma 3.21
Suppose that $d_1 \geq \ldots \geq d_r$ and $d_{r+1} \geq \ldots \geq d_{r+s}$. Then we have

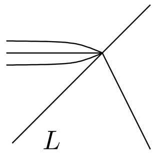
$$\mathcal{N}(u,2,d,\alpha) = \begin{cases} 3^{t_b} \cdot \binom{3t_d+t_c+t_e}{t_c,t_e,3,\ldots,3} \frac{t_c!t_e!}{t_d!} \\ \quad \textit{if } d = (3^{t_a+t_b}, 2^{t_c}, 3^{t_f}, 2^{t_e}, 1^{3t_d+t_c+t_e}) \textit{ and } \alpha = (2^{t_a}, 1^{t_c+t_b}) \\ \quad \textit{with } t_a+t_b+t_c+t_f+t_e+t_d = u \\ 0 \quad \textit{otherwise,} \end{cases}$$

where there are t_d copies of 3 in the above multinomial coefficient.

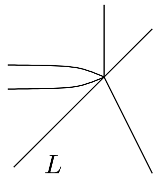
Proof:
Since the technique is exactly as in Lemmas 3.19 and 3.19, we briefly sketch the proof. Each component of an element of $\mathbb{T}\mathcal{S}'(u,1,d,\alpha,\omega)$ has to be mapped to $\mathbb{R}^2$ as depicted in Figure 10, and the result follows from considering all possible partitions of the points $p_1,\ldots,p_{r+s}$. Here, t_i denotes the number of components as in Figure 10i for all i=a,b,c,d,e,f. □

4. Tropical enumerative geometry in the plane X

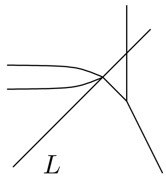
This section is the core of the present paper. We start by setting up a tropical enumerative problem in the tropical surface X (see Example 3.7) that we relate



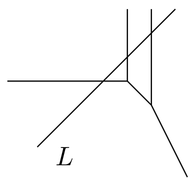
a) $d = (3)$, $\alpha = (2)$, $\mu = 1$



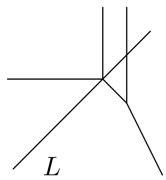
b) $d = (3)$, $\alpha = (1)$, $\mu = 3$



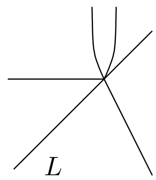
c) $d = (2, 1)$, $\alpha = (1)$, $\mu = 1$



d) $d = (1^3)$, $\alpha = 0$, $\mu = 1$



e) $d = (2, 1)$, $\alpha = 0$, $\mu = 2$



f) $d = (3)$, $\alpha = 0$, $\mu = 3$

FIGURE 10. Components of curves contributing to $\mathcal{N}(u, 2, d, \alpha)$.

to the enumerative geometry of complex Hirzebruch surfaces via a Correspondence Theorem (Section 4.2). This latter is an adaptation of Mikhalkin's Correspondence Theorem in [Mik05], and will be proved in Section 5. In Section 4.3, we prove our main result, Theorem 1.2. We also apply our new method to deduce a formula enumerating *irreducible* curves for some cases. This formula generalizes a result by Abramovich and Bertram (see [Vak00]).

4.1. **Basic tropical enumerative geometry in X.** Here we describe a particular kind of tropical enumerative problems concerning tropical morphisms through point conditions in X, and describe properties of the tropical morphisms that are solutions. Recall that the tropical surface X is made of three 2-dimensional cells, $\sigma_1 = \{x = y \geq z\}$, $\sigma_2 = \{x = z \geq y\}$ and $\sigma_3 = \{y = z \geq x\}$, meeting along the line $L = \mathbb{R}(1, 1, 1)$.

Notation 4.1
Let Δ be a Newton fan only containing vectors in X, but no vectors in L, and let $\chi \in \mathbb{Z}$. We denote by Δ_i the set of directions of Δ in σ_i for $i = 1, 2, 3$, and by d the intersection multiplicity of L with the Newton fan Δ (see Remark 3.10). For the rest of this section, we assume that any direction in Δ_3 has tropical intersection multiplicity 1 with L, i.e. $d = \#\Delta_3$.

Given a configuration ω of $\#\Delta - \chi - d$ points in $\sigma_1 \cup \sigma_2$, we denote by $\mathbb{T}\mathcal{S}(\omega)$ the set of all tropical morphisms $h : C \to X$ with Newton fan Δ, with $\chi_{\text{trop}}(C) = \chi$, and passing through all points in ω.

In the proof of Theorem 1.2 we use a configuration $\omega \subset \sigma_1$, however almost all results from this section still hold without this assumption. The rest of this section is devoted to prove the following proposition describing the elements of $\mathbb{T}\mathcal{S}(\omega)$. Recall that given a tropical morphism $h : C \to X$, we define $C_i = h^{-1}(\sigma_i)$ (see Notation 3.13).

Proposition 4.2
For a generic configuration ω, the set $\mathbb{T}\mathcal{S}(\omega)$ is finite. Moreover any tropical morphism $h : C \to X$ in $\mathbb{T}\mathcal{S}(\omega)$ satisfies the following properties:

1. *there is no edge e of C with $h(e) \subset L$ set theoretically;*
2. *any vertex V of C such that $h(V) \notin L$ is 3-valent; furthermore h is an embedding in a neighborhood of V;*
3. *the tropical curve C is explicit;*
4. *for any point $p \in \omega$, the set $h^{-1}(p)$ consists of a single point which is in the interior of an edge of C;*
5. *C_3 is a union of d ends of C;*
6. *for any vertex V of C such that $h(V) \in L$ one has $ov(V) = 0$, and V is adjacent to exactly one edge of C_1 and C_2;*

If $\omega \subset \sigma_1$ we have in addition:

7. *C_2 is a union of $\#\Delta_2$ trees.*

Note that it follows from the conditions above that C contains no degenerate edges, and that C_3 has exactly d_V ends adjacent to each vertex V with $h(V) \in L$. The proof of Proposition 4.2 will follow from several technical lemmas.

We first recall some basic facts about the structure of the space M^α of all tropical morphisms to $\mathbb{R}^2$ or X with a given combinatorial type α. The space M^α is also called the *(rigid) space of deformations* of α (or of a tropical morphism h of type α). We can naturally identify the space M^α with an unbounded open polyhedron in $\mathbb{R}^{2+\#\text{Edge}^0(C)}$: a tropical morphism $h : C \to \mathbb{R}^2$ or X in M^α is entirely determined by the coordinates of a root vertex of C and the lengths of all bounded edges of C. We cannot vary the lengths of non-degenerate edges independently however, since they have to satisfy the equations that the cycles close up, and that certain vertices and edges are mapped to L if h is a morphism to X. We call the dimension of M^α also the *dimension of α*.

We say that a plane tropical morphism $h : C \to X$ (resp. $h : C \to \mathbb{R}^2$) *contracts a cycle γ of C* if the set-theoretic intersection of X with the (classical) affine span in $\mathbb{R}^3$ of $h(\gamma)$ (resp. if $h(\gamma)$) has dimension at most 1.

Example 4.3
The tropical morphisms to $\mathbb{R}^2$ depicted in Figure 11a and b do not contract the cycle, however the one to $\mathbb{R}^2$ and to X respectively depicted in Figure 11c and d do.

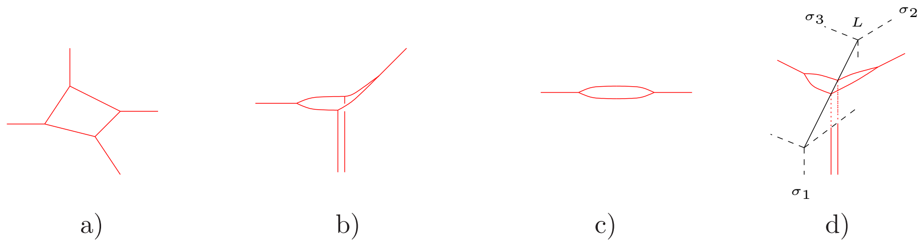


FIGURE 11. Examples of non-contracted and contracted cycles.

Lemma 4.4 (see e.g. [GM07b, Proposition 3.9])
The dimension of a combinatorial type α of tropical morphisms to $\mathbb{R}^2$ with Newton fan δ and topological Euler characteristic χ which does not contract any cycle is at most $\#\delta - \chi$. Moreover $\dim(M^\alpha) = \#\delta - \chi$ if and only if every vertex of α is 3-valent and α does not contain any degenerate edge.

The dimension of a combinatorial type of tropical morphisms to X is much harder to determine. We compute it only in a special situation which will be sufficient for our purposes. Recall that given a tropical morphism $h : C \to X$, an edge e of C is tropically mapped to L if $h(e) \subset L$ and $v(V, e)$ is parallel to $(1, 1, 1)$, and a vertex V of C is tropically mapped to L if either $d_V > 0$ or each edge adjacent to V is tropically mapped to L.

Notation 4.5
Let $h : C \to X$ be a tropical morphism of type α with no vertex V tropically mapped to L with $d_V = 0$. We define

$$\widetilde{C} = C \setminus \{e \in \text{Edge}(C) \mid h(e) \text{ is tropically mapped to } L\}$$

and $\widetilde{h} = h_{|\widetilde{C}}$. Note that $\chi(\widetilde{C}) = \chi(C) + l$, where l is the number of bounded edges of C tropically mapped to L, and that the rigid space of deformations of $\widetilde{h}$ is naturally identified with M^α. We define $\widetilde{C}_i = \widetilde{h}^{-1}(\sigma_i)$ and $\widetilde{h}_i = \widetilde{h}_{|\widetilde{C}_i}$. As explained in Notation 3.13, we may think of $\widetilde{h}_i : \widetilde{C}_i \to \sigma_i \subset \mathbb{R}^2$ as a tropical morphism to the plane $\mathbb{R}^2$. Let $V_1, \ldots, V_k$ be the vertices of $\widetilde{C}$ tropically mapped to L, and y_{ji} for $j = 1, \ldots, k$ and $i = 1, 2, 3$ the number of edges of $\widetilde{C}_i$ adjacent to V_j. We set $y_i = \sum_{j=1}^k y_{ji}$. We denote by χ_i the topological Euler characteristic of $\widetilde{C}_i$, and by v_j the valency of V_j in $\widetilde{C}$. By definition we have

$$v_j = \sum_{i=1}^{3} y_{ji}, \qquad \sum_{j=1}^{k} v_j = y_1 + y_2 + y_3$$

and

$$\chi(\widetilde{C}) = \chi_1 + \chi_2 + \chi_3 - \sum_{j=1}^{k} (v_j - 1) = \chi_1 + \chi_2 + \chi_3 - y_1 - y_2 - y_3 + k. \quad (1)$$

Lemma 4.6
Consider a combinatorial type α of tropical morphisms $h : C \to X$ without any contracted cycle, with no vertex V tropically mapped to L with $d_V = 0$, and which

maps l bounded edges of C tropically to L. Then the dimension of α is less than or equal to $\#\Delta - \chi(C) - l - d$.

Proof:
According to Lemma 4.4, the space of deformations of each morphism $\widetilde{h}_i$ has dimension at most $\#\Delta_i + y_i - \chi_i$. We cannot vary $\widetilde{h}_1$, $\widetilde{h}_2$ and $\widetilde{h}_3$ independently however, since we want to glue them to a single map $\widetilde{h}$. First of all, this imposes some conditions on $\widetilde{h}_1$ itself: for each vertex V_j, we have to require that the y_{j1} adjacent edges meet the same point on L. This yields $y_{j1} - 1$ conditions for each vertex V_j. These conditions are all independent since we assumed that C has no contracted cycle, hence we get $y_1 - k$ conditions altogether for $\widetilde{h}_1$. Since all y_{j2} edges of $\widetilde{C}_2$ adjacent to V_j meet the point $\widetilde{h}_1(V_j)$, we get y_2 conditions for $\widetilde{h}_2$ altogether. Also those are independent because we do not have any contracted cycle. Analogously, we get y_3 independent conditions for $\widetilde{h}_3$. Thus the dimension of α is less than or equal to

$$\begin{aligned}
&\#\Delta_1 + y_1 - \chi_1 + \#\Delta_2 + y_2 - \chi_2 + \#\Delta_3 + y_3 - \chi_3 - y_1 - y_2 - y_3 + k \\
=&\#\Delta - \chi(\widetilde{C}) - y_1 - y_2 - y_3 + 2k \\
=&\#\Delta - \chi(C) - l - \sum_{j=1}^{k} v_j + 2k \\
=&\#\Delta - \chi(C) - l - \sum_{j=1}^{k}(\mathrm{ov}_{V_j} + d_{V_j} + 2) + 2k \\
\leq&\#\Delta - \chi(C) - l - \sum_{j=1}^{k}(d_{V_j} + 2) + 2k \\
=&\#\Delta - \chi(C) - l - d - 2k + 2k \\
=&\#\Delta - \chi(C) - l - d
\end{aligned}$$

where the first equality follows from Equation (1), and the inequality holds since $\mathrm{ov}_{V_j} \geq 0$ for $j = 1, \ldots, k$. □

Lemma 4.7
Consider a combinatorial type α of tropical morphisms $h : C \to X$ as in Notation 4.1 and 4.5. For each $i = 1, 2, 3$ we have

$$\#\Delta_i \geq \chi_i \text{ and } y_i \geq \chi_i.$$

The first inequality is an equality if and only if $\widetilde{C}_i$ is a union of $\#\Delta_i$ trees, the second if and only if $\widetilde{C}_i$ is a union of y_i trees.

Proof:
First note that there cannot be components of C whose image lies in σ_i. If that was the case, it follows from the balancing condition that such a component can only have ends with directions in L which contradicts our general assumptions on Δ. Indeed, if such a component has an end of another direction in σ_i, then it must meet the line L and hence it contains also parts in the other faces of X. Let k_i denote the number of connected components of $\widetilde{C}_i$. Since there are no components

whose image lies in σ_i, we have $k_i \leq y_i$. Since every connected component must contain at least one end, we have $\#\Delta_i \geq k_i$. Any connected component of $\widetilde{C}_i$ has Euler characteristic less than or equal to 1, so $\chi_i \leq k_i$. Moreover we have the equality $\chi_i = k_i$ if and only if $\widetilde{C}_i$ is a union of k_i trees. □

Lemma 4.8
Let $h : C \to X$ be an element of $\mathbb{T}\mathcal{S}(\omega)$ with no vertex V tropically mapped to L with $d_V = 0$, and which does not contract any cycle. Then h satisfies properties (3) and (4) of Proposition 4.2. Furthermore one has the following:

- *no bounded edge of C is tropically mapped to L;*
- *any vertex of C tropically mapped to L is adjacent to exactly one edge of C_1 and C_2;*
- *every vertex V of C_1 or C_2 not tropically mapped to L is 3-valent and h is an embedding in a neighborhood of V;*
- *any degenerate edge of C contained in C_1 or C_2 is adjacent to a vertex of C tropically mapped to L.*

Proof:
We use the notations introduced in 4.1 and 4.5. According to Lemma 4.4 the dimension of the space of deformations of $\widetilde{h}_i$ is at most $\#\Delta_i + y_i - \chi_i$. Assume that r of the $\#\Delta - \chi - d$ point conditions lie in σ_1, the other $\#\Delta - \chi - d - r$ points in σ_2. Then the morphism $\widetilde{h}_1 : \widetilde{C}_1 \to \mathbb{R}^2$ passes through a generic configuration of r points in the plane, and satisfies moreover the gluing conditions that the y_{1j} ends adjacent to V_j in $\widetilde{C}_1$ must meet the same point. As in the proof of 4.6 this gives $b_1 = y_1 - k \geq 0$ extra independent conditions on $\widetilde{h}_1$. Hence we can conclude that $\#\Delta_1 + y_1 - \chi_1 \geq r + b_1$. In the same way we have $\#\Delta_2 + y_2 - \chi_2 \geq \#\Delta - \chi - d - r + b_2$ where $b_2 = y_2 - k \geq 0$. Finally, we have k extra independent gluing conditions for the morphisms h_1 and h_2 to match along L. Altogether we have

$$\begin{aligned}
0 &\geq r + b_1 - \#\Delta_1 - y_1 + \chi_1 + \#\Delta - \chi - d - r + b_2 - \#\Delta_2 - y_2 + \chi_2 + k \\
&= \#\Delta_3 + \sum_V g_V - \chi(\widetilde{C}) + l - d + k - y_1 + \chi_1 + b_1 - y_2 + \chi_2 + b_2 \\
&= \#\Delta_3 + \sum_V g_V - \chi_3 + y_3 - k + l - d + k + b_1 + b_2 \\
&= \sum_V g_V + (y_3 - \chi_3) + l + b_1 + b_2
\end{aligned}$$

where the second equality follows from Equation (1) and the third equality follows since $d = \#\Delta_3$ by the assumption made in Notation 4.1. The five summands above are all non-negative. For the second one, this follows from Lemma 4.7, for the others it is obvious. We can thus conclude that each summand is zero. It follows that $l = 0$, the curve C is explicit, and that $y_1 = y_2 = k$. Hence the tropical morphism h satisfies property (3). Using Lemma 4.7, it follows that C_3 is a union of y_3 trees.

Also, it follows that the dimension of the type of $h_i : C_i \to \mathbb{R}^2$ equals $\#\Delta_i + y_i - \chi_i$ for $i = 1, 2$. By Lemma 4.4, every vertex of C_1 and C_2 not tropically mapped to L is 3-valent, and every bounded edge of C_1 or C_2 is non-degenerate.

Suppose now that C has a 3-valent vertex V_0 not tropically mapped to L in the neighborhood of which h is not an embedding. This means that two edges of C have the same (primitive) direction vector from V_0. Since h_i does not contract any cycle and $y_1 = y_2 = k$, by gluing edges which have the same image by h_i one can produce a tropical morphism $h' : C' \to X$ in $\mathbb{T}\mathcal{S}(\omega)$ with no vertex V tropically mapped to L with $d_V = 0$, which does not contract any cycle, and with a vertex V_1 not tropically mapped to L which is not 3-valent. But we just showed above that this is impossible, hence a contradiction.

We showed that the rigid space of deformation of h has dimension $\#\Delta - \chi - d$, which is exactly the number of independent conditions imposed by ω. In particular h cannot satisfy any further independent condition, like having a vertex or two points of C mapped to a point of ω, i.e. h also satisfies (4) □

Remark 4.9
If we assume that our point conditions lie in σ_1, a modification of the proof of Lemma 4.8 above shows in addition that C_2 is a union of $\#\Delta_2$ trees: In this case, $\#\Delta_1 + y_1 - \chi_1 \geq \#\Delta - \chi - d + b_1$ and

$$\begin{aligned} 0 \geq & \#\Delta - \chi - d + b_1 - \#\Delta_1 - y_1 + \chi_1 \\ = & (\#\Delta_2 - \chi_2) + \sum_V g_V + (y_2 - k) + (y_3 - \chi_3) + b_1 + l, \end{aligned}$$

Here, the first summand is nonnegative due to 4.7 and the third summand is nonnegative since there must be an edge inside σ_2 adjacent to each of the k vertices in L. As before it follows that all summands are zero, and in addition to the results of Lemma 4.8, we can conclude that in this situation C_2 is a union of $\#\Delta_2$ trees.

Corollary 4.10
If $h : C \to X$ is an element of $\mathbb{T}\mathcal{S}(\omega)$, then h does not contract any cycle and no bounded edge of C is tropically mapped to L. In particular, C does not contain any vertex V tropically mapped to L with $d_V = 0$.

Proof:
Suppose that $h : C \to X$ is an element of $\mathbb{T}\mathcal{S}(\omega)$ which contains a vertex V tropically mapped to L with $d_V = 0$. We denote by C_L the union of vertices and edges of C which are tropically mapped to L.

On each connected component of $h^{-1}(L) \cap C_L$ we identify all points having the same image in L. In this way, we produce a tropical morphism $h' : C' \to X$ with the same image as h, with no vertex V tropically mapped to L with $d_V = 0$, and with $b_1(C') \leq b_1(C)$. Note that any vertex of V of C' obviously satisfies $\mathrm{ov}_V \geq 0$. If $b_1(C') < b_1(C)$ we increase the genus of an arbitrary vertex of C' in such a way that $g(C') = g(C)$. If h' does not contract any cycle, then it follows from Lemma 4.8 that no bounded edge of C' is tropically mapped to L. So $C = C'$, $h = h'$, and the corollary is proved in this case.

So we are left to prove the corollary when h contracts a cycle and C does not contain any vertex V tropically mapped to L with $d_V = 0$. Let γ be a contracted cycle of $h : C \to X$. Because of the above, we may assume that $\gamma \cap C_L$ is finite and $h(\gamma \cap C_L) \cap L$ is either empty or a single point. There is a continuous involution on

$\gamma \setminus C_L$ which exists on any morphism with the same combinatorial type as h. We can quotient γ by this involution producing a tropical premorphism $h' : C' \to X$, where $b_1(C') = b_1(C) - 1$. We want to construct a tropical morphism $h'' : C'' \to X$ of genus $g(C)$ out of $h' : C' \to X$. To do so, we have to increase the genus of one vertex of C' by one, making sure that the overvalencies at vertices tropically mapped to L are all nonnegative.

If $h(\gamma \cap C_L) \cap L = \emptyset$ then the overvalency of the vertices of C' mapped to L are the same as those of C, and we increase by one the genus of an arbitrary vertex of C'.

If $h(\gamma \cap C_L) \cap L \neq \emptyset$ then it is a point. Let $V_1, \ldots, V_s$ be the vertices of γ tropically mapped to L. The involution identifies the vertices $V_1, \ldots, V_s$ of C with a single vertex V of C', with overvalency $\text{ov}_V = \sum_{i=1}^s \text{ov}_{V_i} - s' + 2(s-1)$, where s' is the number of pairs of edges in a neighborhood of the V_i that are glued together by the involution. Since $s' = s$, the overvalency can only become negative if $s = 1$. This is the case when the involution identifies two edges adjacent to a single vertex V_1 on L. In this situation we increase the genus of V_1 by one (the corrected overvalency is now positive) thus again producing a morphism $h'' : C'' \to X$ of the same genus. Otherwise, we increase the genus of an arbitrary vertex of C' by one.

We repeat this for any contracted cycle. Eventually, we obtain a tropical morphism $h'' : C'' \to X$ in $\mathbb{T}\mathcal{S}(\omega)$ of genus g, without any contracted cycle, and with $b_1(C'') < b_1(C)$. It follows from Lemma 4.8 that C'' is explicit, a contradiction. □

Lemma 4.11
If $h : C \to X$ is an element of $\mathbb{T}\mathcal{S}(\omega)$, then $\text{ov}(V) = 0$ for any vertex V tropically mapped to L, and C_3 is a union of d ends.

Proof:
By Corollary 4.10 the morphism h does not contract any cycle and does not tropically map any bounded edge of of C to L. By Lemma 4.8, the curve C is explicit, and $y_1 = y_2 = k$. Hence considering the sum $0 \leq \sum_V \text{ov}(V)$ over all vertices with $h(V) \in L$ we obtain

$$\begin{aligned} \textstyle\sum_V \text{ov}(V) &= \textstyle\sum_V (\text{val}(V) - d_V - 2 + 2g_V) \\ &= -d + \textstyle\sum_V \text{val}(V) - 2k \\ &= -d + y_1 + y_2 + y_3 - 2k \\ &= -d + y_3 \\ &\leq 0 \end{aligned}$$

since by definition of $d = C \circ L$ we have $d \geq y_3$. Hence $\sum_V \text{ov}(V) = 0$ and $y_3 = d$ as claimed. □

Proof of Proposition 4.2:
There are finitely many combinatorial types of tropical morphisms in X with Newton fan Δ and of Euler characteristic χ. If $h : C \to X$ is an element of $\mathbb{T}\mathcal{S}(\omega)$ of combinatorial type α, it follows from Lemmas 4.6, 4.8 and Corollary 4.10 that $\dim(M^\alpha) \leq \#\Delta - \chi - d$. Since we fix $\#\Delta - \chi - d$ independent linear conditions, it follows that we have equality and that there is a unique tropical morphism of combinatorial type α in $\mathbb{T}\mathcal{S}(\omega)$. This proves that $\mathbb{T}\mathcal{S}(\omega)$ is finite.

Corollary 4.10 implies that the assumption of Lemma 4.8 are satisfied by all elements of $\mathbb{T}\mathcal{S}(\omega)$, in particular (3) and (4) are satisfied. Suppose that C has a degenerate edge e. By Lemma 4.8, any degenerate component of C is a tree and any edge e of this tree is adjacent to a vertex V of C tropically mapped to L. The other vertex V' adjacent to e is not tropically mapped to L since otherwise e would be tropically mapped to L contradicting Corollary 4.10. Hence V' is a trivalent vertex of C. Now two possibilities can occur:

(a) V' is adjacent to only one degenerate edge of C; in this case, by contracting e we produce a new tropical morphism $h' : C' \to X$ in $\mathbb{T}\mathcal{S}(\omega)$ with a vertex V satisfying $ov(V) > 0$.
(b) V' is adjacent to two degenerate edges; in this case by gluing these two edges as in the proof of Corollary 4.10 we produce a new tropical morphism $h' : C' \to X$ in $\mathbb{T}\mathcal{S}(\omega)$ with a vertex V satisfying $ov(V) > 0$.

Hence both cases contradict Lemma 4.11, and C does not have any degenerate edge. In particular any element of $\mathbb{T}\mathcal{S}(\omega)$ also satisfies (1).

Lemmas 4.11 and 4.8 give (5) and (6). It follows from (5) that there are no vertices of C mapped to in $\sigma_3 \setminus L$. Since we know from Lemma 4.8 that (2) is satisfied for any vertex in $\sigma_1 \cup \sigma_2 \setminus L$, (2) follows.

Remark 4.9 proves (7) for the case when $\omega \subset \sigma_1$. □

4.2. **Relation with enumerative geometry of Hirzebruch surfaces.** One can compute tropically enumerative invariants of Hirzebruch surfaces by enumerating tropical curves in $\mathbb{R}^2$ with Newton fan (see Example 2.3):

$$\delta_0 = \{(1, n)^a, (0, -1)^{an+b}, (-1, 0)^a, (0, 1)^b\}.$$

However to obtain the equation of Theorem 1.2, one has to enumerate tropical curves in X: Theorem 1.2 is based on the deformation of Σ_{n+2} to Σ_n which can be described tropically by the tropical surface X (see Appendix A for more detailed explanations). Hence we are interested in tropical curves with Newton fan

$$\{(1, n, 1)^a, (0, -1, 1)^{an+b}, (-1, 0, 0)^a, (0, 1, 0)^b, (0, 0, -1)^{a(n+1)+b}\}.$$

For convenience later in the formula, we apply the transformation $(x, y, z) \mapsto (x, -y, z)$, i.e. we fix the following Newton fan

$$\Delta = \{(1, -n, 1)^a, (0, 1, 1)^{an+b}, (-1, 0, 0)^a, (0, -1, 0)^b, (0, 0, -1)^{a(n+1)+b}\}.$$

Using Notation 4.1, we have

$$\begin{aligned}
\Delta_1 &= \{(0, 0, -1)^{a(n+1)+b}\},\\
\Delta_2 &= \{(1, -n, 1)^a, (0, -1, 0)^b\} \text{ and}\\
\Delta_3 &= \{(0, 1, 1)^{an+b}, (-1, 0, 0)^a\}.
\end{aligned}$$

Note that here $d = (n+1)a+b$, and $\#\Delta_3 = d$. In particular we are in the situation covered by Section 4.1. As in Section 4.1, let us choose an integer $\chi \in \mathbb{Z}$, and a generic configuration ω of $\#\Delta - \chi - d$ points in $\sigma_1 \cup \sigma_2$.

Following Notation 4.1, we denote by $\mathbb{T}\mathcal{S}(\omega)$ the set of all tropical morphisms $h : C \to X$ passing through ω, with Newton fan Δ, and with C a tropical curve such that $\chi_{\text{trop}}(C) = \chi$. Recall that any element of $\mathbb{T}\mathcal{S}(\omega)$ satisfies the properties $(1) - (6)$ given in Proposition 4.2.

Given an element $h : C \to X$ of $\mathbb{T}\mathcal{S}(\omega)$, we denote by $\text{Vert}_L(C)$ (resp. $\text{Vert}_{\sigma_i}(C)$) the set of vertices of C mapped to L (resp. $\sigma_i \setminus L$). If $V \in \text{Vert}_{\sigma_1}(C) \cup \text{Vert}_{\sigma_2}(C)$, then it follows from Proposition 4.2 that $\text{val}(V) = 3$.

Given $V \in \text{Vert}_{\sigma_i}(C)$, we choose any two of its adjacent edges $e_{V,1}$ and $e_{V,2}$. Note that we have $v(e_{V,j}) = (a_{V,j}, a_{V,j}, b_{V,j})$ if $i = 1$, and $v(e_{V,j}) = (a_{V,j}, b_{V,j}, a_{V,j})$ if $i = 2$ for some $a_{V,j}$ and $b_{V,j}$.

A vertex $V \in \text{Vert}_L(C)$ is adjacent to d_V ends mapped to σ_3, say k_V ends with direction $(-1, 0, 0)$ and l_V ends with direction $(0, 1, 1)$ (pointing away from L). Note that $k_V + l_V = d_V$.

Definition 4.12 (Multiplicity of a tropical morphism in $\mathbb{T}\mathcal{S}(\omega)$)
Let $h : C \to X$ be an element of $\mathbb{T}\mathcal{S}(\omega)$.

We define the multiplicity of a vertex $V \in \text{Vert}_{\sigma_i}(C)$ as

$$\mu_V = \left| \det \begin{pmatrix} a_{V,1} & a_{V,2} \\ b_{V,1} & b_{V,2} \end{pmatrix} \right|.$$

We define the multiplicity of a vertex $V \in \text{Vert}_L(C)$ as

$$\mu_V = \binom{k_V + l_V}{k_V}.$$

We define the multiplicity of h as

$$\mu_h = \prod_{V \in \text{Vert}(C)} \mu_V.$$

We also define the two following numbers

$$\mathbb{T}N_\chi(\Delta, \omega) = \sum_{h \in \mathbb{T}\mathcal{S}(\omega)} \mu_h.$$

and

$$\mathbb{T}N_\chi^{\text{irr}}(\Delta, \omega) = \sum_{h \in \mathbb{T}\widetilde{\mathcal{S}}(\omega)} \mu_h.$$

where $\mathbb{T}\widetilde{\mathcal{S}}(\omega)$ is the set of tropical morphisms $h : C \to X$ in $\mathbb{T}\mathcal{S}(\omega)$ from an irreducible tropical curve C.

The next Theorem is one of the main results of this paper.

Theorem 4.13 (Correspondence Theorem)
Let Δ, χ, and ω be as explained in the beginning of Section 4.2, and let as in Notation 1.1

$$\delta_0 = \{(1, n)^a, (0, -1)^{an+b}, (-1, 0)^a, (0, 1)^b\}.$$

Then we have

$$\mathbb{T}N_\chi(\Delta, \omega) = N_{2\chi}(\delta_0) \quad \textit{and} \quad \mathbb{T}N_\chi^{irr}(\Delta, \omega) = N_{2\chi}^{irr}(\delta_0).$$

We prove this theorem in Section 5 in the case of irreducible curves, from which the case of reducible curves follows immediately. A consequence of the Correspondence Theorem 4.13 is that the numbers $\mathbb{T}N_\chi(\Delta,\omega)$ and $\mathbb{T}N_\chi^{irr}(\Delta,\omega)$ do not depend on the choice of ω, as long as $\omega \subset \sigma_1 \cup \sigma_2$ is generic. We will thus also write $\mathbb{T}N_\chi(\Delta)$ and $\mathbb{T}N_\chi^{irr}(\Delta)$ for ω satisfying the requirements.

4.3. **Proof of Theorem 1.2.** We still fix Δ, χ, and ω as in the beginning of Section 4.2.

Notation 4.14
For the rest of the paper, we suppose that $\omega \subset \sigma_1$ and that the points in ω have very low z-coordinate compared to the x and y-coordinates.

To prove Theorem 1.2, we need the following lemma, which describes how tropical morphisms in $\mathbb{T}\mathcal{S}(\omega)$ can meet the line L. We still use Notation 3.13. Remember that any element of $\mathbb{T}\mathcal{S}(\omega)$ satisfies the properties $(1)-(7)$ of Proposition 4.2.

Lemma 4.15
Let $h : C \to X$ be an element of $\mathbb{T}\mathcal{S}(\omega)$, and V be a vertex of C mapped to the line L. Then all possibilities of how h can look like in a neighborhood of V are depicted in Figure 12.

Recall that C_3 only consists of ends of C (Proposition 4.2(5)), and that any connected component of C_2 contains exactly one end of C by 4.2(7). In cases a, b and c, the connected component of C_2 containing V also just consists of one end. In case d, this component might contain other vertices than V, i.e. the part in σ_2 in picture d can continue and contain more vertices on L. In particular, $(-\alpha+k, -\alpha-l, -\alpha+k)$ does not need to be the direction of an end of C.

Proof:
By property (6) in Proposition 4.2, every vertex V_i of C mapped into L has exactly one adjacent edge mapped in σ_1. We denote by $(-\alpha_i, -\alpha_i, \beta_i)$ the direction of this edge (pointing towards L), with $-\alpha_i < \beta_i$. Then by 4.2(5) some ends of C adjacent to V_i are mapped to σ_3, say k_i ends with direction $(-1, 0, 0)$ and l_i ends with direction $(0, 1, 1)$ (pointing away from L). Finally, by 4.2(6) again, exactly one edge adjacent to V_i is mapped to σ_2. By the balancing condition this edge has direction $(-\alpha_i + k_i, -\alpha_i - l_i, \beta_i - l_i)$ (pointing away from L). Since $k_i + l_i = d_{V_i} = \beta_i + \alpha_i$, we get $-\alpha_i + k_i = \beta_i - l_i$.

Let us consider one connected component of C_2, call it $\tilde{C}$ and assume it meets L at the vertices $V_1, \ldots, V_r$. Note that the sum of the intersection multiplicities of $\tilde{C}$ at the V_i equals the intersection multiplicity of the end of $\tilde{C}$ with L. If the end of $\tilde{C}$ is of direction $(0, -1, 0)$, then it intersects L with multiplicity 1, if it is of direction $(1, -n, 1)$, then it intersects with multiplicity $n+1$. So we have $\sum_{i=1}^r (k_i + l_i)$ equals 1 in the first case, or $n + 1$ in the second case.

First, let us consider the case when the end of $\tilde{C}$ is of direction $(0, -1, 0)$. Since all the k_i and l_i are nonnegative numbers, and $(k_i + l_i) > 0$ for each i, it follows that $r = 1$ and $\tilde{C}$ is in fact just an end, so $(-\alpha_1 + k_1, -\alpha_1 - l_1, -\alpha_1 + k_1) = (0, -1, 0)$. There are two possibilities how the vertex V_1 can look like: Either $k_1 = 1$, then $l_1 = 0$, $\alpha_1 = 1$ and $\beta_1 = 0$; thus h is as depicted in Figure 12a in a neighborhood

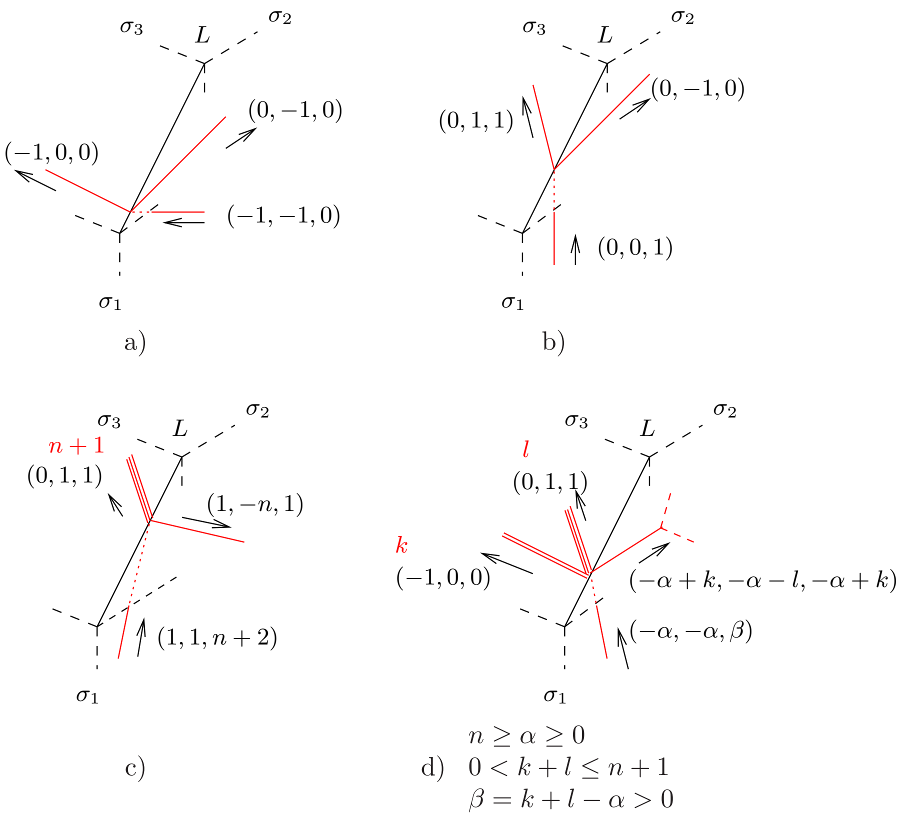


FIGURE 12. Four ways to hit L.

of V. Or $k_1 = 0$, then $l_1 = 1$, $\alpha_1 = 0$, and $\beta_1 = 1$; thus h is as depicted in Figure 12b in a neighborhood of V.

Now consider the case when the end of $\tilde{C}$ is of direction $(1, -n, 1)$. By the balancing condition, we then have $\big(\sum(-\alpha_i + k_i), \sum(-\alpha_i - l_i), \sum(-\alpha_i + k_i)\big) = (1, -n, 1)$. Since we choose the configuration $\omega \subset \sigma_1$ very far down from L (Notation 4.14), we may assume that the "slopes relative to L" of the edges of C_1 meeting L, i.e. the values $\frac{-\alpha_i+\beta_i}{-\beta_i-\alpha_i}$, decrease from left to right. If we assume the vertices are ordered from left to right, i.e. V_1 is the vertex most left on L, V_r the most right, then we have $\frac{-\alpha_i+\beta_i}{-\beta_i-\alpha_i} \geq \frac{-\alpha_j+\beta_j}{-\beta_j-\alpha_j}$ for $i < j$. If the relative slopes are not ordered like this for two edges, then the infinite continuations of these two edges intersect, and change their order after intersecting. Since we assume that the points are so far down, we can assume that all these changes of orders happen before the edges meet L, and thus the slopes are ordered as above. An example is depicted in Figure 13, where the slopes relative to L from the most left edge meeting L to the most right are $1, 1, 0, -1, -1, -3, -3$.

Note that since $\beta_i + \alpha_i > 0$, we have

$$\frac{-\alpha_i + \beta_i}{-\beta_i - \alpha_i} < 1 \iff \beta_i > 0, \quad \text{and} \quad \frac{-\alpha_i + \beta_i}{-\beta_i - \alpha_i} < -1 \iff \alpha_i < 0.$$

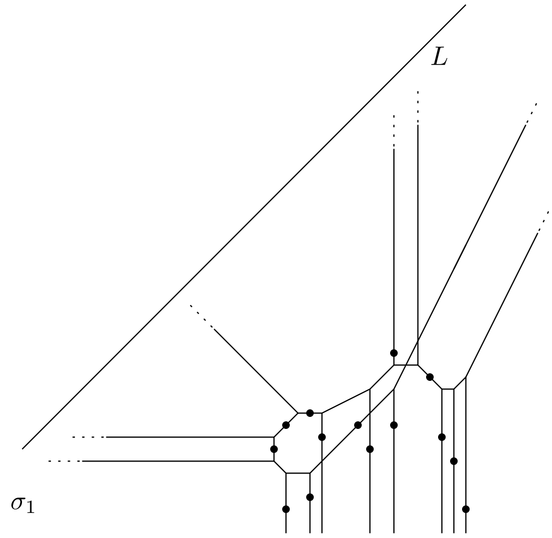


FIGURE 13. The slopes relative to L from the left to the right are $1, 1, 0, -1, -1, -3, -3$.

In particular, if $\alpha_i < 0$ (resp. $\beta_i > 0$) for some i then also $\alpha_j < 0$ (resp. $\beta_j > 0$) for all $j > i$.

Suppose that $\alpha_r < 0$. Denote by T the smallest subtree of $\tilde{C}$ containing the end of $\tilde{C}$ and all the vertices V_j, $j \geq i_0$, where i_0 is the smallest index such that $\alpha_{i_0} < 0$. Orient the edges in $\tilde{C}$ such that they point towards the end of $\tilde{C}$ (see Figure 8 where the subtree T is marked with thick edges). If $T = \tilde{C}$, then $i_0 = 1$ and all the $-\alpha_j + k_j$, $j = 1 \ldots r$, are positive. If $T \neq \tilde{C}$, there are two-valent vertices of T, i.e. in $\tilde{C}$ two of the adjacent edges belong to T and the third does not. Since the end of $\tilde{C}$ belongs to T, the third edge then has to point toward the vertex. Thus, it needs to connect to some vertices V_j, $j < i_0$, behind, and since these vertices lie to the left of the V_j, $j \geq i$, it needs to have a positive x-coordinate (in the orientation as before). Hence any edge in T and any edge adjacent to T has a positive x-coordinate. Every edge of $\tilde{C}$ is of direction $\sum_{m \in I}(-\alpha_m + k_m)$ for some subset $I \subset \{1, \ldots, r\}$. Edges adjacent to T correspond to disjoint subsets of $\{1, \ldots, i_0 - 1\}$ whose union equals $\{1, \ldots, i_0 - 1\}$. These edges thus hand us a way to group the summands of $\sum_{j=1}^{i_0-1}(-\alpha_j + k_j)$ in such a way that the sum of each group is positive, even though the single summands $-\alpha_j + k_j$ might be negative. The summands $-\alpha_j + k_j$ for $j \geq i$ are positive, too. For example in Figure 8 we have $(-\alpha_3 + k_3) < 0$, but $(-\alpha_1 + k_1) + (-\alpha_3 + k_3) > 0$.

Thus we grouped the whole sum $\sum_{j=1}^{r}(-\alpha_j + k_j)$ into positive summands, and in total we get 1. Since by assumption we have $-\alpha_r + k_r > 0$, we deduce that $r = 1$, $\alpha_1 = -1$, $k_1 = 0$ and $\tilde{C}$ is just a end, namely the end of direction $(1, -n, 1)$. Thus $-\alpha_1 - l_1 = -n$, $l_1 = n + 1$ and $\beta_1 = -\alpha_1 + k_1 + l_1 = n + 2$. In C_1, we thus have an edge meeting L with direction $(1, 1, n + 2)$. It follows that h is as depicted in Figure 12c in a neighborhood of V.

From now on we assume that $\alpha_i \geq 0$ for all vertices V_i in $\tilde{C}$. The ordering of the slopes also shows us that for the left-most vertex, V_1, we must have $-\alpha_1 + k_1 = \beta_1 - l_1 > 0$, since otherwise there would be no way to connect V_1 to the end of $\tilde{C}$. Since $l_1 \geq 0$ it follows that $\beta_1 > 0$, and so that $\beta_i > 0$ for $i = 1 \ldots r$. Consider the y-coordinate $-\alpha_i - l_i$ of the edge of $\tilde{C}$ adjacent to V_i. Since $\alpha_i \geq 0$ and $l_i \geq 0$ it is non-positive. The sum $\sum_{i=1}^{r}(-\alpha_i - l_i)$ equals $-n$. Since each summand is non-positive, we can conclude that each summand is bigger or equal to $-n$ and thus also $0 \leq \alpha_i \leq n$. From the above, we know that $\sum(k_i + l_i) = n + 1$, so in particular each summand $k_i + l_i$ has to be smaller or equal to $n + 1$. Thus h is as depicted in Figure 12d in a neighborhood of V. □

Definition 4.16
For an element $h : C \to X$ in $\mathbb{T}\mathcal{S}(\omega)$, denote by $C' \subset C$ the union of all connected components of $C \cap (\sigma_2 \cup \sigma_3)$ containing a vertex of type d as in Lemma 4.15. Let $h' : C' \to \mathbb{R}^2$ be the composition $h' = h \circ \pi$, where π denotes the projection to the first two coordinates of $\mathbb{R}^3$. The map h' is a tropical morphism, and we call it the *roof* of $h : C \to X$.

Note that it follows immediately from Lemma 4.15 that the roof of any element $h : C \to X$ in $\mathbb{T}\mathcal{S}(\omega)$ is in $\mathbb{T}\mathcal{S}'(a - m, n, d, \alpha, \omega')$ (see Definition 3.14) for values m, n, d and α coming from a fan $\delta \vdash \delta_0$ (see 1.1), and ω' determined by $h_1 : C_1 \to \mathbb{R}^2$.

Proof of Theorem 1.2:
As before (see Notation 1.1 and Section 4.2), let

$$\delta_0 = \{(1, n)^a, (0, -1)^{an+b}, (-1, 0)^a, (0, 1)^b\}, \text{ and}$$

$$\Delta = \{(1, -n, 1)^a, (0, 1, 1)^{an+b}, (-1, 0, 0)^a, (0, -1, 0)^b, (0, 0, -1)^{a(n+1)+b}\}.$$

It follows from the Correspondence Theorem 4.13 that

$$N_{2\chi}(\delta_0) = \mathbb{T}N_\chi(\Delta) = \mathbb{T}N_\chi(\Delta, \omega)$$

where $\omega \subset X$ is a configuration of points as before, i.e. in general position in σ_1 and very far down from L (see Notation 4.14). Let $h : C \to X$ be in $\mathbb{T}\mathcal{S}(\omega)$, i.e. a morphism to X contributing to $\mathbb{T}N_\chi(\Delta, \omega)$. We first want to show how we can split the information of h into data as required by the right hand side of the equation. We keep using Notation 3.13 and 4.1. By abuse of notation, we often do not distinguish between the tropical morphism $h_1 : C_1 \to \mathbb{R}^2$ to the plane and the restriction of $h_1 : C_1 \to \sigma_1$: if we speak about directions of edges, we use two coordinates x and z in both cases. In the first case, these denote the two coordinates of $\mathbb{R}^2$, in the second case, this is a shortcut for the direction (x, x, z) in σ_1.

Lemma 4.15 tells us what Newton fan δ of $h_1 : C_1 \to \mathbb{R}^2$, viewed as a plane curve by prolonging the edges that meet L, has:

- $a(n + 1) + b$ ends of direction $(0, -1)$, since these are just the ends of C in σ_1,
- $m \leq a$ ends of direction $(1, n + 2)$ which become vertices as in Figure 12c when meeting L,
- A ends of direction $(-1, 0)$ which become vertices as in Figure 12a when meeting L,

- U ends of direction $(0,1)$, B of them become vertices as in Figure 12b when meeting L, $U-B$ become vertices as in Figure 12d.
- ends of direction $(-\alpha_i,\beta_i)$ satisfying $n \geq \alpha_i \geq 0$, $0 < \beta_i$ and $1 < \beta_i$ if $\alpha_i = 0$ which become vertices as in Figure 12d.

Hence $h_1 : C_1 \to \mathbb{R}^2$ has Newton fan

$$\delta = \{(1,n+2)^m, (0,-1)^{a(n+1)+b}, (-1,0)^A, (-\alpha_1,\beta_1),\ldots,(-\alpha_r,\beta_r),(0,\beta_{r+1}),\ldots,(0,\beta_{r+s}),(0,1)^U\}$$

i.e. with $\delta \vdash \delta_0$.

We must have $A+B=b$ since this is the total number of ends with direction $(0,-1,0)$ of h. Since the total number of ends of direction $(0,1)$ in δ is U, and since B of these become vertices as in Figure 12b, we have $U-B=U+A-b$ ends of direction $(0,1)$ that are adjacent to the roof C' of C. We clearly have $\chi(C) = \chi(C_1) + \chi(C') - \#C_1 \cap C'$, so since any irreducible component of C' is a tree we get

$$\chi(C_1) = \chi(C) - (a-m) + (r+s+U+A-b) = \chi - (a+b-m-r-s-A-U).$$

By Lemma 4.15, the roof $h' : C' \to \mathbb{R}^2$ of $h : C \to X$ is an element of the set $\mathbb{T}\mathcal{S}'(a-m,n,d,\alpha,\omega')$ (see Definition 3.14) with

$$d = (\beta_1+\alpha_1,\ldots,\beta_r+\alpha_r,\beta_{r+1},\ldots,\beta_{r+s},1^{U+A-b}) \text{ and } \alpha = (\alpha_1,\ldots,\alpha_r)$$

and where ω' is determined by $h_1 : C_1 \to \mathbb{R}^2$ and the choice of $B = b-A$ ends of C_1 which become vertices as in Figure 12b when meeting L and accordingly are not part of the roof. We define

$$\mu_{h_1} = \prod_{V\in \mathrm{Vert}_{\sigma_1}(C)} \mu_V$$

where μ_V has been defined in Definition 4.12. So by definition we have

$$\mu_h = \prod_{i=1}^{r} \gcd(\alpha_i,\beta_i) \cdot \prod_{i=1}^{s} \beta_{r+i} \cdot \mu_{h_1}\mu_{h'}.$$

Conversely given a Newton fan δ as above, we denote by $\mathbb{T}\mathcal{S}''(\omega)$ the set of all tropical morphisms $h_1 : C_1 \to \mathbb{R}^2$ with Newton fan δ, Euler characteristic $\chi' = \chi - (a+b-m-r-s-A-U)$, and passing through all points in ω. According to the Correspondence Theorems in [Mik05] and [Shu12] (see also [GM07a]), we have

$$\sum_{h_1 \in \mathbb{T}\mathcal{S}''(\omega)} \mu_{h_1} = N_{2\chi'}(\delta).$$

Now choose any element $h_1 : C_1 \to \mathbb{R}^2$ of $\mathbb{T}\mathcal{S}''(\omega)$, and any set of $b-A$ ends of C_1 with direction $(0,1)$. These $b-A$ ends define a set $\mathbb{T}\mathcal{S}'(a-m,n,d,\alpha,\omega')$, and given any of its element $h' : C' \to \mathbb{R}^2$, one can reconstruct a unique tropical morphism $h : C \to X$ in $\mathcal{C}(\omega)$ reversing the construction above. □

In the following, we demonstrate that one can, with a little more care, also use our methods to count irreducible curves. We generalize to any genus a formula previously proved by Abramovich and Bertram for rational curves (see [Vak00]).

Given a tuple $(l_1, \ldots, l_k)$ of positive integers, we denote by $\tau_{l_1,\ldots,l_k}$ the number of its symmetries, i.e. if $(l_1, \ldots, l_k)$ contains exactly n_i times the entry i then

$$\tau_{l_1,\ldots,l_k} = \prod_i n_i!.$$

Proposition 4.17
Let $n, b, g \geq 0$ be integers, and

$$\delta_{n,b} = \{(1,n)^2, (0,-1)^{2n+b}, (-1,0)^2, (0,1)^b\}.$$

Then

$$\begin{aligned} N^{\mathrm{irr}}_{2-2g}(\delta_{n,b}) = N^{\mathrm{irr}}_{2-2g}(\delta_{n+2,b-2}) + \\ \sum_{l_1+\ldots+l_{g+1}=g+1}^{n+1} \binom{2(n+b)+3}{n+1-\sum l_i} \cdot \frac{(b+g-1)! \cdot \prod_{i=2}^{g+1} l_i^2}{(b-1)!} \cdot \\ \cdot \left(\frac{(b+g)l_1^2}{\tau_{l_1,\ldots,l_{g+1}}} + \frac{1}{\tau_{l_2,\ldots,l_{g+1}}} \cdot \binom{l_1}{2} \right). \end{aligned}$$

Proof:
To prove this equation, we mainly apply the techniques used in the proof of Theorem 1.2, taking into account irreducibility issues.

First of all, it follows from the Correspondence Theorem 4.13 that $N^{\mathrm{irr}}_{2-2g}(\delta_{n,b}) = \mathbb{T}N^{\mathrm{irr}}_{1-g}(\Delta)$ where

$$\Delta = \{(1,-n,1)^2, (0,1,1)^{2n+b}, (-1,0,0)^2, (0,-1,0)^b, (0,0,-1)^{2(n+1)+b}\}.$$

Just as in Notation 4.14, we choose a generic configuration of $2(n+b)+3+g$ points ω very far down in σ_1. As in the proof of Theorem 1.2, for any element $h : C \to X$ of $\mathbb{T}\mathcal{S}^{\mathrm{irr}}(\omega)$, we have to understand the contributions of $h_1 : C_1 \to \mathbb{R}^2$ and the roof $h' : C' \to \mathbb{R}^2$. Let δ be the Newton fan of h_1. Since $a = 2$ in our case, there are not many possibilities for δ, the Newton fan of h_1:

$$\begin{aligned} \delta = \quad & \{(1, n+2)^m, (0,-1)^{2(n+1)+b}, (-1,0)^A, \\ & (-\alpha_1, \beta_1), \ldots, (-\alpha_r, \beta_r), (0, \beta_{r+1}), \ldots, (0, \beta_{r+s}), (0,1)^U\} \end{aligned}$$

with $m \leq 2$ (recall that $\delta \vdash \delta_{n,b}$, see Notation 1.1).

Let us assume first that $m = 2$. Recall that $\mathcal{N}(0, n, d, \alpha) \neq 0$ if and only if $d = \alpha = 0$, in which case it is equal to 1. This implies that $r = s = 0$, which in its turn gives $A = 2$ and $U = b - 2$. Hence the contribution of all elements of $\mathbb{T}\mathcal{S}^{\mathrm{irr}}(\omega)$ such that $m = 2$ is equal to $N^{\mathrm{irr}}_{2-2g}(\delta_{n+2,b-2})$.

Let us assume that $m = 0$. In this case, the Newton polygon of h_1 is the segment with endpoints $(0,0)$ and $(0, 2n+2+b)$. In particular the space of deformations of h_1 has dimension $2n+2+b < 2(n+b)+3+g$, so $h_1(C_1)$ cannot pass through all points in ω since this latter configuration is generic.

Hence we are left to study the case $m = 1$. Let us first assume $A = 1$, which is equivalent to $r = 0$. So the roof h' looks like in Figure 7. Because of its Newton polygon (see Figure 14a), the tropical morphism h_1 has one irreducible rational component $h_0 : C_0 \to \mathbb{R}^2$ with Newton polygon depicted in Figure 14b, and all the other irreducible components have the segment $[0,1]$ as Newton polygon, i.e. they are vertical lines of weight one. Since C is irreducible and of genus g, the roof of C has to connect $g+1$ vertical ends of C_0 with all the other irreducible components of C_1. We depict, in a floor diagram style (see [BM08], [BMa]), in Figure 14c how the morphism $h_1 : C_1 \to \mathbb{R}^2$ looks like. The roof of C connects $g+1$ vertical ends of C_0

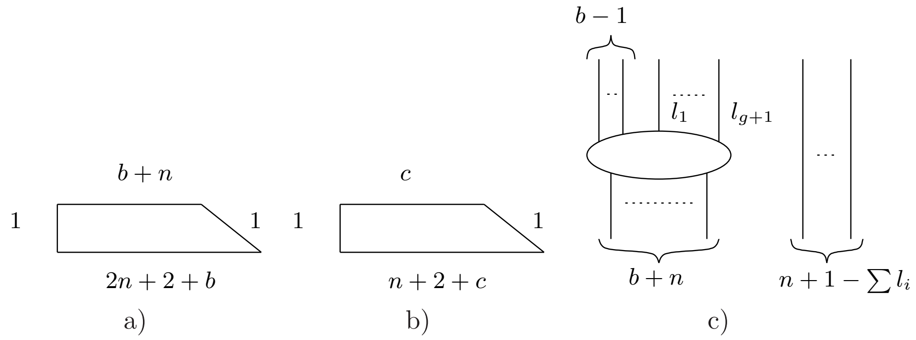


FIGURE 14. The case $m = A = 1$.

with weights $l_1, \ldots, l_{g+1}$. C_0 contains $b - A = b - 1$ ends of direction $(0,1)$ which become vertices of type b as in Figure 12b. Therefore we have $c = b - 1 + \sum l_i$, and there are $n + b - c = n + 1 - \sum l_i$ irreducible components which are vertical lines. There are $\binom{2(n+b)+3}{n+1-\sum l_i}$ distinct choices for the points of ω through which pass the vertical lines. The number of tropical morphisms $h_0 : C_0 \to \mathbb{R}^2$ passing through the remaining points times the number of ways to choose the ends l_i attached to the roof equals

$$\frac{(b+g)!}{(b-1)!\tau_{l_1,\ldots,l_{g+1}}}.$$

In any case we have $\mu_{h_1} = \prod l_i$ and the roof h' contributes 1, so altogether the contribution of all tropical morphisms in $\mathbb{T}\mathcal{S}^{\mathrm{irr}}(\omega)$ such that $m = A = 1$ is equal to

$$\sum_{l_1+\ldots+l_{g+1}=g+1}^{n+1} \binom{2(n+b)+3}{n+1-\sum l_i} \cdot \frac{(b+g)!}{(b-1)!\tau_{l_1,\ldots,l_{g+1}}} \cdot \prod_{i=1}^{g+1} l_i^2.$$

Finally, it remains to consider summands with $m = 1$ and $A = 0$, i.e. $r = 1$. We denote $(-\alpha_1, \beta_1) = (-1, l_1 - 1)$, so the roof h' looks like in Figure 15. Since the end of C_2 has direction $(1, -n-2)$, we have $k = 2$. Just as before the roof of C connects g vertical ends of C_0 with weights $l_2, \ldots, l_{g+1}$, and all the other irreducible components of C_1 are just vertical lines of weight one. As before there are $n + 1 - \sum l_i$ such components. There are $\binom{2(n+b)+3}{n+1-\sum l_i}$ distinct choices for the points of ω through which pass the vertical lines. The number of tropical morphisms $h_0 : C_0 \to \mathbb{R}^2$ passing through the remaining points times the number of ways to choose the ends l_i attached to the roof equals

$$\frac{(b+g-1)!}{(b-1)!\tau_{l_2,\ldots,l_{g+1}}}.$$

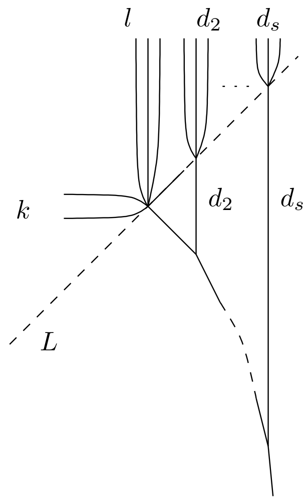


FIGURE 15. The case $m = 1$, $A = 0$.

We have $\mu_{h_1} = \prod_{i=2}^{g+1} l_i$ and $\mu_{h'} = \binom{l_1}{2}$, so altogether the contribution of all tropical morphisms in $\mathbb{T}\widetilde{\mathcal{S}}(\omega)$ such that $m = 1$ and $A = 0$ is equal to

$$\sum_{l_1+\ldots+l_{g+1}=g+1}^{n+1} \binom{2(n+b)+3}{n+1-\sum l_i} \cdot \frac{(b+g-1)!}{(b-1)!\tau_{l_2,\ldots,l_{g+1}}} \cdot \prod_{i=2}^{g+1} l_i^2 \cdot \binom{l_1}{2}.$$

This completes the proof. □

Example 4.18
In the case $g = 0$, Proposition 4.17 reduces to

$$N_2^{\mathrm{irr}}(\delta_0) = N_2^{\mathrm{irr}}(\delta) + \sum_{l=1}^{n+1} \binom{2(n+b)+3}{n+1-l} \cdot \left(b \cdot l^2 + \binom{l}{2} \right).$$

which has been first proved by Abramovich and Bertram (see [Vak00]).

5. Proofs of Correspondence theorems

Here we prove Theorems 4.13 and 3.17 relating the enumeration of complex algebraic and tropical curves. Our proof is a mild adaptation of the proof of [Mik05, Theorem 1] in the framework of tropical morphisms and their approximation established in [BBM, Section 6]. We begin this section by recalling this latter framework. Then we reduce Theorems 4.13 and 3.17 to a Correspondence Theorem relating the enumeration of tropical curves in $\mathbb{R}^2$ to the enumeration of complex curves in $(\mathbb{C}^*)^2$ with a fixed Newton fan δ and having an ordinary multiple point of maximal multiplicity at a fixed point on a toric divisor of $\mathrm{Tor}(\Pi_\delta)$. We adapt techniques from [Mik05, Section 8] to our situation, and are eventually left to solve some easy local enumerative problems.

5.1. **Phase-tropical geometry.** Here we recall definitions and results from [BBM, Section 6] that we need later. We start to define the *phase* of a point, and of a tropical morphism. Roughly speaking, the phase of a tropical variety Y is a choice, in a compatible way, of an algebraic variety $\mathcal{Y}_p$ of dimension $\dim Y$ for each point $p \in Y$.

The notion of a phase-tropical limit is based on the degeneration of the standard complex structure on $(\mathbb{C}^*)^n$ via the following self-diffeomorphism of $(\mathbb{C}^*)^n$:

$$\begin{array}{rccc} H_t : & (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^n \\ & (z_i) & \longmapsto & \left(|z_i|^{\frac{1}{\log t}} \frac{z_i}{|z_i|}\right) \end{array}.$$

We also define the two following maps:

$$\begin{array}{rccc} \mathrm{Log} : & (\mathbb{C}^*)^n & \longrightarrow & \mathbb{R}^n \\ & (z_i) & \longmapsto & (\log(|z_i|)) \end{array} \quad \text{and} \quad \begin{array}{rccc} \mathrm{Arg} : & (\mathbb{C}^*)^n & \longrightarrow & (S^1)^n \\ & (z_i) & \longmapsto & (\arg(z_i)) \end{array}.$$

As usual, all definitions are particularly easy in the case of points.

Definition 5.1 ([BBM, Definitions 6.1 and 6.4])
Let $p \in \mathbb{R}^n$ be a point. A *phase* of p is the choice of a point $\phi(p) \in (S^1)^n$.

Let (p_{t_j}) be a sequence of points in $(\mathbb{C}^*)^n$ such that $\lim_{j \to +\infty} H_{t_j}(p_{t_j})$ exists as a point in $(\mathbb{C}^*)^n$. The *phase-tropical limit* of the sequence (p_{t_j}) is defined as the point $p = \mathrm{Log}(\lim_{j \to +\infty} H_{t_j}(p_{t_j}))$ enhanced with the phase $\mathrm{Arg}(\lim_{j \to +\infty} H_{t_j}(p_{t_j}))$.

Next, we define *phase-tropical morphisms* and the *phase-tropical limit* of a sequence of algebraic maps from Riemann surfaces. A *pluriharmonic map* $\Phi : S \to (S^1)^n$ is a map from a punctured Riemann surface S which is the composition of an algebraic map $\Phi_0 : S \to (\mathbb{C}^*)^n$ with the map $\mathrm{Arg} : (\mathbb{C}^*)^n \to (S^1)^n$. Such an algebraic map Φ_0 is called an *algebraic lift* of Φ. Clearly, two algebraic lifts of Φ differ by a multiplicative translation in $(\mathbb{C}^*)^n$ by a vector in $(\mathbb{R}_{>0})^n$.

Given a pluriharmonic map $\Phi : S \to (S^1)^n$ we can naturally associate a map $\Phi^\varepsilon : S^1 \to (S^1)^n$ for each of the punctures ε of S. Let us denote by $\overline{S}$ the compact Riemann surface obtained from S by performing a real blow-up at each puncture of S. That is to say we replace each puncture ε of S with a *boundary circle* b_ε of length 2π, oriented as a boundary component of $\overline{S}$, the metric on b_ε being given by the conformal structure of S at ε. (see [MO07, Section 6.2] or [BBM, Section 6.1]). The map $\Phi^\varepsilon : b_\varepsilon \to (S^1)^n$ is defined as the limit of the map $\Phi_{|l}$ where l is a small loop around ε converging to ε.

The map Φ is proper at ε if and only if Φ cannot be extended at ε to a pluriharmonic map, i.e. an algebraic lift of Φ does not send any neighborhood of ε into a compact subset of $(\mathbb{C}^*)^n$. In this case the map Φ^ε is a covering of some degree $w \geq 1$ of a geodesic on the flat torus $(S^1)^n = (\mathbb{R}/2\pi)^n$ (see [MO07, Section 6.2] or [BBM, Section 6.1]). Hence it is a dilation of factor w if $\Phi^\varepsilon(b_\varepsilon)$ is equipped with the metric, of total length 2π, induced by the natural flat metric on $(\mathbb{R}/2\pi)^n$.

If Φ is not proper in a neighborhood of a puncture ε, then it has a removable singularity, and Φ^ε maps the whole circle b_ε to a point.

Recall that a stable Riemann surface S is a, maybe reducible, nodal complex algebraic curve such that any of its irreducible component is a punctured Riemann surface; the total number of nodes and punctures on a component S_0 of S is at least 3 if S_0 is rational, and at least 1 if S_0 is elliptic. If $\mathcal{I}$ is the graph of intersection of the irreducible components of S (i.e. the dual graph), the genus of S is equal to $b_1(\mathcal{I}) + \sum_{S_0} g(S_0)$. We denote by S° the Riemann surface obtained from S by removing all nodes.

In the following definition, we identify the group $H_1((S^1)^n, \mathbb{Z})$ with $\mathbb{Z}^n$ via the map Arg (recall that since $\mathbb{C}^*$ is canonically oriented, the group $H_1(\mathbb{C}^*, \mathbb{Z})$ is canonically identified with $\mathbb{Z}$, and therefore $H_1((\mathbb{C}^*)^n, \mathbb{Z})$ is canonically identified with $\mathbb{Z}^n$).

Finally, we say that a tropical morphism $h : C \to \mathbb{R}^n$ is *minimal* if $v(V, e) \neq 0$ for any edge e of C (see Definition 3.4). The next definition only deals with minimal tropical morphisms. For a definition in a more general situation we refer to the forthcoming paper [Mik].

Definition 5.2 ([BBM, Definition 6.5])
Let $h : C \to \mathbb{R}^n$ be a minimal tropical morphism. A *phase ϕ of h* consists of the following the data

- for each vertex $V \in \mathrm{Vert}^0(C)$, a proper pluriharmonic map
$$\Phi_V : S_V \to (S^1)^n$$
where S_V is a stable Riemann surface of genus g_V with k punctures, equipped with a one-to-one correspondence $\varepsilon \leftrightarrow e$ between the punctures of S_V and the edges of C adjacent to V, such that for each edge e, the homology class $[\Phi_V^e(b_e)] \in H_1((S^1)^n, \mathbb{Z}) = \mathbb{Z}^n$ satisfies
$$[\Phi_V^e(b_e)] = v(V, e) \in \mathbb{Z}^n;$$
- for each edge $e \in \mathrm{Edge}^0(C)$ adjacent to $V, V' \in \mathrm{Vert}^0(C)$, an orientation-reversing isometry
$$\rho_e : b_e^V \to b_e^{V'} \tag{2}$$
between the two boundary circles of the punctures corresponding to e, such that $\Phi_V^e = \Phi_{V'}^e \circ \rho_e$;
- for each vertex $V \in \mathrm{Vert}^0(C)$ and each node κ of S_V, an orientation-reversing isometry
$$\rho_\kappa : b_{\varepsilon'} \to b_{\varepsilon''} \tag{3}$$
where ε' and ε'' are the two punctures of S_V° corresponding to the node κ.

Note that in this latter case, both boundary circles $b_{\varepsilon'}$ and $b_{\varepsilon''}$ are mapped to the same point in $(S^1)^n$, in particular one has $\Phi_V^{\varepsilon'} = \Phi_V^{\varepsilon''} \circ \rho_\kappa$. We denote by (h, ϕ) a *phase-tropical morphism*, i.e. a tropical morphism h equipped with a phase ϕ.

Remark 5.3
There exist slight formal differences between Definition 5.2 and [BBM, Definition 6.5]. However these differences only come from minor differences in the presentation we chose in this paper compared to the one chosen in [BBM]. In this latter tropical curves are not allowed to contain degenerate edges, and pluriharmonic maps Φ_V might be non-proper (in the language of [BBM] Φ_V might have *essential* boundary

circles). In the present paper, essential boundary circles are replaced by degenerate edges, and the maps Φ_V are now required to be proper.

By definition, a pluriharmonic map Φ_V is associated to each vertex V of a phase-tropical morphism $(h : C \to \mathbb{R}^n, \phi)$. If S_V is smooth then we set $\hat{S}_V = S_V$, and $\hat{\Phi}_V = \Phi_V$. If S_V is nodal, then we denote by $\hat{S}_V$ the topological oriented surface obtained by replacing each node κ with the corresponding boundary circle, i.e. either side of the isometry (3). By construction we have a natural continuous map $\hat{S}_V \to S_V$ contracting each boundary circle to the corresponding node of S_V. Furthermore the pluriharmonic map Φ_V naturally induces a continuous map $\hat{\Phi}_V : \hat{S}_V \to (S^1)^n$.

If W_c is a degenerate component of C, we denote by $\hat{S}_{W_c}$ the topological oriented surface obtained by gluing all surfaces $\hat{S}_V$, with V ranging over all vertices of W_c, along all boundary circles corresponding to edges of W_c using isometries (2). We define $\widetilde{S}_{W_c}$ as the surface $\hat{S}_{W_c}$ with these boundary circles removed. By construction the surface $\widetilde{S}_{W_c}$ is the disjoint union of the surfaces $\hat{S}_V$ over all vertices V of W_c. Furthermore all pluriharmonic maps $\hat{\Phi}_V$ with V a vertex of W_c naturally induce a continuous map $\hat{\Phi}_{W_c} : \hat{S}_{W_c} \to (S^1)^n$.

For each non-degenerate edge e of C adjacent to V, we write $v(e, V) = (v_1, \dots, v_n)$, and we associate to e the Riemann surface $S_e = \mathbb{C}^*$ and the pluriharmonic map

$$\begin{array}{cccc} \Phi_e : & \mathbb{C}^* & \longrightarrow & (S^1)^n \\ & re^{i\theta} & \longmapsto & (v_1\theta, \dots, v_n\theta) \end{array}.$$

An algebraic lift of Φ_e is given by the map $z \to (z^{v_1}, \dots, z^{v_n})$, which might be thought as the complexification of the map $\Phi_V^e : b_e^V \to (S^1)^n$: since b_e^V (resp. each coordinate circle of $(S^1)^n$) is oriented, the tangent space of b_e^V (resp. $(S^1)^n$) can naturally be identify with $\mathbb{C}^*$ (resp. $(\mathbb{C}^*)^n$). Note that $\Phi_e(\mathbb{C}^*) = \Phi_V^e(b_e^V)$.

A connected open subset of C is said to be *admissible* if any degenerate component of C is either disjoint or contained in W, and if W contains two vertices of C then these two vertices belong to the same degenerate component. In particular W contains at most one degenerate component, and if not, at most one vertex. Given W such an admissible connected open subset of C, we denote by $S_W = \widetilde{S}_W = \hat{S}_V$ and $\Phi_W = \hat{\Phi}_V$ if V is the unique vertex of C in W, by $S_W = \widetilde{S}_W = S_e$ and $\Phi_W = \Phi_e$ if W is contained in the non-degenerate edge e of C, and by $S_W = \hat{S}_{W_c}$, $\widetilde{S}_W = \widetilde{S}_V$, and $\Phi_W = \hat{\Phi}_{W_c}$ if W contains the degenerate component W_c.

Let $h : C \to \mathbb{R}^n$ be a minimal tropical morphism. A convex open subset $U \subset \mathbb{R}^n$ is called *h-admissible* if $h(C) \cap U$ is connected, if U contains at most one point p such that $h^{-1}(p)$ contains a vertex of C, and if no vertex of C is mapped to the boundary of U.

Finally, we call a *positive multiplication* in $(\mathbb{C}^*)^n$ a map of the form $\tau(z_1, \dots, z_n) = (a_1 z_1, \dots, a_n z_n)$ with $a_1, \dots, a_n > 0$. In next definition we use the standard flat metric on $(S^1)^n$.

Definition 5.4 ([BBM, Definition 6.5])
Let $f_{t_j} : S_{t_j} \to (\mathbb{C}^*)^n$ be a sequence of algebraic maps from punctured Riemann surfaces S_{t_j}, with $t_j \to +\infty$ when $j \to +\infty$. We say that a phase-tropical morphism

$(h : C \to \mathbb{R}^n, \phi)$ is the *phase-tropical limit* of f_{t_j} if for any choice of h-admissible open set $U \subset \mathbb{R}^n$ and all sufficiently large t_j there is a 1-1 correspondence between connected components W_{t_j} of $f_{t_j}^{-1}(\text{Log}_{t_j}^{-1}(U))$ and connected components W of $h^{-1}(U)$ with the following properties of the corresponding components.

- There exists an open embedding $\Xi_{t_j}^W : W_{t_j} \to S_W$ and, for any connected component S of $\widetilde{S}_W$, an algebraic lift $\Psi_S : S \to (\mathbb{C}^*)^n$ of $\Phi_W|_S$ and a sequence (τ_{t_j}) of positive translations in $(\mathbb{C}^*)^n$ such that for any $z \in S$
$$\lim_{t_j \to +\infty} \tau_{t_j} \circ f_{t_j} \circ (\Xi_{t_j}^W)^{-1}(z) = \Psi_S(z).$$
In particular, we require that $z \in \Xi_{t_j}^W(W_{t_j})$ for large t_j.
- For any edge e connecting vertices V and V', any point $z \in b_e^V$, any $\eta > 0$ and a sufficiently large t_j there exist
  (1) a point $z_\eta \in S_V$ and a point $z'_\eta \in S_{V'}$;
  (2) a path $\gamma_\eta \subset \bar{S}_V$ connecting z_η and z and a path $\gamma'_\eta \subset \bar{S}_{V'}$ connecting z'_η and $z' = \rho_e(z)$ (see Equation (2) in Definition 5.2) such that the diameter of $\Phi_V(\gamma_\eta) \subset (S^1)^n$ and that of $\Phi_{V'}(\gamma'_\eta) \subset (S^1)^n$ are less than η;
  (3) a path $\gamma_{t_j} \subset S_{t_j}$ connecting $(\Xi_{t_j}^v)^{-1}(z_\eta)$ and $(\Xi_{t_j}^{v'})^{-1}(z'_\eta)$ such that the diameter of $\text{Arg}(f_{t_j}(\gamma_{t_j})) \subset (S^1)^n$ is less than η.

It follows from Definition 5.4 that if $(h : C \to \mathbb{R}^n, \phi)$ is the phase-tropical limit of $f_{t_j} : S_{t_j} \to (\mathbb{C}^*)^n$, then $h(C)$ is the limit, in the Hausdorff metric on compact sets of $\mathbb{R}^n$, of $\text{Log}_{t_j}(f_{t_j}(S_{t_j}))$. In the case of curves mapped to $(\mathbb{C}^*)^2$, each curve $f_{t_j}(S_{t_j})$ has a polynomial equation $P_{t_j}(x,y) = 0$, and the equation of the tropical limit $h(C) \subset \mathbb{R}^2$ can be deduced from the sequence of polynomials $P_{t_j}(x,y)$.

Proposition 5.5 ([Mik04b, Theorem 6.4])
Let $P_{t_j}(x,y) = \sum_{i,j} a_{i,j,t_j} x^i y^j$ be a sequence of complex polynomials, such that one of the sequence a_{i_0,j_0,t_j} is the constant sequence equal to 1, and the absolute value of any coefficients a_{i,j,t_j} satisfies $|a_{i,j,t_j}| = O_{j\to+\infty}(1)$. Suppose in addition that $\text{Log}_{t_j}(\{P_{t_j}(x,y) = 0\})$ converges, in the Hausdorff metric on compact sets of $\mathbb{R}^2$, to a tropical curve $C \subset \mathbb{R}^2$. Then C is given by the tropical polynomial
$$\text{“}\sum_{i,j} \lambda_{i,j} x^i y^j\text{”}$$
where $\lambda_{i,j} = \inf\{\lambda \mid a_{i,j,t_j} = o_{j\to+\infty}(t_j^\lambda)\}$.

Example 5.6
We consider the following family of algebraic maps
$$\begin{array}{cccc} f_t : & \mathbb{C}^* \setminus \{1, \frac{t}{t-1}\} & \longrightarrow & (\mathbb{C}^*)^2 \\ & z & \longmapsto & \left(\frac{z}{1-z}, \frac{t}{(t-1)z-t}\right) \end{array}.$$

The image of the map f_t is given by the polynomial $P_t(x,y) = 1 + x + y + t^{-1}xy$ in $(\mathbb{C}^*)^2$, so it converges to the embedded tropical curve C' given by the polynomial “$0 + x + y + (-1)xy$”. This tropical curve has two vertices $(0,0)$ and $(1,1)$, one bounded edge of direction $(1,1)$, and four ends of direction $(-1,0), (0,-1), (1,0),$

and $(0,1)$ (see Figure 16). Now one computes easily that the family (f_t) converges tropically to the phase-tropical morphism $(h : C \to \mathbb{R}^2, \phi)$ where

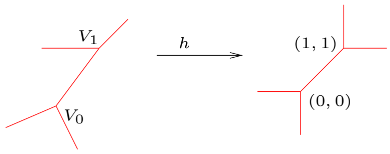


FIGURE 16. A tropical limit.

- h is the unique parameterization of C' from a rational tropical curve C with four ends and with all weights equal to 1;
- if V_0 is the vertex of C mapped to $(0,0)$, then S_{V_0} is the curve in $(\mathbb{C}^*)^2$ given by the equation $1 + x + y = P_{+\infty}(x,y)$ and Φ_{V_0} is the restriction of the argument map to S_{V_0};
- if V_1 is the vertex of C mapped to $(1,1)$, then S_{V_1} is the curve in $(\mathbb{C}^*)^2$ given by the equation $x + y + xy = \lim_{t\to+\infty} \frac{1}{t} P_t(tx, ty)$ and Φ_{V_1} is the restriction of the argument map to S_{V_1}.

Note that there is a unique possibility for the map ρ_e corresponding to the bounded edge e of C.

Example 5.7
Proposition 5.5 implies that a phase tropical morphism (h, ϕ) which is the phase-tropical limit of a sequence of algebraic maps $f_{t_j} : S_{t_j} \to (\mathbb{C}^*)^n$ with a fixed Newton fan δ might have a Newton fan different from δ. For example, the the image of map $f_t(z) = (-1-e^{-t}z, z)$ in $(\mathbb{C}^*)^2$ satisfies the equation $1+x+e^{-t}y = 0$, and the family (f_t) has a tropical limit with Newton polygon the segment $[(0,0),(1,0)] \subset \mathbb{R}^2$.

We define the *Euler characteristic of a phase-tropical morphism* $(h, C \to \mathbb{R}^n, \phi)$ by

$$\chi(h) = \sum_{V \in \mathrm{Vert}^0(C)} \chi(S_V).$$

In particular one has $2\chi_{\mathrm{trop}}(C) \leq \chi(h) + \#\mathrm{Edge}^\infty(C)$, with equality if and only if all surfaces S_V are smooth. The next lemma is an immediate consequences of Definition 5.4.

Lemma 5.8
Let $f_{t_j} : S_{t_j} \to (\mathbb{C}^*)^n$ be a sequence of non-constant algebraic maps which converges, as a tropical limit, to a phase-tropical morphism $(h : C \to \mathbb{R}^n, \phi)$. Suppose that the Riemann surfaces S_{t_j} are connected and of constant Euler characteristic χ for t_j large enough. Then $\chi(h) \geq \chi$, and if equality holds then C is connected.

The following compactness result is fundamental for proving Correspondence Theorems in the phase-tropical framework.

Proposition 5.9 ([BBM, Proposition 6.8])
Let $f_t : S_t \to (\mathbb{C}^*)^n$ be a family of algebraic maps with a fixed genus and Newton fan, defined for all sufficiently large positive parameter $t >> 1$. Then there exists

a phase-tropical morphism (f,ϕ) and a sequence $t_j \to +\infty$ such that f_{t_j} converges to (f,ϕ) in the sense of Definition 5.4.

Now we prove that the tropical limit of a sequence of algebraic maps to the surface $\mathcal{X}$ in $(\mathbb{C}^*)^3$ with equation $x+y+z=0$ is a tropical morphism to X in the sense of Definition 3.11. It is well known that X is the limit, for the Hausdorff metric on compact sets of $\mathbb{R}^n$, of $\operatorname{Log}_t(\mathcal{X})$ when $t \to +\infty$ (see for example [Mik04a]). Note that $\mathcal{X}$ is invariant under any multiplicative translation by an element of $(\mathbb{C}^*)^n$ of the form $(\lambda,\ldots,\lambda)$.

The next proposition is a particular case of a result proved by Mikhalkin and the first author in the forthcoming paper [BMb]. Since the latter is not available yet, we provide a proof for the sake of completeness.

We use the following notation in the proof of Proposition 5.10. Let us consider a sequence of algebraic maps $f_j : S_j \to (\mathbb{C}^*)^3$ from a family of punctured Riemann surfaces such that the sequence of compactified maps $\overline{f}_j : \overline{S}_j \to \mathbb{C}P^3$ converges to some map $\overline{f}_\infty : \overline{S}_\infty \to \mathbb{C}P^3$. Note that the Riemann surface $\overline{S}_\infty$ might be reducible, and some of its connected components might be mapped to the toric boundary divisors of $\mathbb{C}P^3$. We denote by $f_\infty : S_\infty \to (\mathbb{C}^*)^3$ the restriction of $\overline{f}_\infty$ to $(\mathbb{C}^*)^3$. We emphasize that the Riemann surface S_∞ might be disconnected.

Proposition 5.10 ([BMb])
Let $f_{t_j} : S_{t_j} \to (\mathbb{C}^*)^3$ be a sequence of algebraic maps which converges, as a tropical limit, to a phase-tropical morphism $(h : C \to \mathbb{R}^3, \phi)$. Suppose in addition that $f_{t_j}(S_{t_j}) \subset \mathcal{X}$ for t_j large enough. Then h is a tropical morphism to X.

Proof:
Since $h(C)$ and X are respectively the limit, in the Hausdorff metric on compact sets of $\mathbb{R}^3$, of $\operatorname{Log}_{t_j}(f_{t_j}(S_{t_j}))$ and $\operatorname{Log}_{t_j}(\mathcal{X})$, we have $h(C) \subset X$.

Let V be a vertex of C. If $\tau = (\tau_{t_j})$ is a sequence of positive translation in $(\mathbb{C}^*)^3$ we denote by $f^\tau_{t_j}$ the sequence of maps $\tau_{t_j} \circ f_{t_j}$. By Definition 5.4 there exists a sequence $\tau_{t_j}(x,y,z) = (a_{t_j,1}x, a_{t_j,2}y, a_{t_j,3}z)$ of positive translations in $(\mathbb{C}^*)^3$ for which there exists a connected Riemann surface $S_V \subset S_\infty$ such that $f^\tau_{\infty\ |S_V} : S_V \to (\mathbb{C}^*)^n$ is an algebraic lift of the phase $\Phi_V : S_V \to (S^1)^3$ of $h : C \to \mathbb{R}^3$ at the vertex V. Note that $\lim \operatorname{Log}_{t_j}(a_{t_j,1}, a_{t_j,2}, a_{t_j,3}) = h(V)$.

There are only three possibilities, up to a positive translation, for the limit $\mathcal{X}^\tau_\infty$ of $\tau_{t_j}(\mathcal{X})$: it has equation either $x+z=0$, $x+y=0$, $y+z=0$, or $x+y+z=0$. Moreover this latter arises if and only if $f(V) = (u,u,u)$ and the sequence $(t_j^{-u}a_{t_j,1}, t_j^{-u}a_{t_j,2}, t_j^{-u}a_{t_j,3})$ is contained in a compact set of $(\mathbb{C}^*)^3$. Moreover, since $f_{t_j}(S_{t_j}) \subset \mathcal{X}$ for all t_j large enough, we have $f^\tau_{\infty\ |S_V}(S_V) \subset \mathcal{X}^\tau_\infty$. In particular this implies that $h : C \to X$ is a tropical premorphism, i.e. h satisfies the second condition of Definition 3.8.

Let V be a vertex of C with $d_V > 0$. In particular, $h(V) = (u,u,u)$ with $u \in \mathbb{R}$ and the corresponding sequence $(t_j^{-u}a_{t_j,1}, t_j^{-u}a_{t_j,2}, t_j^{-u}a_{t_j,3})$ is contained in a compact set of $(\mathbb{C}^*)^3$. After composing the map f_{t_j} with the multiplicative translation by $(t_j^{-u}a_{t_j,1}, t_j^{-u}a_{t_j,2}, t_j^{-u}a_{t_j,3})$, and the map h with the additive translation by $(-u,\ldots,-u)$, we may suppose that $u=0$ and $(a_{t_j,1}, a_{t_j,2}, a_{t_j,3}) = (1,1,1)$. From what we said above, there exists an algebraic lift $f_{\infty\ |S_V} : S_V \to (\mathbb{C}^*)^n$ of the phase $\Phi_V : S_V \to (S^1)^3$ such that $f_\infty(S_V) \subset \mathcal{X}$.

Let us consider the plane $\mathcal{P} \simeq (\mathbb{C}^*)^2 \subset (\mathbb{C}^*)^3$ with equation $z = 1$. Then the curve $S_0 = \mathcal{P} \cap \mathcal{X}$ is the Riemann sphere punctured in three points, and the map $(x, y, z) \mapsto ((\frac{x}{z}, \frac{y}{z}, 1), z)$ provides an algebraic isomorphism between $\mathcal{X}$ and $S_0 \times \mathbb{C}^*$. The restriction on $f_\infty(S_V)$ of the projection to the first factor $\mathcal{X} \to S_0$ gives a holomorphic map $\pi : S_V \to S_0$, which can be extended to all punctures of S_V corresponding to an edge of C entirely mapped to L. In this way we obtain a proper holomorphic map $\overline{\pi} : \overline{S}_V \to S_0$, which has degree d_V by [ST08, Theorem 1.1]. Smoothing nodes of $\overline{S}_V$ if there are any, we may even further suppose that $\overline{S}_V$ is a non-singular Riemann surface of genus g_V with k_V punctures, where k_V is the number of edges adjacent to V which are not entirely mapped to L. Hence it follows from the Riemann-Hurwitz formula that

$$\chi(S_V) = d_V \chi(S_0) - \iota$$

where $\iota \geq 0$ is the sum of ramification indices of $\overline{\pi}$ over all points of $\overline{S}_V$. Since we have $\chi(\overline{S}_V) = 2 - 2g_V - k_V$ and $\chi(S_0) = -1$, we obtain

$$k_V - d_V - 2 + 2g_V \geq 0.$$

which means exactly that f is a tropical morphism to X. □

5.2. **The proof of Theorem 4.13.** Let us first recast notations from previous sections. We denote by X (resp. $\mathcal{X}$) the tropical hypersurface in $\mathbb{R}^3$ (resp. in $(\mathbb{C}^*)^3$) defined by the tropical polynomial "$x+y+z$" (resp. the equation $x+y+z = 0$). The tropical surface X is made of three half-planes $\sigma_1 = \{x = y \geq z\}$, $\sigma_2 = \{x = z \geq y\}$ and $\sigma_3 = \{y = z \geq x\}$ meeting along the line $L = \mathbb{R}(1, 1, 1)$. We have fixed the following Newton fan

$$\Delta = \{(1, -n, 1)^a, (0, 1, 1)^{an+b}, (-1, 0, 0)^a, (0, -1, 0)^b, (0, 0, -1)^{a(n+1)+b}\}$$

as well as an integer $\chi \in \mathbb{Z}$ and a generic configuration ω of $\#\Delta_2 + \#\Delta_3 - \chi$ points in $\sigma_1 \cup \sigma_2$, where $\Delta_i \subset \Delta$ consists of elements of Δ contained in σ_i.

Let us fix the following two additional Newton fans

$$\delta_z = \{(1, -n)^a, (0, 1)^{an+b}, (-1, 0)^a, (0, -1)^b\},$$

and

$$\delta_x = \{(-n, 1)^a, (1, 1)^{an+b}, (-1, 0)^b, (0, -1)^{a(n+1)+b}\}.$$

Note that δ_z is the image of δ_0 (see Theorem 4.13) under the map $(x, y) \mapsto (x, -y)$, in particular $N^{irr}_{2\chi}(\delta_0) = N^{irr}_{2\chi}(\delta_z)$. We define the projections

$$\begin{array}{cccc} \pi_z : & (\mathbb{C}^*)^3 & \longrightarrow & (\mathbb{C}^*)^2 \\ & (x, y, z) & \longmapsto & (x, y) \end{array} \quad \text{and} \quad \begin{array}{cccc} \pi_x : & (\mathbb{C}^*)^3 & \longrightarrow & (\mathbb{C}^*)^2 \\ & (x, y, z) & \longmapsto & (y, z) \end{array}.$$

Note that $\delta_z = \pi_z(\Delta)$, $\delta_x = \pi_x(\Delta)$, and $\#\delta_z = \#\Delta_2 + \#\Delta_3$. Furthermore, we choose a configuration $\omega^{\mathbb{C}}$ of $\#\delta_z - \chi$ points in $\mathcal{X}$. We denote by $\mathcal{S}(\Delta, \omega^{\mathbb{C}})$ the set of all irreducible algebraic curves in $\mathcal{X}$ of Euler characteristic $2\chi - \#\Delta$, with Newton fan Δ, and passing through $\omega^{\mathbb{C}}$. We define $N^{irr}_{2\chi}(\Delta) = \#\mathcal{S}(\Delta, \omega^{\mathbb{C}})$.

The strategy to prove the Correspondence Theorem is as follows: we first "put the algebraic enumerative problem into three-space", i.e. we prove in Lemma 5.11 that $N^{irr}_{2\chi}(\Delta) = N^{irr}_{2\chi}(\delta_z)$. Note that this implies in particular that $N^{irr}_{2\chi}(\Delta)$ does not depend on ω. As usual, we have to prove two statements for a Correspondence Theorem: first we have to show that the algebraic curves considered in our enumerative problem (i.e. now the curves in $\mathcal{S}(\Delta, \omega^{\mathbb{C}})$) degenerate to the curves in the tropical enumerative problem, i.e. to elements in $\mathbb{T}\mathcal{S}(\omega)$. Second, we have to show that the number of algebraic curves degenerating to a fixed tropical curve equals the tropical multiplicity. The first part is Lemma 5.12. For the second part, we use the projection π_x to "put the algebraic enumerative problem back into the plane", but differently. Our intention to use this different projections is that then we are able to apply known techniques for Correspondence Theorems for plane curves.

Contrary to the situation where we project with π_z however, we do not obtain an enumerative problem that involves only simple point conditions. Instead, we obtain curves with a multiple point (see Lemma 5.14). We then relate this new plane algebraic enumerative problem to our tropical curves in X and their projections by π_x. Compared to the existing Correspondence Theorems for plane curves, our situation differs since we have vertices to which several edges of the same direction are adjacent (the projections of vertices in L). We compute the number of algebraic preimages for these vertices locally in Lemma 5.16.

Lemma 5.11
For a generic configuration of points $\omega^{\mathbb{C}}$, the set $\mathcal{S}_{2\chi}(\Delta, \omega^{\mathbb{C}})$ is finite and

$$N^{irr}_{2\chi}(\Delta) = N^{irr}_{2\chi}(\delta_z).$$

Proof:
There clearly exists a bijection between curves in $\mathcal{S}(\Delta, \omega^{\mathbb{C}})$ and irreducible complex algebraic curves in $(\mathbb{C}^*)^2$ of Euler characteristic $2\chi - \#\delta_z$, with Newton fan $\delta_z = \pi_z(\Delta)$, and passing through $\pi_z(\omega^{\mathbb{C}})$. □

Recall that we chose a tropically generic configuration ω of $\#\delta_z - \chi$ points in $\sigma_1 \cup \sigma_2 \subset X$. By Proposition 4.2 this implies in particular that for any point $p \in \omega$ and any morphism $h : C \to X$ in $\mathbb{T}\mathcal{S}(\omega)$ there exists a unique edge e_p of C such that $p \in h(e_p)$. Let us equip ω with a phase structure ϕ_ω in $\mathrm{Arg}(\mathcal{X})$, i.e. we equip each point p in ω with a phase $\phi_p \in \mathrm{Arg}(\mathcal{X})$. Let us choose an approximation $(\omega^{\mathbb{C}}_t)_{t>0}$ of (ω, ϕ_ω) by generic configurations in $\mathcal{X}$. That is to say for each $t > 0$, we choose a point $p_t \in \mathcal{X}$ for each $p \in \omega$ in such a way that the configuration $\omega^{\mathbb{C}}_t$ formed by those points is generic, and that (p, ϕ_p) is the tropical limit of (p_t).

Proposition 5.12
Let (h, ϕ) be an accumulation point of the sequence of sets $\mathcal{S}(\Delta, \omega^{\mathbb{C}}_t)$, in the sense of a tropical limit. Then h is an element of $\mathbb{T}\mathcal{S}(\omega)$. Moreover for any $p \in \omega$, one has $\phi_p \in \Phi_{e_p}(S_{e_p})$.

Proof:
Let $(h : C \to \mathbb{R}^3, \phi)$ be such an accumulation point. Since $f_t(S_t) \subset \mathcal{X}$ for any element $f_t : S_t \to \mathcal{X}$ of $\mathcal{S}(\Delta, \omega^{\mathbb{C}}_t)$, it follows from Proposition 5.10 that h is a tropical morphism to X. Since in addition $\omega^{\mathbb{C}}_t \subset f_t(S_t)$, we clearly have $\omega \subset h(C)$ and $\phi_p \in \Phi_{e_p}(S_{e_p})$.

Next we show that the Newton fan of h is equal to Δ. Let us first look at the projection to the (x, y)-coordinates. According to Lemma 5.11, each curve

$\pi_z(f_{t_j}(S)) \subset (\mathbb{C}^*)^2$ is given by an equation $P_{t_j}(x,y) = 0$. Moreover, up to rescaling the coefficients of P_{t_j}, we may suppose that the biggest absolute value of the coefficients of P_{t_j} is equal to one. We denote by $|\delta_z|$ the linear system on $\mathrm{Tor}(\Pi_{\delta_z}) = \Sigma_n$ defined by the fan δ_z. It is naturally a projective space of dimension $N = \mathbb{Z}^2 \cap \Pi_{\delta_z} - 1$. We denote by $\mathcal{V}_{\delta_z,2\chi}$ the closure in $\mathbb{C}P^N$ of the set of all nodal complex irreducible algebraic curves in $(\mathbb{C}^*)^2$ with Newton fan δ_z, and whose normalization has Euler characteristic 2χ.

Passing through a point in $(\mathbb{C}^*)^2$ imposes a linear condition on curves in $|\delta_z|$. Hence curves in $|\delta_z|$ which pass through all points in $\omega_{t_j}^{\mathbb{C}}$ form a linear subspace $\mathcal{L}_{t_j} \subset |\delta_z|$. By construction all points $\omega_{t_j}^{\mathbb{C}}$ have a tropical limit in $(\mathbb{C}^*)^3$, so the coefficients of the equations defining $\mathcal{L}_{t_j}$ may also be chosen to have a tropical limit in $\mathbb{C}^*$, i.e. they are all equivalent to some function ct_j^{λ}. For each t_j, elements of $\mathcal{S}(\delta_z, \omega_{t_j}^{\mathbb{C}})$ correspond precisely to intersections of $\mathcal{V}_{\delta_z,2\chi}$ with the linear space $\mathcal{L}_{tj}$. Since equations defining $\mathcal{V}_{\delta_z,2\chi}$ do not depend on t, it follows from the analytic dependency of the root of a polynomial with respect to its coefficients that all coefficients of P_{t_j} also have a tropical limit in $\mathbb{C}^*$. Hence according to Lemma 5.5, the tropical curve $\lim \mathrm{Log}_{tj}(\{P_{t_j} = 0\})$ has Newton polygon Π_{δ_z}.

It follows from what we just proved that the Newton fan of C has less elements than Δ, since C might have ends of weight at least 2. We also have $\chi(C) \leq 2\chi$ according to Lemma 5.8. Since ω is generic, it follows from Proposition 4.2 that $\chi(C) = 2\chi$, and that the Newton fan of C has as many elements than Δ, and so is equal to Δ. □

The projection π_x relates the number $N_{2\chi}^{irr}(\Delta)$ to another enumerative invariant of some toric surface. This relation will allow us to relate the multiplicity of a tropical curve in $\mathbb{T}\mathcal{S}(\omega)$ to an actual number of complex curves in $\mathcal{S}(\Delta, \omega^{\mathbb{C}})$. Together with Lemma 5.11 this will imply Theorem 4.13. Let us denote by $\mathcal{L}$ the compactification in $\mathrm{Tor}(\Pi_{\delta_x})$ of the line in $(\mathbb{C}^*)^2$ defined by the equation $y+z=0$, and let us denote by $\mathcal{E}$ the toric divisor of $\mathrm{Tor}(\Pi_{\delta_x})$ corresponding to the elements $(1,1)$ of δ_x. Let us denote by $\mathcal{S}(\delta_x, \omega^{\mathbb{C}})$ the set of all irreducible algebraic curves in $(\mathbb{C}^*)^2$ with Newton fan δ_x, of Euler characteristic $2\chi - \#\delta_x$, passing through all points in $\pi_x(\omega^{\mathbb{C}})$, and whose closure in $\mathrm{Tor}(\Pi_{\delta_x})$ has an ordinary multiple point of multiplicity $an+b$ at the point $\mathcal{L} \cap \mathcal{E}$.

Lemma 5.13
If $\omega^{\mathbb{C}}$ is generic, then the set $\mathcal{S}(\delta_x, \omega^{\mathbb{C}})$ is finite. Moreover if $S \in \mathcal{S}(\delta_x, \omega^{\mathbb{C}})$ and D is a branch of S such that the closure of D in $Tor(\Pi_{\delta_x})$ intersects $\mathcal{L}$, then this intersection is transverse.

Proof:
We define $N = \#\left(\Pi_{\delta_x} \cap \mathbb{Z}^2\right) - 1$. The space $\mathcal{V}_{\delta_x,2\chi}$ of irreducible curves in $(\mathbb{C}^*)^2$ with Newton fan δ_x and Euler characteristic $2\chi - \#\delta_x$ has dimension $\#\delta_x - \chi$ and is naturally a quasiprojective variety in the linear system $|\Pi_{\delta_x}| = \mathbb{C}P^N$ on $\mathrm{Tor}(\Pi_{\delta_x})$. Passing through a generic configuration of points $\omega^{\mathbb{C}}$ imposes $\#\delta_z - \chi$ linearly independent conditions on curves in $|\Pi_{\delta_x}|$, and having a point of multiplicity $an+b$ at $\mathcal{L} \cap \mathcal{E}$ imposes $an+b$ extra linearly independent conditions. Hence the set $\mathcal{S}(\delta_x, \omega^{\mathbb{C}})$ is the intersection of $\mathcal{V}_{\delta_x,2\chi}$ with a linear space of complementary

codimension. For such a generic linear space, the intersection will be finite and transverse. So for a generic configuration $\omega^{\mathbb{C}}$, the set $\mathcal{S}(\delta_x,\omega^{\mathbb{C}})$ is finite, and any of its elements cannot satisfy any further independent condition, like for example having $\mathcal{L}$ as a tangent. □

Lemma 5.14
If $\omega^{\mathbb{C}}$ is generic, then the projection π_x establishes a bijection between the sets $\mathcal{S}(\Delta,\omega^{\mathbb{C}})$ and $\mathcal{S}(\delta_x,\omega^{\mathbb{C}})$.

Proof:
The surface $\mathcal{X}$ is the image of the map

$$\begin{array}{cccc} \iota: & (\mathbb{C}^*)^2\setminus\mathcal{L} & \longrightarrow & (\mathbb{C}^*)^3 \\ & (y,z) & \longmapsto & (-y-z,y,z) \end{array}.$$

In particular, any irreducible curve S in $(\mathbb{C}^*)^2$ intersecting $\mathcal{L}$ in finitely many points has a birational lift $\iota(S\setminus\mathcal{L})$ in $\mathcal{X}$. Suppose now that S has Newton fan δ_x. Then a puncture p of S corresponding to a vector $(1,1)$ of its Newton fan will be a puncture of $\iota(S\setminus\mathcal{L})\subset(\mathbb{C}^*)^3$ corresponding to a vector $(1-l,1,1)$ where l is the order of contact of the closure of S and $\mathcal{L}$ at the point $\mathcal{L}\cap\mathcal{E}$ in $\mathrm{Tor}(\Pi_{\delta_x})$. Hence if $\omega^{\mathbb{C}}$ is generic, it follows from Lemma 5.13 that $\iota(S\setminus\mathcal{L})\in\mathcal{S}(\Delta,\omega^{\mathbb{C}})$ if and only if $S\in\mathcal{S}(\delta_x,\omega^{\mathbb{C}})$. □

Hence it remains to compute, given an element $h\in\mathbb{T}\mathcal{S}(\omega)$, how many elements of $\mathcal{S}(\Delta,\omega_t^{\mathbb{C}})$ have their amoeba $\mathrm{Log}_t(f_t(S_t))$ contained in a small neighborhood of $h(C)$. To do so, we work with the projection π_x in order to apply results from [Mik05, Section 8.2]. In this latter paper, everything is stated in terms of curves in $(\mathbb{C}^*)^2$ given by an equation. However, the generalization to maps from abstract curves to $(\mathbb{C}^*)^2$ is straightforward.

Given a tropical morphism $h:C:\to\mathbb{R}^2$, we denote by $w(e)$ the weight of the edge e of C for h (see Definition 3.4).

Proposition 5.15
Let $h:C\to X$ be a tropical morphism in $\mathbb{T}\mathcal{S}(\omega)$ equipped with a phase ϕ such that $\phi_p\in\Phi_{e_p}(S_{e_p})$ for all points p in ω. Then there exists a sequence $t_j\to+\infty$ such that exactly

$$\prod_{p\in\omega} w(e_p)$$

elements of $\mathcal{S}(\Delta,\omega_{t_j}^{\mathbb{C}})$ converge tropically to (h,ϕ).

Proof:
The tropical morphism h composed with the projection π_x induces a tropical morphism $\overline{h}:\overline{C}\to\mathbb{R}^2$, where $\overline{C}$ is obtained from C by contracting all ends e with $v(e)=(-1,0,0)$ (i.e. C is an open tropical modification of $\overline{C}$, see [BBM, Section 2.1]). Moreover, the weight of an edge of $\overline{C}$ is the same as the weight of the corresponding edge of C. Hence according to Lemma 5.14, it is equivalent to prove that there exist a sequence $t_j\to+\infty$ such that exactly $\prod_{p\in\pi_x(\omega)} w(e_p)$ elements of $\mathcal{S}(\delta_x,\omega_{t_j}^{\mathbb{C}})$ converge tropically to $(\overline{h},\pi_x(\phi))$.

The proof of this latter statement follows exactly along the lines of [Mik05, Section 8.2]. The only point to check is that the tropical morphism $\overline{h} : \overline{C} \to \mathbb{R}^2$ is *regular* in the sense of [Mik05, Definition 2.22]. Let $\widetilde{h} : \widetilde{C} \to \mathbb{R}^2$ be the tropical morphism to $\mathbb{R}^2$ such that $\widetilde{h}(\widetilde{C}) = \overline{h}(\overline{C})$ (set theoretically), where $\widetilde{C}$ is obtained from $\overline{C}$ by identifying all ends of $\overline{C}$ mapped to $\pi_x(L)$ and adjacent to a common vertex of $\overline{C}$. Clearly, the morphism $\overline{h} : \overline{C} \to \mathbb{R}^2$ uniquely determines $\widetilde{h} : \widetilde{C} \to \mathbb{R}^2$, and their spaces of deformation are canonically isomorphic.

It follows from Proposition 4.2 that the tropical curve $\widetilde{C}$ is 3-valent and $\widetilde{h}$ is an immersion, so by [Mik05, Proposition 2.23], the dimension of the space of deformations of $\widetilde{h}$ is equal to

$$\text{Edge}^\infty(\widetilde{C}) + g(\widetilde{C}) - 1 = \text{Edge}^\infty(\overline{C}) + g(\overline{C}) - 1 - \sum_{V \in \text{Vert}^0(\overline{C})} (\text{val}(V) - 3).$$

Thus $\overline{h}$ is regular, and all proofs in [Mik05, Section 8.2] apply literally. The fact that we require the complex morphisms to pass through a (unique) point $an + b$ times instead of requiring to pass through $an + b$ distinct points in general position does not make any difference since these conditions provides independent equations. □

Now we have to compute, given a tropical morphism $f : C \to \mathbb{R}^2$, the number of phases ϕ we can endow f with, such that $\phi_p \in \Phi_{e_p}(S_{e_p})$ for all points p in ω. Again, this can be done by a straightforward adaptation of [Mik05, Section 8.2]. There is only one local computation needed here which is not covered by [Mik05], and that we perform now.

We first fix some notation. Let T_0 be the triangle with vertices $(0,0), (l_1, 0)$, and (wl_2, wl_3) where l_1, l_2, l_3 and w are four positive integers such that $\gcd(l_2, l_3) = 1$. We denote by $\mathcal{E}_1$ (resp. $\mathcal{E}_2$ and $\mathcal{E}_3$) the toric divisor of $\text{Tor}(T_0)$ corresponding to $[(0,0); (l_1, 0)]$ (resp. $[(0,0); (wl_2, wl_3)]$ and $[(l_1, 0); (wl_2, wl_3)]$). Finally we choose a point p_1 on $\mathcal{E}_1 \setminus (\mathcal{E}_2 \cup \mathcal{E}_3)$, and a point p_2 on $\mathcal{E}_2 \setminus (\mathcal{E}_1 \cup \mathcal{E}_3)$.

Note that the restriction to the case of T_0 in the following lemma does not cause any loss of generality. Indeed let T be a triangle with vertices in $\mathbb{Z}^2$, and choose any two points q_1 and q_2 on two different toric divisors of $\text{Tor}(T)$ such that neither q_1 or q_2 is the intersection point of two toric divisors. Then there exists a unique choice of l_1, l_2, l_3 and w and a unique toric isomorphism $\text{Tor}(T) \to \text{Tor}(T_0)$ mapping q_1 to p_1 and q_2 to p_2, and sending the linear system defined by T on $\text{Tor}(T)$ to the linear system defined by T_0 on $\text{Tor}(T_0)$.

Lemma 5.16
With the above notation, up to re-parameterization of $\mathbb{C}P^1$, there exist exactly $\frac{1}{w}\binom{wl_3}{l_1}$ algebraic maps $f : \mathbb{C}P^1 \to Tor(T)$ such that

- $f^{-1}(\mathcal{E}_1) = f^{-1}(p_1)$ *and consists of l_1 distinct points on $\mathbb{C}P^1$;*
- $f^{-1}(\mathcal{E}_2) = f^{-1}(p_2)$ *and consists of a single point;*
- $f^{-1}(\mathcal{E}_3)$ *consists of a single point.*

Proof:
Since $\gcd(l_2, l_3) = 1$ there exist two integers u and v such that $vl_2 - ul_3 = 1$. We fix on $(\mathbb{C}^*)^2$ the unique choice of coordinates (x, y) such that $p_1 = \psi(p_2) = (1, 0)$

where ψ is the automorphism of $(\mathbb{C}^*)^2$ given by $\psi(x,y) = (x^{l_2}y^{l_3}, x^u y^v)$. We also choose a coordinate system on $\mathbb{C}P^1$ such that $f^{-1}(p_2) = \{0\}$, $f^{-1}(\mathcal{E}_3) = \{\infty\}$, and $f(1) = p_1$.

Hence all curves we are counting may be parameterized by a map of the form

$$\begin{array}{rccc} f: & \mathbb{C}^* \setminus \{1\} \setminus \{z \mid Q_2(z) = 0\} & \longrightarrow & (\mathbb{C}^*)^2 \\ & z & \longmapsto & \left(z^{wl_3}, \frac{(z-1)Q_2(z)}{z^{wl_2}}\right) \end{array}$$

where $Q_2(z)$ is a complex polynomial of degree $l_1 - 1$. From the conditions imposed on f, the roots of Q_2 must be a subset of $l_1 - 1$ elements of the set of all (wl_3)-th roots of unity distinct from 1. Hence we have $\binom{wl_3-1}{l_1-1}$ distinct possibilities for the roots of Q_2, and any such choice determines Q_2 up to a multiplicative constant. The equation that this constant satisfies is given by the condition $f(0) = p_2$. Since we have $\psi \circ f(z) = \left((z-1)^{l_3}Q_2(z)^{l_3}, z^w(z-1)^v Q_2(z)^v\right)$ this condition translates to $(-1)^{l_3}Q(0)^{l_3} = 1$. Hence there exists exactly l_3 distinct polynomials $Q_2(z)$ once the set of its roots is chosen.

In conclusion there exist exactly $l_3\binom{wl_3-1}{l_1-1}$ admissible functions f with the chosen coordinates system on $\mathbb{C}P^1$. Since there exist l_1 possibilities to choose a point q in $f^{-1}(p_1)$ in order to fix the coordinate system on $\mathbb{C}P^1$, the number of maps f up to re-parameterization of $\mathbb{C}P^1$ is $\frac{l_3}{l_1}\binom{wl_3-1}{l_1-1} = \frac{1}{w}\binom{wl_3}{l_1}$. □

In the next proposition we apply this computation to vertices of the tropical curve mapped to L. After the projection with π_x, these vertices are adjacent to l edges of the same direction, corresponding to the l_1 preimages of p_1 here. From this computation, we obtain the binomial coefficients $\binom{k+l}{l}$ that appear in the definition of tropical multiplicity of a vertex mapped to L.

Proposition 5.17
Let $h : C \to X$ be a tropical morphism in $\mathbb{T}\mathcal{S}(\omega)$. Then there exist exactly

$$\frac{\mu_h}{\prod_{p\in e} w(e)}$$

possibilities to choose a phase ϕ for h such that $\phi_p \in \Phi_{e_p}(S_{e_p})$ for all points p in ω.

Proof:
As in the proof of Proposition 5.15, we can equivalently compute how many possibilities there are to phase $\overline{h} : \overline{C} \to \mathbb{R}^2$ in a coherent way with $\pi_x(\phi_\omega)$. Again we follow the lines of [Mik05]. As in [Mik05, Lemma 4.20], any connected component of $C \setminus \mathcal{P}$ is a tree containing exactly one end not mapped to L, otherwise the set $\mathbb{T}\mathcal{S}(\omega)$ would not be finite. Hence we may reconstruct all possible phases step by step as in [Mik05, Section 8.2], each step consisting of solving a simple enumerative problem in a toric surface. The only local computation we need here which is not covered by [Mik05] is done in Lemma 5.16. □

Now Theorem 4.13 follows immediately from Propositions 5.12, 5.15, and 5.17.

5.3. **The proof of Theorem 3.17.** The proof of Theorem 3.17 follows exactly the same strategy as the proof of Theorem 4.13: we reformulate the definition of the numbers $\mathcal{N}(u,n,d,\alpha)$ in terms of enumeration of curves in some toric surface having prescribed intersections with the toric divisors; then we prove a Correspondence Theorem for this latter enumerative problem by applying the methods from [Mik05] and [Shu12]. As in the proof of Theorem 4.13, we first "lift" our enumerative problem to three-space and then project it back to the plane using the different projection π_x. In this way, we can make use of known techniques.

We fix a coordinate system (x,y) on $(\mathbb{C}^*)^2$ such that $(x,\frac{1}{y})$ is a standard coordinate system on Σ_n. Let S_0 be the curve of bidegree $(1,1)$ in Σ_n with equation $x+y=0$, and let F_0 be the curve with equation $x=\infty$. We define

$$\begin{array}{cccc} \iota': & (\mathbb{C}^*)^2\setminus\{x+y=0\} & \longrightarrow & (\mathbb{C}^*)^3 \\ & (x,y) & \longmapsto & (x,y,-x-y) \end{array}.$$

The next lemma is a straightforward computation.

Lemma 5.18
Let C be an algebraic curve in Σ_n of bidegree $(a,0)$, and let D be a branch of C such that $(D\circ S_0)_{p_0}=d+\alpha(n+1)$ and $(D\circ F_0)_{p_0}=\alpha$. If C' denotes the restriction of C to the torus orbit corresponding to the coordinate system (x,y) fixed above, then the element of the Newton fan of $\iota'(C'\setminus\{x+y=0\})$ corresponding to D is $(\alpha,\alpha,\alpha-d)$.

Let δ be the Newton fan

$$\delta=\{(1,1)^{un-\sum\alpha_i},(-n,1)^u,(\alpha_1,\alpha_1-d_1),\dots,(\alpha_s,\alpha_s-d_s),\\ (0,-d_{r+1}),\dots,(0,-d_{r+s})\}.$$

We denote by $\mathcal{L}$ the closure in $\mathrm{Tor}(\Pi_\delta)$ of the line in $(\mathbb{C}^*)^2$ with equation $y+z=0$, and by $\mathcal{E}$ the toric divisor of $\mathrm{Tor}(\Pi_\delta)$ corresponding to elements $(1,1)$ of δ. Given an algebraic curve C in $(\mathbb{C}^*)^2$, we denote by $\overline{C}$ its closure in $\mathrm{Tor}(\Pi_\delta)$.

Using the projection δ_x and the techniques from Lemma 5.14 and Lemma 5.18, we obtain the following corollary.

Corollary 5.19
The number $\mathcal{N}(u,n,d,\alpha)$ is equal to the number of algebraic curves C in $(\mathbb{C}^*)^2$ with Newton fan δ and such that

- *$\overline{C}$ has u connected components, whose normalization are all rational;*
- *for each element (α_i,α_i-d_i) or $(0,-d_i)$ of δ, the intersection of $\overline{C}$ with the corresponding toric divisor is fixed;*
- *$\overline{C}$ has an ordinary multiple point of multiplicity $un-\sum\alpha_i$ at the point $\mathcal{L}\cap\mathcal{E}$.*

Now the proof of Theorem 4.13 extends literally to Theorem 3.17, using refinements of [Mik05] from [Shu12] to enumerate curves with prescribed intersections with toric divisors.

6. Concluding remarks

We discuss some of the possible extensions of results and methods presented in this paper.

(1) Although Theorem 4.13 assumes that configurations ω are contained in the two faces σ_1 and σ_2, Theorem 1.2 is obtained just by considering configurations ω contained in σ_1. It should be possible to generalize Theorem 1.2 for any configuration $\omega \subset \sigma_1 \cup \sigma_2$. This would also require to enlarge the family of $(1,1)$-relative invariants considered here.
(2) It would also be interesting to relate enumerative invariants of Σ_n and Σ_{n+2k} when $k \geq 2$. According to Appendix A, one possible way would be to study enumerative geometry of the tropical surface X_k in $\mathbb{R}^3$ given by the polynomial "$x^k + y + z$". In this case the assumption we made throughout Section 3.2, i.e. that $d = \#\Delta_i$ for some i, fails. In particular the study of enumerative geometry of X_k requires more care for $k \geq 2$.
(3) Related to the previous remark is the question of determining the multiplicity of a tropical morphism to X. In general, the multiplicity of a vertex tropically mapped to the line L should be expressed in terms of *triple Hurwitz numbers* weighted by some binomials coefficients. Although all those numbers are in principle computable, no nice general formula is known yet. In the particular case treated in this paper, the corresponding Hurwitz numbers are very simple: it is the number of rational maps $\mathbb{C}P^1 \to \mathbb{C}P^1$ of degree d with a prescribed pole and zero of maximal order. In particular we could perform easily all computations keeping hidden the Hurwitz numbers aspect. However for more general enumerative problems in X, these Hurwitz numbers will show up naturally.
(4) More generally, the study of enumerative geometry of general tropical surfaces, or even tropical varieties of any dimension, is of great interest. So far, little is known about this problem. In this case all Hurwitz numbers will come into the game, not only the triple ones mentioned above. A generalization of Proposition 5.15 in nice cases should be a consequence of general results about the realization of regular phase-tropical curves in the forthcoming paper [Mik].

Appendix A. Hirzebruch surfaces and their deformations

In this appendix we translate to the tropical setting Kodaira deformation of Hirzebruch surfaces. We include it since it provides a justification for the strategy of the proof of Theorem 1.2, however the present paper is formally independent from this appendix. As a consequence, we only sketch the following proofs, and we assume that the reader is well acquainted with tropical geometry.

We first recall Kodaira deformation of Hirzebruch surfaces before turning to tropical deformations.

A.1. **Kodaira deformation of Hirzebruch surfaces and deformation to the normal cone.** As explained in Section 2.1, two Hirzebruch surfaces Σ_n and $\Sigma_{n'}$ are not biholomorphic if $n \neq n'$. However, if n and n' have the same parity, one can deform one of the two surfaces to the other one.

Theorem A.1 (Kodaira, see [Kod86])
Let $n, k \geq 0$ be two integer numbers. There exists a complex manifold $X_{n,k}$ of dimension 3 equipped with a submersion $\phi_{n,k} : X_{n,k} \to \mathbb{C}$ such that

$$\forall t \neq 0,\ \phi_{n,k}^{-1}(t) = \Sigma_n, \quad \textit{and} \quad \phi_{n,k}^{-1}(0) = \Sigma_{n+2k}.$$

Note that this implies that Σ_n and Σ_{n+2k} are diffeomorphic. We will prove Theorem A.1 in Section A.2 in the tropical language. However, our proof translates literally to the complex setting.

Kodaira deformation of Hirzebruch surfaces can actually be reduced to a standard procedure in both complex (deformation to the normal cone, see [Ful84]) and symplectic (symplectic sum, see [Ler95] and [IP04], stretching the neck, see [EGH00]) geometries.

Proposition A.2 (see [Ful84])
Let S be a nonsingular complex surface, let C be a nonsingular algebraic curve in S, and let $X'_{S,C}$ be the blow-up of the curve $C \times \{0\}$ in the complex 3-fold $S \times \mathbb{C}$. Then the exceptional divisor of $X'_{S,C}$ is isomorphic to $\mathbb{P}(\mathcal{N}_{C/S} \oplus \mathbb{C})$, where $\mathcal{N}_{C/S}$ is the normal bundle of C in S.

The first Chern class of $\mathcal{N}_{C/S}$ is the self-intersection of C in S. In particular, if C is rational of self-intersection m, then $\mathbb{P}(\mathcal{N}_{C/S} \oplus \mathbb{C}) = \Sigma_{|m|}$.

We denote by $\phi'_{(S,C)} : X'_{S,C} \to \mathbb{C}$ the obvious (holomorphic) projection on the $\mathbb{C}$ factor. Then Proposition A.2 implies that $\phi'_{(S,C)}$ is a submersion over $\mathbb{C}^*$, that

$$\forall t \neq 0,\ \phi'^{-1}_{(S,C)}(t) = S,$$

and that $\phi'^{-1}_{(S,C)}(0)$ is the union of S and $\mathbb{P}(\mathcal{N}_{C/S} \oplus \mathbb{C})$ intersecting transversely along C.

Suppose now that $S = \Sigma_n$ and C is a smooth rational curve of bidegree $(1, k)$. Since C has self-intersection $n + 2k$ in Σ_n, we get from Proposition A.2 that

$$\phi'^{-1}_{(\Sigma_n,C)}(0) = \Sigma_n \cup \Sigma_{n+2k}.$$

In this case, it turns out that the two complex 3-folds, $X_{n,k}$ from Theorem A.1 and $X'_{\Sigma_n,k}$ from Proposition A.2, are related by a blow-up: one can contract the copy of Σ_n in $\phi'^{-1}_{(\Sigma_n,C)}(0)$ to obtain $X_{n,k}$.

Proposition A.3
The complex manifold $X'_{\Sigma_n,C}$ is the blow-up bl of $X_{n,k}$ along the exceptional curve E_{n+2} of Σ_{n+2k}, and

$$\phi'_{(\Sigma_n,C)} = \phi_{n,k} \circ bl.$$

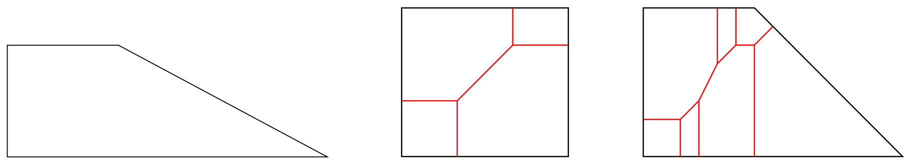
We prove the tropical analogue of Proposition A.3 in Section A.2. Once again, our tropical proof translates literally to the complex setting.

A.2. **Tropical Hirzebruch surfaces.** The construction of any non-singular toric variety can be performed exactly in the same way in tropical and algebraic geometry. In particular, the tropical Hirzebruch surface of degree $n \geq 0$, denoted by $\mathbb{T}\Sigma_n$ is constructed by taking two copies of $\mathbb{T}\times\mathbb{T}P^1$ glued along $\mathbb{T}^*\times\mathbb{T}P^1$ via the tropical isomorphism

$$\begin{array}{rccll}\psi: & \mathbb{T}^*\times\mathbb{T}P^1 & \longrightarrow & \mathbb{T}^*\times\mathbb{T}P^1 & \\ & (x_1,y_1) & \longmapsto & (“\frac{1}{x_1}”, “\frac{y_1}{x_1^n}”) & = (-x_1, y_1-nx_1)\end{array}$$

As in the complex setting, $\mathbb{T}\Sigma_0 = \mathbb{T}P^1\times\mathbb{T}P^1$, and $\mathbb{T}\Sigma_1$ is $\mathbb{T}P^2$ blown up at $[-\infty:0:-\infty]$.

The map ψ sends the vector $(1,n)$ to the vector $(-1,0)$. For this reason, the tropical surface $\mathbb{T}\Sigma_n$ is usually represented by a quadrangle with two horizontal edges, one vertical edge, and one edge of slope $-\frac{1}{n}$, see Figure 17a. More generally, once a linear system is fixed, the tropical moment map provides an homeomorphism from any non-singular toric tropical variety to any Newton polygon corresponding to the linear system. This homeomorphism is also given by the Veronese embedding corresponding to the chosen linear system.



a) $\mathbb{T}\Sigma_n$ b) A curve of bidegree $(1,1)$ in $\mathbb{T}\Sigma_0$ c) A curve of bidegree $(1,2)$ in $\mathbb{T}\Sigma_1$

FIGURE 17. Tropical toric surfaces and embedded curves.

Let us denote by B_n the tropical curve in $\mathbb{T}\Sigma_n$ defined by the tropical polynomial “y_1”, and by F_n the curve defined by the polynomial “x_1” (in the coordinate system defined above on $\mathbb{T}\Sigma_n$). That is to say, the curve B_n is the lowest horizontal edge, and F_n is the left vertical edge. Note that $B_n^2 = n$ and $F_n^2 = 0$. The tropical Picard group of $\mathbb{T}\Sigma_n$ is the free abelian group of rank two generated by B_n and F_n, and a tropical 1-cycle C in $\mathbb{T}\Sigma_n$ is said to have bidegree (a,b) if it is linearly equivalent to $aB_n + bF_n$. Equivalently, C is of bidegree (a,b) if and only if $C\circ B_n = an+b$ and $C\circ F_n = a$.

Example A.4
We depicted in Figure 17b a tropical curve of bidegree $(1,1)$ in $\mathbb{T}\Sigma_0$, and a tropical curve of bidegree $(1,2)$ in $\mathbb{T}\Sigma_1$ in Figure 17c.

The exceptional divisor E_n of $\mathbb{T}\Sigma_n$ is the upper horizontal edge, defined by the rational function “$\frac{1}{y_1}$”, and represents the class $B_n - nF_n$ in $\mathrm{Pic}(\mathbb{T}\Sigma_n)$. In particular, one has $E_n^2 = -n$.

A.3. **Deformation of tropical Hirzebruch surfaces.** Let C be a non-singular tropical curve in $\mathbb{T}\Sigma_n$ of bidegree $(1,k)$. By the genus formula C is rational. Moreover we have $C^2 = n+2k$.

We start by describing the deformation of $\mathbb{T}\Sigma_n$ to the normal cone of C. Recall that the sign “$=$” between two tropical varieties means “isomorphic up to tropical modifications” (see [Mik06]). Note that we have to introduce tropical modifications at this point since Kodaira deformation of Hirzebruch surfaces in non-toric. Indeed the exceptional divisor is a toric divisor, so any Hirzebruch surface is torically rigid.

Theorem A.5
There exists a non-singular tropical variety $\mathbb{T}X'$ *of dimension 3 and a tropical morphism* $\Phi' : \mathbb{T}X' \to \mathbb{T}P^1$ *such that*

$$\forall t \neq -\infty,\ \Phi'^{-1}(t) = \mathbb{T}\Sigma_n, \quad \textit{and} \quad \Phi'^{-1}(-\infty) = \mathbb{T}\Sigma_n \cup \mathbb{T}\Sigma_{n+2k}.$$

Moreover, the intersection curve of the two latter surfaces is C *in* $\mathbb{T}\Sigma_n$*, and the exceptional section* E_{n+2k} *in* $\mathbb{T}\Sigma_{n+2k}$*.*

Proof:
Let Π' be the polytope in $\mathbb{R}^3$ with vertices (see Figure 18)

$(0,0,0),\ (2n+2k,0,0),\ (2k,2,0),\ (0,2,0),\ (0,0,1),\ (n+k,0,1),\ (1,k,1),\ (0,1,1).$

The polytope Π' defines a non-singular tropical toric variety $\text{Tor}(\Pi')$ of dimension 3. If (x,y,z) are the coordinates in the dense $\mathbb{R}^3$-orbit of $\text{Tor}(\Pi')$, then the map $(x,y,z) \to (x,y)$ induces a tropical morphism $\pi : \text{Tor}(\Pi') \to \mathbb{T}\Sigma_n$ whose fibers are $\mathbb{T}P^1$.

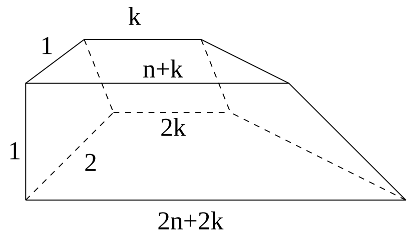


FIGURE 18. The polytope Π'.

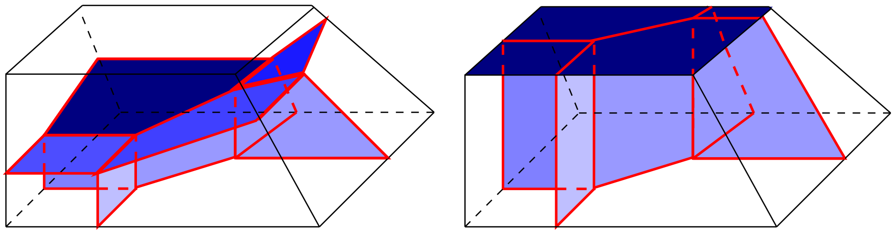
We fix a tropical polynomial $p(x,y)$ defining the curve C in $\mathbb{T}\Sigma_n$, and we define $\mathbb{T}X'$ as the hypersurface in $\mathbb{T}P^1 \times \text{Tor}(\Pi')$ defined by the tropical polynomial

$$\text{“}tz + p(x,y)\text{”}$$

where t is the coordinate in $\mathbb{T}P^1$.

The tropical variety $\mathbb{T}X'$ is non-singular, and there is a natural tropical morphism $\Phi' : \mathbb{T}X' \to \mathbb{T}P^1$ whose fiber over t_0 is the tropical hypersurface in $\text{Tor}(\Pi')$ defined by the tropical polynomial “$t_0 z + p(x,y)$” (see Figure 19a in the case $n = 0$ and $k = 1$). If $t_0 \neq -\infty$, then $\Phi'^{-1}(t_0)$ is nonsingular, and the morphisms $\pi_{|\Phi'^{-1}(t_0)} : \Phi'^{-1}(t_0) \to \mathbb{T}\Sigma_n$ is a tropical modification of $\mathbb{T}\Sigma_n$ along C. In particular $\Phi'^{-1}(t_0) = \mathbb{T}\Sigma_n$. The hypersurface $\Phi'^{-1}(-\infty)$ is the union of the tropical surface S_0 defined by $p(x,y)$ with the surface S_1 in $\text{Tor}(\Pi')$ defined by the rational function “$\frac{1}{z}$” (i.e. the upper horizontal face of $\text{Tor}(\Pi')$), see Figure 19b. The surfaces S_0 and S_1 intersect along a tropical curve E.

It is clear from Π' that $S_1 = \mathbb{T}\Sigma_n$, so it remains to prove that $S_0 = \mathbb{T}\Sigma_{n+2k}$. First, the morphisms π restricts to a tropical morphism $S_0 \to C$ whose fibers are $\mathbb{T}P^1$. It follows from elementary tropical intersection theory that $C^2 = n + 2k = -E^2$ in S_0. For example, by adding a vertical edge to each tropical intersection points of



a) $\Phi'^{-1}(t_0)$ with $t_0 \neq -\infty$ b) $\Phi'^{-1}(-\infty)$

FIGURE 19. The deformation process.

C in $\mathbb{T}\Sigma_n$, we see that the self-intersection of C in $\mathbb{T}\Sigma_n$ and S_0 are the same, i.e. $S_0 = \mathbb{T}\Sigma_{n+2k}$ (see Figure 20). Note that for the same reasons, we have $E^2 = -n-2k$ in S_0. □

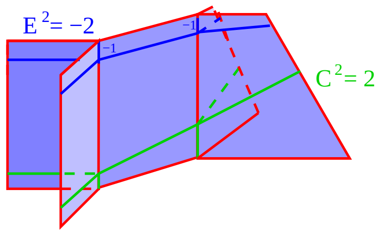


FIGURE 20. Self intersection of C and E in S_0.

Example A.6
We depict in Figure 19 this degeneration process when $n = 0$, $k = 1$, and C is the tropical curve depicted in Figure 17b.

In the next lemma, we use notation introduced in the proof of Theorem A.5.

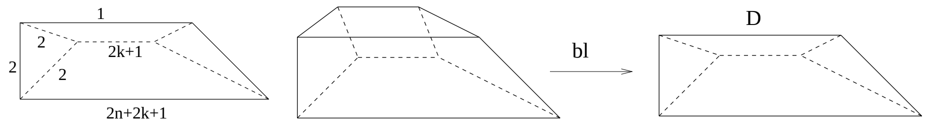
Lemma A.7
One can blow down S_1 to E in $\mathbb{T}X'$, i.e. there exists a non-singular tropical variety $\mathbb{T}X$ of dimension 3 and a tropical blow down $bl : \mathbb{T}X' \to \mathbb{T}X$ which contracts the surface S_1 to a curve isomorphic (up to tropical modifications) to E.

Proof:
This is an immediate consequence of the fact that one can blow down the surface S_1 in $\mathrm{Tor}(\Pi')$. Indeed, let Π be the polytope with vertices (see Figure 21a)

$$(0,0,0),\ (2n+2k+1,0,0),\ (2k+1,2,0),\ (0,2,0),\ (0,0,1),\ (1,0,1).$$

The tropical 3-fold $\mathrm{Tor}(\Pi)$ is non-singular, and the existence of the desired blow-up $\mathrm{Tor}(\Pi') \to \mathrm{Tor}(\Pi)$ can be observed directly at the polytopes Π and Π' (see Figure 21b). □

As an immediate corollary, we obtain the tropical version of Kodaira deformation of Hirzebruch surfaces.



a) Π b) Toric blow up of the curve D

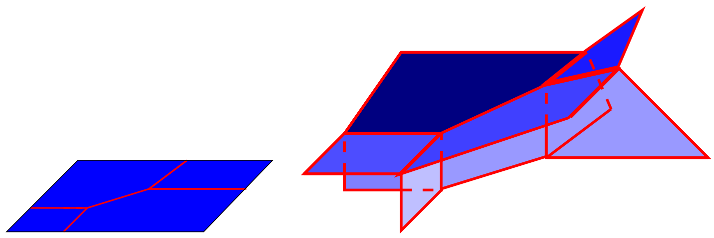
FIGURE 21. The blow up $\mathrm{Tor}(\Pi') \to \mathrm{Tor}(\Pi)$.

Corollary A.8
There exists a non-singular tropical variety $\mathbb{T}X$ of dimension 3 and a tropical morphism $\Phi : \mathbb{T}X \to \mathbb{T}P^1$ such that

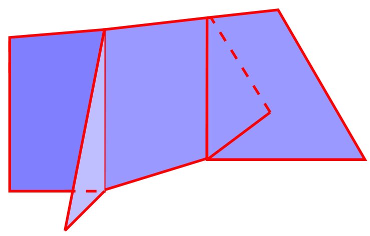
$$\forall t \neq -\infty,\ \Phi^{-1}(t) = \mathbb{T}\Sigma_n, \quad \text{and} \quad \Phi^{-1}(-\infty) = \mathbb{T}\Sigma_{n+2k}.$$

Proof:
Take $\mathbb{T}X$ as in Lemma A.7, and $\Phi : \mathbb{T}X \to \mathbb{T}P^1$ such that $\Phi' = \Phi \circ \mathrm{bl}$. □

Example A.9
We depict in Figure 22 this deformation when $n = 0$, $k = 1$, and C is the tropical curve depicted in Figure 17b.



a) $\Phi^{-1}(+\infty)$ b) $\Phi^{-1}(t_0)$ with $t_0 \neq \pm\infty$



$\Phi^{-1}(-\infty)$

FIGURE 22. Tropical Kodaira deformation.

Remark A.10
This section explains why we prove Theorem 1.2 by enumerating tropical curves in X. This latter surface (after the change of coordinates $(x, y, z) \mapsto (x, -y, z)$) is the part contained in the $\mathbb{R}^3$-orbit of $\mathrm{Tor}(\Pi')$ of the degeneration process described in

the proof of Theorem A.5, where C is the tropical curve of bidegree $(1,1)$ in $\mathbb{T}\Sigma_n$ defined by the tropical polynomial "$1+xy$". Since X is the open part of a tropical modification of $\mathbb{T}\Sigma_n$, counting carefully tropical curves in X or $\mathbb{T}\Sigma_n$ should give the same numbers.

On the other hand, we choose the configuration ω inside the face σ_1 of X which degenerates to $\mathbb{T}\Sigma_{n+2}$, replacing the condition $t \to -\infty$ by the condition that *points in ω have very low z-coordinates*. In particular it is natural that the parts in σ_1 far from L of the tropical curves we are counting look like curves in $\mathbb{T}\Sigma_{n+2}$.

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