ON MAXIMALLY INFLECTED HYPERBOLIC CURVES

AUBIN ARROYO, ERWAN BRUGALLÉ, AND LUCIA LÓPEZ DE MEDRANO

Abstract. In this note we study the distribution of real inflection points among the ovals of a real non-singular hyperbolic curve of even degree. Using Hilbert's method we show that for any integers d and r such that $4 \leq r \leq 2d^2 - 2d$, there is a non-singular hyperbolic curve of degree $2d$ in $\mathbb{R}^2$ with exactly r line segments in the boundary of its convex hull. We also give a complete classification of possible distributions of inflection points among the ovals of a maximally inflected non-singular hyperbolic curve of degree 6.

1. Introduction

The fact that a non-singular real algebraic curve in $\mathbb{RP}^2$ of degree δ has at most $\delta(\delta-2)$ real inflection points was proved by Klein in 1876, see [Kle76]; see also [Ron98], [Sch04] and [Vir88]. A non-singular real plane algebraic curve is called *maximally inflected* if it has $\delta(\delta-2)$ distinct real inflection points. The existence of maximally inflected curves of any degree was also proved by Klein. Possible distributions of inflection points of a maximally inflected real algebraic curve in $\mathbb{RP}^2$ are subject to non-trivial obstructions that mainly remain mysterious (see for example [KS03] and [BLdM12]). In this note we focus on the case of maximally inflected hyperbolic curves of even degree.

Let $X = \mathbb{R}^2$ or $\mathbb{RP}^2$. A real algebraic curve C in X is said to be *hyperbolic* if there exists a point $p \in X \setminus C$ such that any real line through p intersects C only in real points. The topology of a non-singular hyperbolic curve is easy to describe. An embedded circle O in X is called an *oval* if its complement in X has two connected components. In this case, one of them, called the *interior* of O and denoted by $\mathrm{Int}\,(O)$, is homeomorphic to a disk, while the other is called the *exterior* of O and denoted by $\mathrm{Ext}\,(O)$. An oval O is said to be contained in another oval $\tilde{O}$ if $O \subset \mathrm{Int}\,(\tilde{O})$. A non-singular hyperbolic curve C of degree $2d$ is a set of d nested ovals, i.e. the inclusion relation among its ovals is a total ordering. The oval of C containing all the others is called the *outer oval* of C. We define the *interior* and the *exterior* of C, denoted by $\mathrm{Int}\,(C)$ and $\mathrm{Ext}\,(C)$, as the interior and exterior of its outer oval respectively.

1.1. Line segments in the boundary of the convex hull of a hyperbolic curve in $\mathbb{R}^2$.

Denote by $\mathrm{Hull}\,(S)$ the convex hull of a compact set $S \subset \mathbb{R}^2$. Of course, if S is not convex then the boundary of $\mathrm{Hull}\,(S)$ contain some line segments. Denote by $s(S)$ the number of line segments in $\partial\mathrm{Hull}\,(S)$.

In the case of a non-singular hyperbolic curve C of degree $2d$ in $\mathbb{R}^2$, the number $\mathrm{s}\,(C)$ is related to the number of inflection points contained in the outer oval O of C. More

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precisely, for each line segment l in $\partial\mathrm{Hull}\,(C)$, there are at least two inflection points of C in the connected component of the closure $O \setminus \partial\mathrm{Hull}\,(C)$ with the same endpoints than l. Klein Inequality then implies that:

$$\mathrm{s}\,(C) \leq 2d(d-1).$$

Next theorem shows, in particular, that this upper bound is sharp, answering to a question posed by De Loera, Sturmfels, and Vinzant in [DLSV12].

Theorem 1.1. *Let d and r be two positive integers such that $4 \leq r \leq 2d(d-1)$. Then there exists a non-singular maximally inflected hyperbolic curve C of degree $2d$ in $\mathbb{R}^2$ with $s\,(C) = r$.*

The proof of Theorem 1.1 is given in Section 2, and is based on Hilbert's method of construction of real algebraic curves. See [Vir89] for an exposition of this method in modern terms. For the reader's convenience, we break the proof of Theorem 1.1 into two parts: in Section 2.1 we explain how to adapt Hilbert's method to construct non-singular maximally inflected hyperbolic curves of even degree with all real inflection points contained in the outer oval; In Section 2.2 we adapt this construction to prove Theorem 1.1.

1.2. **Hyperbolic curves of degree 6.** Recall that a hyperbolic curve of degree 6 has three nested ovals. Bezout's Theorem implies that there are no inflection points in the smallest oval of C. Next theorem gives then a complete classification of possible distributions of inflection points among the ovals of a non-singular maximally inflected hyperbolic curve of degree 6 in $\mathbb{RP}^2$.

Theorem 1.2. *Let C be a non-singular maximally inflected hyperbolic curve of degree 6 in $\mathbb{RP}^2$. Then, the outer oval of C contains at least 6 real inflection points. Moreover, for any $0 \leq k \leq 8$, there exists such a hyperbolic curve with exactly $6 + 2k$ real inflection points on the outer oval.*

The proof of this theorem will be given in Section 3 and it combines two main ingredients: First we use the tropical methods developed in [BLdM12] to construct real algebraic curves with a prescribed position of their real inflection points. Then we rely on Orevkov's braid theoretical method to prove that the outer oval of C cannot contain less than 6 inflection points. Note that our proof of this latter fact uses in a crucial way that we deal with curves of degree 6. The existence of a non-singular maximally inflected hyperbolic curve of degree $2d > 6$ with an outer oval not containing any inflection points remains an open problem.

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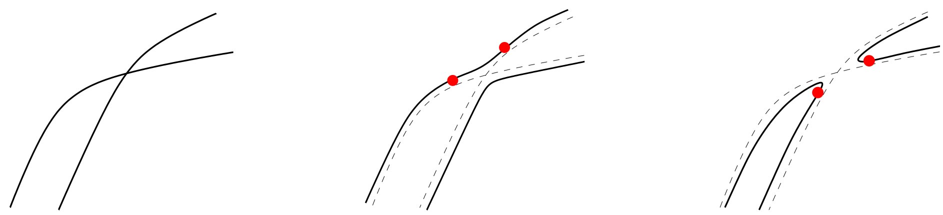
2. Construction of inflected curves

Hilbert's method can be adapted to construct non-singular maximally inflected hyperbolic curves of even degree with all real inflection points contained in the outer oval. The proof of Theorem 1.1, given in Section 2.2, will consider perturbations of certain curves with generic nodes; these curves are described in Section 2.1. Recall that a *node* of an algebraic curve C in $\mathbb{CP}^2$ is a point in the curve C which is the transverse intersection of two non-singular local branches of C. Such a node is called *generic* in the case that both branches of C at the point have intersection multiplicity 2 with their tangent at the node. If C is a real curve with a node $p \in \mathbb{RP}^2$, then either p is the intersection of two real branches or is the intersection of two complex conjugated branches of C. For our construction we shall focus only on nodes of the former type.

The fact that Hilbert's method produces maximally inflected curves is based on the following observation (see Figure 1).

Proposition 2.1 (see [Ron98, Proposition 3.1])**.** *Let C be a real algebraic curve in $\mathbb{RP}^2$ having a generic node $p \in \mathbb{RP}^2$ with two real branches. Then any real perturbation of C has exactly two real inflection points on a neighborhood of p.*



a) b) c)

Figure 1. Smoothing of a generic node and the creation of two real inflection points.

An real algebraic curve C in $\mathbb{RP}^2$ is said to be *generically inflected* if it is non-singular or nodal, and any of its real inflection point is a non-singular point at which C has intersection multiplicity 3 with its tangent. Given a generically inflected real algebraic curve C, we denote by $I(C)$ the set of its non-singular real inflection points, and by $N(C)$ the set of its real generic nodes. It follows from Klein Inequality and Proposition 2.1 that if C has degree d,

$$(*) \qquad \#I(C) + 2\#N(C) \leq d(d-2).$$

A generically inflected curve C is called *maximally inflected* if equality holds in (*).

Next Lemma is an immediate consequence of Proposition 2.1.

Lemma 2.2. *Let C be a generically inflected real algebraic curve in $\mathbb{RP}^2$. Then for any non-singular real deformation C' of C, one has*

$$\#I(C') = \#I(C) + 2\,\#N(C).$$

In particular, any non-singular real deformation of a maximally inflected real curve is maximally inflected.

2.1. **Construction of hyperbolic curves with a maximally inflected outer oval.** For the rest of this section we fix a non-singular real ellipse C_0 in $\mathbb{R}^2$. An oval O of a maximally inflected non-singular curve C is said to be maximally inflected, if $I(C) \subset O$. Let us consider the following properties for a non-singular maximally inflected hyperbolic curve C of degree $2d$:

- $(\mathcal{P})$: C has a maximally inflected outer oval and $\text{Int}\,(C) \cap \text{Int}\,(C_0)$ is convex;
- $(\mathcal{P}')$: $C \cap C_0$ is a set of $4d$ distinct real points contained in the outer oval of C, none of them being an inflection point of C.

Next we proceed by induction to construct a family $(C_d)_{d\geq 1}$ of real algebraic curves of degree $2d$ in $\mathbb{R}^2$. Given two real algebraic curves C and L in $\mathbb{R}^2$, respectively, defined by equations $P(x,y) = 0$ and $Q(x,y) = 0$, a *perturbation of C by L* is a real algebraic curve defined by the equation $P(x,y) + \varepsilon Q(x,y) = 0$, with $\varepsilon \ll 1$ a real number. Note that two perturbations of C by L using parameters with opposite signs have a priori different topology in $\mathbb{R}^2$. The family $(C_d)_{d\geq 1}$ is constructed as follows:

(1) We choose for C_1 any non-singular ellipse in $\mathbb{R}^2$ intersecting C_0 in 4 real points.
(2) Suppose that the curve C_{d-1} is constructed, and does not contain C_0 as a component. Consider L_d a union of $2d$ real lines in $\mathbb{R}^2$ intersecting C_0 in a set P_d of $4d$ distinct real points such that $P_d \subset \text{Ext}\,(C_{d-1})$, and that any connected component of $C_0 \cap \text{Ext}\,(C_{d-1})$ contains an even number of points of P_d. These two conditions ensure that there exists a perturbation of $C_{d-1} \cup C_0$ by L_d producing a non-singular hyperbolic curve, that we define to be C_d.

Proposition 2.3. *If C_{d-1} satisfies properties $(\mathcal{P})$ and $(\mathcal{P}')$, and if C_d is a perturbation of $C_{d-1} \cup C_0$ by L_d, then C_d also satisfies properties $(\mathcal{P})$ and $(\mathcal{P}')$.*

Proof. By assumption we have:

$$\#I(C_{d-1}) + 2\,\#N(C_0 \cup C_{d-1}) = 2(d-1)(2(d-1)-2) + 8(d-1) = 2d(2d-2).$$

Hence $C_0 \cup C_{d-1}$ is a maximally inflected real hyperbolic curve. Property $(\mathcal{P}')$ implies that this latter curve is generically inflected. Lemma 2.2 implies that C_d is maximally inflected. Moreover, property $(\mathcal{P})$ implies that $I(C_{d-1}) \subset \text{Ext}\,(C_0)$ and that each node of $C_0 \cup C_{d-1}$ produces two real inflection points of C_d on the outer oval of C_d (see Figure 1b). Hence C_d has a maximally inflected outer oval, and $I(C_d) \subset \text{Ext}\,(C_0)$.

To prove that $\text{Int}\,(C_d) \cap \text{Int}\,(C_0)$ is convex, it is enough to show that $\text{Int}\,(C_d)$ is locally convex at each point of the outer oval of C_d contained in $\text{Int}\,(C_0)$. However this is true for points close to P_d, since C_d is a perturbation of C_0 in a neighborhood of a point in P_d. Moreover, this is true for all points since $I(C_{d-1}) \subset \text{Ext}\,(C_0)$; and hence $\text{Int}\,(C_d) \cap \text{Int}\,(C_0)$ is convex.

By construction we have $C_d \cap C_0 = P_d$ which has cardinality $4d$. Since $I(C_{d-1}) \cap C_0 = \emptyset$, there exists a small neighbourhood U of $I(C_{d-1})$ such that $U \cap C_0 = \emptyset$. Let U' be a neighbourhood of P_d such that $C_0 \cap C_d \cap U' = \emptyset$. If C_d is a perturbation of $C_0 \cup C_{d-1}$, then $I(C_d) \subset U \cup U'$, so C_d satisfies property $(\mathcal{P}')$ $\square$

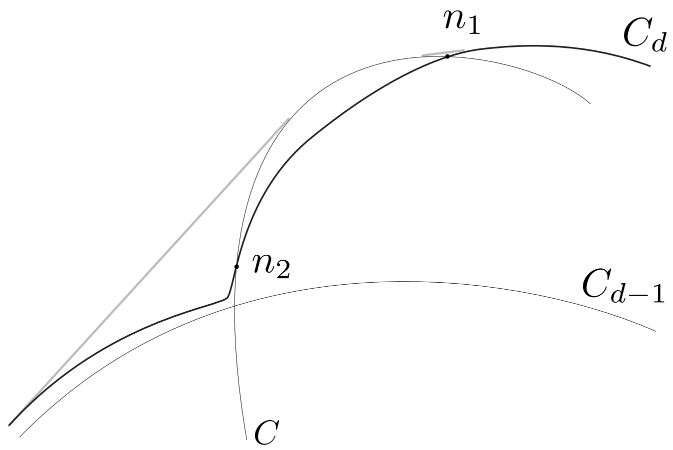


FIGURE 2. Gray lines correspond to line segments in $\partial\mathrm{Hull}\,(C_0 \cup C_d)$. The node n_1 is a point in P_d containted in $\partial\mathrm{Hull}\,(C_{d-1} \cup C_0)$. The node n_2 is a point of P_d contained in the interior of $\mathrm{Hull}\,(C_{d-1} \cup C_0)$.

Note that C_1 obviously satisfies properties $(\mathcal{P})$ and $(\mathcal{P}')$. Hence Proposition 2.3 ensures the existence of a family $(C_d)_{d\geq 1}$ containing only non-singular hyperbolic curve in $\mathbb{R}^2$ with a maximally inflected outer oval.

2.2. **Proof of Theorem 1.1.** Here we explain how to control the numbers $\mathrm{s}\,(C_d)$ in the previous construction (see section 1.1 for the definition of $\mathrm{s}\,(C)$). For this, at each step we require the additional condition that the set P_d is disjoint from the endpoints of line segments in $\mathrm{Hull}\,(C_{d-1} \cup C_0)$ (see item (2) in section 2.1 for the definition of P_d). We denote by r_d the number of points in P_d contained in $\partial\mathrm{Hull}\,(C_{d-1} \cup C_0)$. Note that we may freely choose the number r_d, $d \geq 2$, between 0 and $4d$.

Proposition 2.4. *If C_d is a perturbation of C_{d-1}, then $s\,(C_d) = s\,(C_{d-1} \cup C_0)$ and $s\,(C_d \cup C_0) = s\,(C_{d-1}) + r_d$.*

Proof. By continuity, we have $\mathrm{s}\,(C_d) \geq \mathrm{s}\,(C_{d-1} \cup C_0)$. If this inequality were strict, there would exists a line segment in $\partial\mathrm{Hull}\,(C_d)$ coming from the perturbation of a real point of C_{d-1} at which C_{d-1} has intersection multiplicity at least 4 with its tangent. However the curve C_{d-1} is maximally inflected, and in particular is generically inflected. Hence all this together prove that $\mathrm{s}\,(C_d) = \mathrm{s}\,(C_{d-1} \cup C_0)$.

There are two types of line segments contained in $\partial\mathrm{Hull}\,(C_d \cup C_0)$: those which are a perturbation of a line segment contained in $\partial\mathrm{Hull}\,(C_{d-1} \cup C_0)$, and those which come from the perturbation of $C_{d-1} \cup C_0$ in a neighborhood U_p of a point $p \in P_d \cap \partial\mathrm{Hull}\,(C_{d-1} \cup C_0)$; see Figure 2. The set $\partial\mathrm{Hull}\,((C_d \cup C_0) \cap U_p)$ contains a line segment. Hence the set $\partial\mathrm{Hull}\,(C_d \cup C_0)$ contains exactly r_d line segments more than $\partial\mathrm{Hull}\,(C_{d-1} \cup C_0)$. ☐

Now, Theorem 1.1 follows from the fact that we can choose the number r_d between 0 and $4d$ freely, and because $\mathrm{s}\,(C_1) = 0$ and $\mathrm{s}\,(C_1 \cup C_0) = 4$.

Example 2.5. Let us give an explicit equation of a degree 6 polynomial defining a non-singular hyperbolic curve of degree 6 in $\mathbb{R}^2$ with 12 line segments on the boundary of its convex hull. For that, consider three numbers $\varepsilon_1 = 1.58$, $\varepsilon_2 = -5.5 \times 10^{-3}$ and $\varepsilon_3 = -10^{-10}$, and denote by $C := C(x,y) = x^2+y^2-1$. Consider the following polynomial:

$$f(x,y) := C^3 + C^2\left(y^2 - a^2x^2\right)\varepsilon_1 + C\left(y^2 - x^2\right)\left(y^2 - b^2x^2\right)\varepsilon_2 + \varepsilon_3,$$

where $a = \tan(\pi/12)$ and $b = \tan(5\pi/12)$. Figure 3 is a numerical plot of the zero locus of f. In the construction of this example we used two lines through the origin at angles $\pm\frac{\pi}{12}$ in the first step, and four lines through the origin at angles $\pm\frac{\pi}{4}, \pm\frac{5\pi}{12}$ in the second step (all angles are measured with respect to the positive x-axis). The plot in Figure 3 is

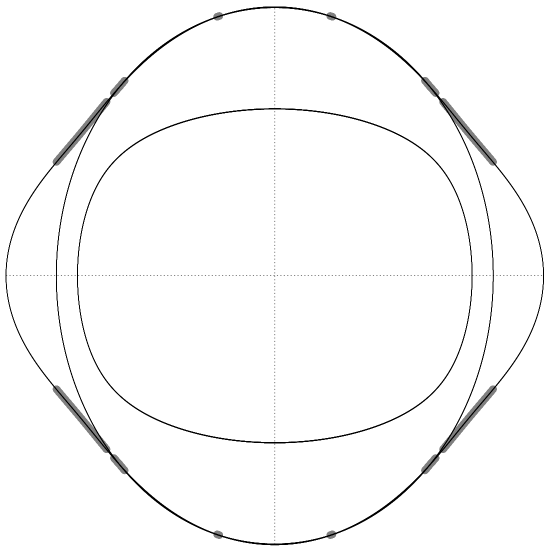


FIGURE 3. A non-singular hyperbolic curve of degree 6 in $\mathbb{R}^2$ with exactly 12 line segments in the boundary of its convex hull. Shaded regions on the outer oval (three in each quadrant) correspond to points in the curve contained in the interior of the convex hull of the curve.

obtained by finding numerically the roots of the real one variable polynomial defined by the intersection of the curve with lines through the origin.

3. Maximally inflected hyperbolic curves of degree 6

Now we turn to the study all possible arrangements of the 24 real inflection points of a non-singular maximally inflected hyperbolic sextic C in $\mathbb{RP}^2$. We denote by O_1, O_2, O_3 the three ovals of C in such a way that O_1 contains O_2 which in its turn contains O_3. By Bezout Theorem, the oval O_3 does not contain any real inflection point. Moreover, each oval O_1 and O_2 contains an even number of real inflection points.

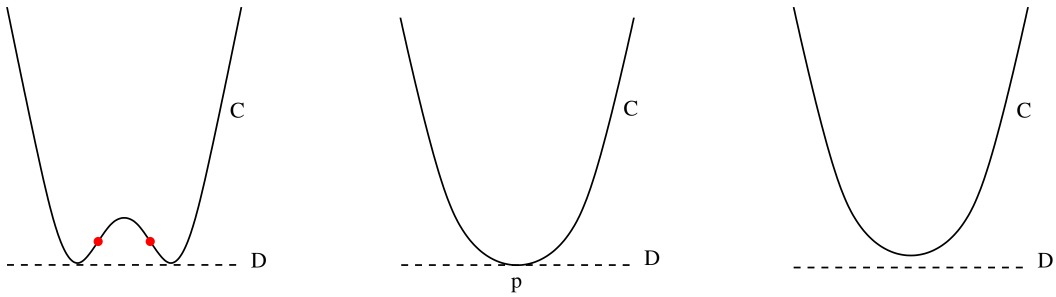
3.1. **Obstructions.** Here we prove that the oval O_2 cannot contain more than 18 real inflection points. Proposition 3.1 is a straightforward consequence of Lemmas 3.3 and 3.2, that we prove in the rest of this section.

Proposition 3.1. *Let C be a non-singular maximally inflected algebraic hyperbolic curve of degree 6 in $\mathbb{RP}^2$. Then the oval O_1 contains at least 6 real inflection points.*

Proof. Let $2n$ be the number of real inflection points of C contained in O_2. According to Lemma 3.2 the curve C has n real bitangents which are tangent to C at two points on O_2. Hence we get $n \leq 9$ by Lemma 3.3. □

Let C_0 and C_1 be two non-singular real algebraic curves in $\mathbb{RP}^2$ of degree at least 3, such that there exists a continuous path $(C_t)_{0\leq t\leq 1}$ of non-singular real algebraic curves in $\mathbb{RP}^2$ from C_0 to C_1. A real line D is said to be *deeply tangent* to the curve C_t if there exists a point $p \in D \cap C_t$ such that the order of contact of C_t and D is at least 4.

Recall that if C_0 and C_1 are generic, then they do not have any deep tangent lines. Moreover Klein proved in [Kle76] (see also [Ron98]) that in this case, it is always possible to choose the family (C_t) such that appearance and disappearance of real inflections points when t varies occur only when passing through a curve C_{t_0} with a unique deep tangent D with order of contact exactly 4 with C_{t_0} at a unique point p (see Figure 4b). Then for $t = t_0 + \varepsilon$ with $-1 \ll \varepsilon \ll 1$ a non-zero real number, the line D deforms to a real bitangent of C_t tangent to C_t in two real points close to p (see Figure 4a), and deforms to a real bitangent of C_t tangent to C_t to two complex conjugated points for $t = t_0 - \varepsilon$ (see Figure 4c). The curve $C_{t_0+\varepsilon}$ has two real inflection points close to p, that disappear when passing through C_{t_0}.



a) $C = C_{t_0+\varepsilon}$ b) $C = C_{t_0}$ c) $C = C_{t_0-\varepsilon}$

FIGURE 4. Appearance/disappearance of real inflection points

Lemma 3.2. *Let C be a non-singular maximally inflected hyperbolic curve of degree 6 in $\mathbb{RP}^2$ such that the oval O_i contains exactly $2n$ real inflection points. Then the curve C has exactly n real bitangents which are tangent to C at two points on O_i.*

Proof. Let C_0 be the non-singular hyperbolic real sextic in $\mathbb{CP}^2$ obtained by a generic perturbation of the union of 3 nested disjoint ellipses. By construction, C_0 does not have any real inflection points, and does not have any real bitangent tangent to two real points. Since we are dealing with curves of degree 6, it follows from Bezout Theorem that the number of real bitangents of a non-singular real algebraic curve of degree 6 is invariant under deformation in the space of non-singular real algebraic curve of degree 6. Since the space of non-singular hyperbolic sextics in $\mathbb{RP}^2$ is connected (see [Nui68]), the result follows from the description of appearance/disappearance of real inflection points given above. □

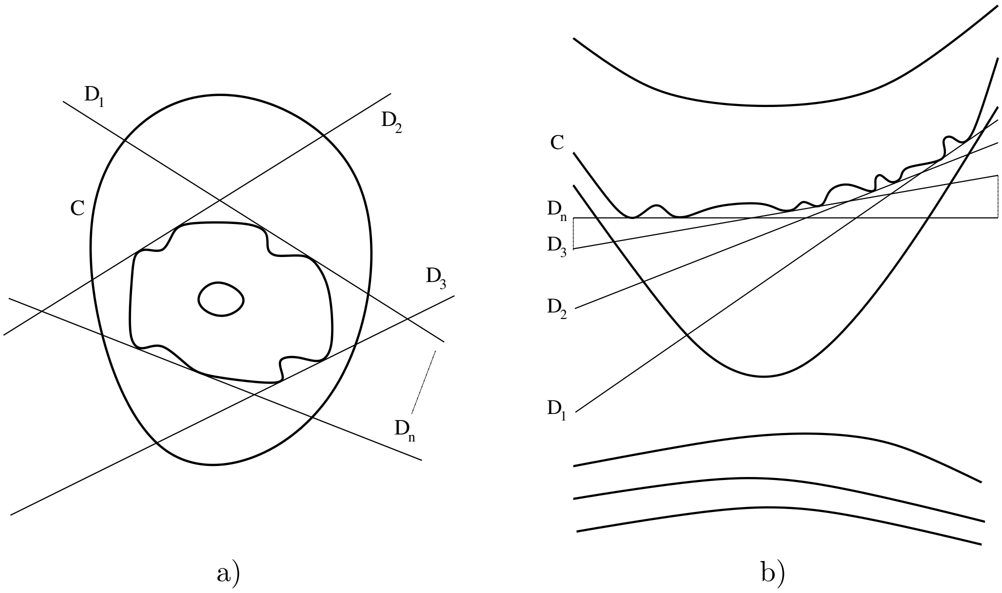


FIGURE 5

Lemma 3.3. *Let C be a non-singular maximally inflected hyperbolic curve of degree 6 in $\mathbb{RP}^2$ with exactly n real bitangents tangent to C at two points on O_2 (see Figure 5a). Then $n \leq 9$.*

Proof. The proof uses the symplectic techniques developed by Orevkov to study the topology of real plane curves. For the sake of shortness, we do not recall this technique here. We refer instead to [Ore02, Section 2.1] for the definition of a real pseudoholomorphic curve in $\mathbb{CP}^2$, and to [Ore99, Section 3] for the definition of a $\mathcal{L}_p$ scheme and for an account of the braid theoretical methods in the study of real plane curves. Note nevertheless that any real algebraic curve is a real pseudoholomorphic curve.

Choose a point p inside the oval O_3. Applying [Ore99, Proposition 3.6] if necessary, this implies that the $\mathcal{L}_p$-scheme depicted in Figure 5b is realizable by a reducible real pseudoholomorphic curve of degree $6+n$, whose irreducible components are a sextic C' and l real pseudoholomorphic lines $D_1, \dots, D_n$ intersecting C' transversely in two points and tangent to C' in two points.

The braid associated to this $\mathcal{L}_p$-scheme is

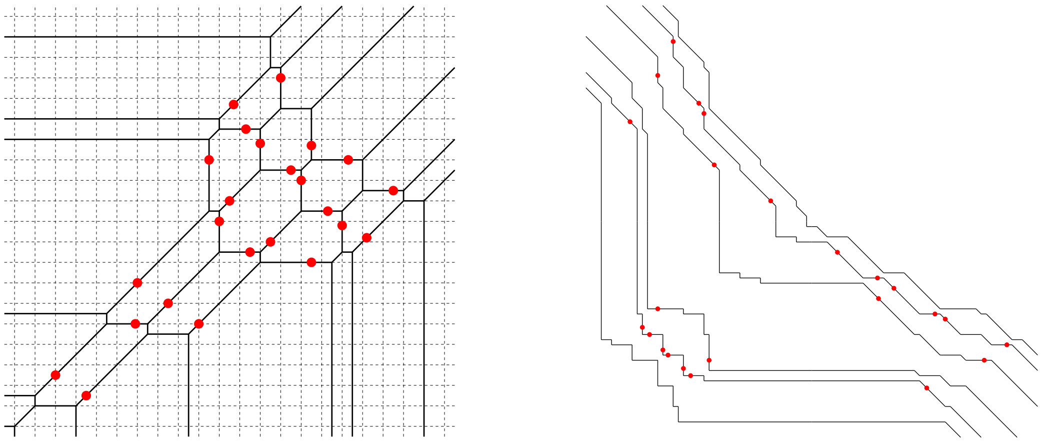
$$b_n = \prod_{i=4}^{n+3} \left[\left(\prod_{j=i}^{n+3} \sigma_j^{-1} \right) \sigma_{n+4}^{-4} \right] \cdot \prod_{j=4}^{n+3} \sigma_j^{-1} \cdot \prod_{i=1}^{n+5} \left(\prod_{j=i}^{n+5} \sigma_j \right).$$

Computing the Alexander polynomial p_{10} of b_{10}, we see that $\pm i$ is a simple root of p_{10}. On the other hand the sum of exponents of b_{10} equals 15, which is the number of strings of b_{10} minus 1. Hence the Murasugi-Tristram inequality [Ore99, Section 2.4] together with [Ore99, Lemma 2.1] implies that the $\mathcal{L}_p$-scheme depicted in Figure 5b with $n \geq 10$ is not realizable by a real pseudoholomorphic curve of degree $6+n$. $\square$

3.2. **Constructions.** We end the proof of Theorem 1.2 by showing that all distributions of the 24 inflection points on the ovals O_1 and O_2 which are not forbidden by Proposition 3.1 are realizable.

Proposition 3.4. *For any integer* $0 \leq k \leq 9$*, there exists a non-singular maximally inflected hyperbolic curve* $C(k)$ *of degree 6 in* $\mathbb{RP}^2$ *such that* O_1 *contains exactly* $6 + 2k$ *real inflection points.*

Proof. The curve $C(9)$ has been constructed in Section 2.1. The curve $C(k)$ with $k = 1, 3, 5$ and 7 is constructed similarly. Consider two real conics C_0 and C_1 intersecting into 4 real points. Choose $\frac{k-1}{2}$ lines intersecting C_0 in $k-1$ points in $\mathrm{Ext}\,(C_0 \cup C_1)$, and $\frac{7-k}{2}$ lines intersecting C_0 in $7-k$ points in $\mathrm{Int}\,(C_0 \cup C_1)$. Denote by L the union of these lines. We obtain a non-singular maximally inflected quartic C_2 by perturbing $C_0 \cup C_1$ with L. By perturbing $C_0 \cup C_2$ to a non-singular maximally inflected curve of degree 6, we obtain the curve $C(k)$.

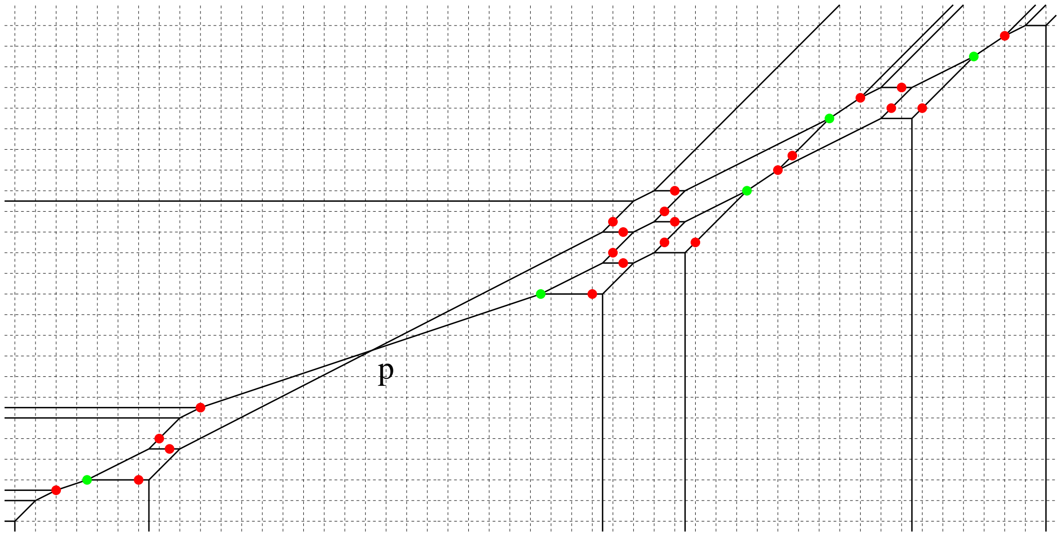


a) b)

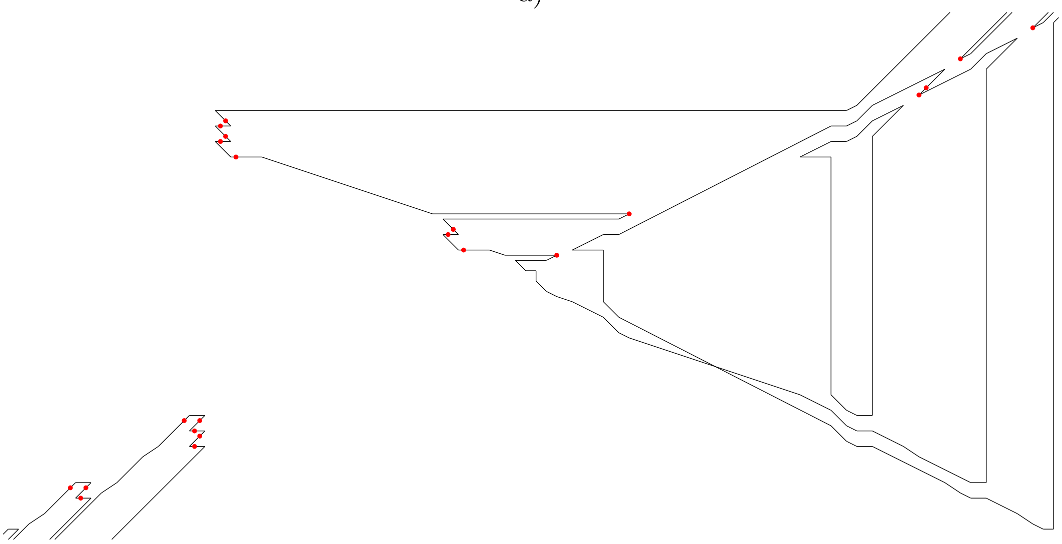
FIGURE 6. Patchworking of the curve $C(0)$

We construct the curves $C(k)$ with k even by patchworking. We refer to [BLdM12] for the definition of tropical inflection points of tropical plane curves, and for the construction of real algebraic curves with a controlled position of their real inflection points by patchworking. The tropical curve depicted in Figure 6a together with the patchworking depicted in Figure 6b produce the curve $C(0)$. The red dots represent inflection points in Figure 6 (note that all tropical inflection points have complex multiplicity 3, and real multiplicity 1 in this case). Since their position on the tropical curve heavily depend on the length of its edges, we depicted $\mathbb{Z}^2 \subset \mathbb{R}^2$ by the intersection points of the doted lines in Figure 6a. By varying the length of the edges of the tropical curve depicted in Figure 6a, we construct similarly the curves $C(2)$, $C(4)$, and $C(6)$.

The construction of the curve $C(8)$ requires a mild adaptation of the method presented in [BLdM12]. Consider the tropical curve $\mathbb{T}C(8)$ depicted in Figure 7a. Note that $\mathbb{T}C(8)$ has a node: all its vertices are 3-valent, except one 4-valent vertex p which is the transversal intersection of two edges of $\mathbb{T}C(8)$. The method used in [BLdM12] to locate tropical



a)



b)

FIGURE 7. Patchworking of the curve $\widetilde{C}(8)$

inflexion points and to compute their real and complex multiplicities only uses local data about tropical curves. Since the two edges of $\mathbb{T}C(8)$ intersecting at p do not have any common direction with an edge of a tropical line, this implies that all computations performed in [BLdM12] to locate and compute the multiplicities of tropical inflexion points of $\mathbb{T}C(8)$ distinct from p still holds in this case. In particular, red dots in Figure 7a represent tropical inflexion points distinct from p with real multiplicity 1, and green dots those of real multiplicity 0. Using Shustin's version [Shu98] of Viro's Patchworking Theorem to construct singular curves, we can approximate the real tropical curve depicted in Figure 7b by a family $(\widetilde{C}_t)_{t\gg 1}$ of real hyperbolic sextic with one generic node. Up to a translation in $\mathbb{R}^2$, the local equation of $\widetilde{C}_t$ at the node is

$$(x - \alpha y^2)(x + \beta y^3) + o_{t\to+\infty}(t^{-1})$$

with α and β two positive real numbers. Proposition 2.1 implies that smoothing this node such that the resulting curve is hyperbolic produces the curve $C(8)$. $\square$

Remark 3.5. If one smoothes "tropically" the node of the tropical curve $\mathbb{T}C(8)$ (i.e. perturbing p into two 3-valent vertices before patchworking), the two additional real inflection points appear on the oval O_2 in the positive quadrant of $\mathbb{R}^2$ instead of appearing on the oval O_1 in the quadrant $\{x > 0, y < 0\}$. This is a manifestation of the totally discontinuous topology induced on a field by a non-archimedean valuation.

On the other hand, one could use original Viro's Patchworking to construct directly the non-singular curve $C(8)$. In this case, one should use the chart (in the sense of [Vir84]) at the node p of $\mathbb{T}C(8)$ given by a suitable perturbation of the curve $(x - y^2)(x + y^3)$.

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