A bound on the number of jump of real algebraic curves in Σ_n

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Abstract

Inspired by the work of Orevkov, I study subresultants of a polynomial and extend to any degree a restriction Orevkov proved for real algebraic trigonal curves. This restriction is not fulfilled by real pseudoholomorphic curves.

1 Subresultants of a polynomial

Here I define subresultants of a polynomial. This is a particular case of subresultants of 2 polynomials, but I won't need the general definition here.
The standard definition of subresultants is by means of determinant of some Sylvester matrix (see [BPR03] for example). However, inspired by H. Hong (see [Hon]), I define it in terms of roots of the polynomial, as it is more convenient for my purpose.

Remark. All the polynomials I consider are monic, i.e.their leading coefficient is 1.

Definition 1.1 *Let $P(X)$ be a monic polynomial of degree d whose roots are $\omega_1, \ldots, \omega_d$ (multiple roots are repeated in this sequence). The jth subresultant of P is defined as*

$$sr_j(P) = \sum_{|J|=j} \left(\prod_{i \notin J} \prod_{k \notin J, k<i} (\omega_i - \omega_k)^2 \right)$$

where J is a subset of $\{1, \ldots, d\}$.

Examples. $sr_0(P)$ is the usual discriminant of P, and $sr_{d-1} = 1$.

For any subset J of $\{1, \ldots, d\}$, I put $P_J(X) = \prod_{i \notin J}(X - \omega_i)$.

Proposition 1.2 *One has*

$$sr_j(P) = \sum_{|J|=d-j} sr_0(P_J).$$

□

From now on, all polynomials are real.

The next two properties follows directly from the definition.

Proposition 1.3 *The polynomial P has d modulo 4 real roots if and only if $sr_0(P) \geq 0$. Moreover, $sr_0(P) = 0$ if and only if P has a multiple root.* □

Proposition 1.4 *If all the roots of P are real, then $sr_j(P) \geq 0$ for any j. Moreover, all these inequalities are strict as soon as all the roots of P are distinct.* □

2 Sign of the first subresultant

Here I extract the sign of $sr_1(P)$ when all roots of P but 2 are real and the real part of the 2 imaginary roots is a real root of P.

Lemma 2.1 *Let $r_1, \ldots, r_m$ and q be some real numbers. Then*

$$\prod_{k=1}^{m} r_k^2 (r_k + iq)^2 \quad + \quad \prod_{k=1}^{m} r_k^2 (r_k - iq)^2 \quad + \quad 4\prod_{k=1}^{m} (r_k^2 + q^2)^2 \quad \geq \quad 0.$$

Moreover, if $q \neq 0$, then this inequality is strict.

Proof. Denote by (E) the lefthandside summand. As $r_k^2 + q^2 \geq r_k^2$, one has

$$\begin{aligned} (E) &\geq \left(\prod_{k=1}^{m} r_k^2\right)\left(\prod_{k=1}^{m}(r_k + iq)^2 \quad + \quad \prod_{k=1}^{m}(r_k - iq)^2 \quad + \quad 4\prod_{k=1}^{m}(r_k^2 + q^2)\right) \\ &\geq \left(\prod_{k=1}^{m} r_k^2\right)\left(\prod_{k=1}^{m}(2r_k)^2 \quad + \quad 2\prod_{k=1}^{m}(r_k^2 + q^2)\right). \end{aligned}$$

All the terms in the righthandside are positive, so the result follows. □

Proposition 2.2 *Let $P(X)$ be a monic real polynomial of degree d whose roots are $\omega_1, \ldots, \omega_d$ (multiple roots are repeated in this sequence). Suppose that ω_i is real for $i \geq 3$, that ω_1 and ω_2 are not real and that $\Re(\omega_1) = \omega_3$. Then $sr_1(P) \leq 0$, and $sr_1(P) < 0$ if P has no multiple roots.*

Proof. The polynomials $P_{\{j\}}$ with $j \geq 4$ are real with only 2 nonreal roots, so $sr_0(P_{\{j\}}) \leq 0$ when $j \geq 4$.
Put $\omega_1 = \alpha + i\beta$. Then, one has

$$sr_0(P_{\{1\}}) + sr_0(P_{\{2\}}) + sr_0(P_{\{3\}}) =$$

$$-\beta^2 \prod_{j=4}^{d}\prod_{k>j}(\omega_j - \omega_k)^2 \left(\prod_{j=4}^{d}(\alpha - \omega_k)^2(\alpha - \omega_k + i\beta)^2 + \prod_{j=4}^{d}(\alpha - \omega_k)^2(\alpha - \omega_k - i\beta)^2 + 4\prod_{j=4}^{d}\left((\alpha - \omega_k)^2 + \beta^2\right)^2\right).$$

Hence, according to Lemma 2.1, $sr_0(P_{\{1\}}) + sr_0(P_{\{2\}}) + sr_0(P_{\{3\}}) \leq 0$, and the result follows from Proposition 1.2. □

3 Application to real algebraic curves

Here I generalize to any degree a restriction for trigonal curves due to Orevkov (see [Ore03]).

Let $n \in \mathbb{N}^*$ and let $P(X, Y) = Y^d + \sum_{i=1}^{d} a_i(X) Y^{d-i}$ be a real polynomial where $a_i(X)$ is a polynomial of degree ni. So P define a real algebraic curve of bidegree $(d, 0)$ in Σ_n. One can consider the subresultants of P considered as a polynomial in Y. Hence, one obtains polynomials in X.

Proposition 3.1 *One has $d^\circ(sr_j(P)(X)) = n(d - j)(d - j - 1)$.* □

Definition 3.2 *Let $x_1 < x_2$ be two real numbers such that $P(x_i, Y)$ has $d-2$ simple real roots and a double real root y_i, and that $P(x, Y)$ has exactly 2 nonreal roots for any x in $]x_1, x_2[$.*
One says that P has a jump in $[x_1, x_2]$ if the number of real roots of $P(x_1, Y)$ below y_1 is not the same as the number of real roots of $P(x_2, Y)$ below y_2.

Proposition 3.3 *The polynomial $P(X, Y)$ has no more than $n(d-1)(d-2)/2$ jumps.*

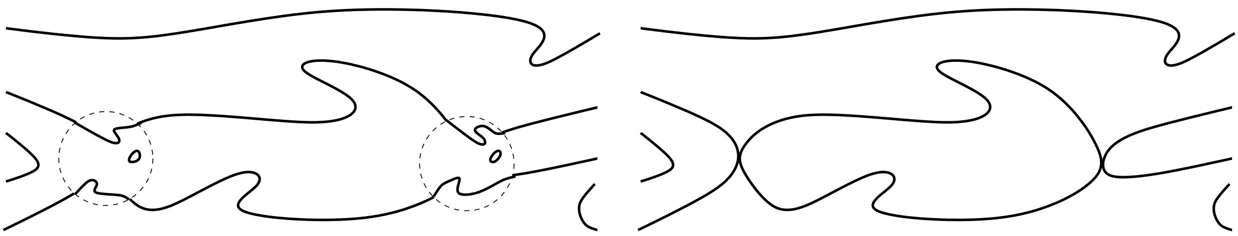
Proof. According to Proposition 1.4, one has $sr_1(P)(x_i) > 0$. Moreover, by assumption, there exists some number x in $]x_1, x_2[$ such that $P(x, Y)$ has two nonreal roots whose real part is a root of $P(x, Y)$. Hence, according to Proposition 2.2, one has $sr_1(P)(x) < 0$. Hence the polynomial $sr_1(P)(X)$ has at least 2 real roots on $]x_1, x_2[$. The result follows from the degree of $sr_1(P)(X)$. □

Remarks.

- This inequality is sharp for any integers n and d : there exists a real algebraic curve of degree d in Σ_n with exactly $n(d-1)(d-2)/2$ jumps. These curves can be easily constructed by T-construction.

- This bound is not true for real pseudoholomorphic curves. As we already know some counterexamples for $d = 3$, one can easily construct counterexamples for any degree $d \geq 3$.

- In Figure 1a) is depicted a pseudoholomorphic counterexample of degree 5 in Σ_1. The curve has 8 jump as the algebraic bound is 6. I believe that this example does not come from a counterexample of degree 3 (provided that this sentence has any meaning). This curve can be obtain as a perturbation of the real algebraic T-curve with two A_3 singularities depicted in Figure 1b).



a) b)

Figure 1:

References

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