BEHAVIOR OF WELSCHINGER INVARIANTS UNDER MORSE SIMPLIFICATIONS

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Abstract. We relate Welschinger invariants of a rational real symplectic 4-manifold before and after a Morse simplification (i.e deletion of a sphere or a handle of the real part of the surface). This relation is a consequence of a real version of Abramovich-Bertram formula which computes Gromov-Witten invariants by means of enumeration of J-holomorphic curves with a non-generic almost complex structure J. In addition, we give some qualitative consequences of our study, for example the vanishing of Welschinger invariants in some cases.

1. Introduction

On a rational symplectic 4-manifold (X, ω), genus 0 Gromov-Witten invariants can be computed by enumerating *irreducible* J-holomorphic rational curves on X, realizing a fixed homology class $d \in H_2(X, \mathbb{Z})$, and passing through a configuration of $c_1(X)d - 1$ points, where J is a generic almost complex structure on X tamed by ω ([12]). Now suppose that J is midly non-generic, i.e. X contains a unique irreducible J-holomorphic curve E with $E^2 < -1$, and moreover E is a smooth rational curve with $E^2 = -2$. In this situation, one can still compute Gromov-Witten invariants of (X, ω) by enumerating J-holomorphic curves on (X, ω), but now also taking into account *reducible* curves with some components mapped isomorphically to E. Abramovich and Bertram first proved this when (X, ω, J) is the second Hirzebruch ruled surface ([1]), Vakil extended later this proof to the case of any weak Del Pezzo surface ([14]), and eventually Ionel and Parker symplectic sum formula ([6]) provides a proof in the general case.

Results of this note are based on real versions of this Abramovich-Bertram type formula. A real structure $c : X \to X$ on a rational symplectic 4-manifold (X, ω) is an involution such that $c^*\omega = -\omega$. The set $\mathbb{R}X = Fix(c)$ is called the real locus of X. Welschinger invariants provide real analogues of Gromov-Witten invariants in genus 0 for real rational symplectic 4-manifolds ([15]).

Suppose that (X, ω, c) contains a real smooth rational symplectic curve E with $E^2 = -2$, and let $(X^\#, \omega^\#)$ be the symplectic sum of (X, ω) with $S^2 \times S^2$ along E, where E realizes the diagonal class in $H_2(S^2 \times S^2, \mathbb{Z})$. There exist two real structures c_+ and c_- on $S^2 \times S^2$ for which E is real, which give rise to two different real structures $c_+^\#$ and $c_-^\#$ on $(X^\#, \omega^\#)$ satisfying (with the convention that $\chi(\emptyset) = 0$)

$$\chi(\mathbb{R}X_+^\#) = \chi(\mathbb{R}X) = \chi(\mathbb{R}X_-^\#) - 2.$$

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One may interpret this construction as follows: blow-down the real (-2)-curve E to a nodal real 4-manifold, and smooth the node in two different ways.

The real symplectic manifold $(X^\#, \omega^\#, c_+^\#)$ is in fact a deformation of (X, ω, c) and in this case one can immediatly extract a real version of Abramovich-Bertram formula from the complex one without decomposing $(X^\#, \omega^\#, c_+^\#)$ into a symplectic sum, as it has already been noticed by several people ([3], [4], [11], [13]). This is not true for $(X^\#, \omega^\#, c_-^\#)$, and one of the main results of this note is a real version of Abramovich-Bertram formula also in this case. These two different real versions of Abramovich-Bertram formula allows one to compare Welschinger invariants of $(X^\#, \omega^\#, c_\pm^\#)$. This can be thought as a generalization of the invariant θ introduced by Welschinger in [15], and has several consequences (e.g. vanishing results) concerning Welschinger invariants.

Detailed proofs of the statements announced in this note will appear in [5].

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2. Welschinger invariants

Let (X, ω, c) be a real rational symplectic 4-manifold, and let J be an almost complex structure on X tamed by ω which is J-antiholomorphic. Recall that the *mass* $m(C)$ of a real rational J-holomorphic curve C in (X, ω, c) is the number of solitary real nodes of $\mathbb{R}C$ in $\mathbb{R}X$ (i.e. nodes locally given over $\mathbb{R}$ by the equation $x^2 + y^2 = 0$). Let us fix a homology class d in $H_2(X, \mathbb{Z})$, an integer $0 \leq r \leq c_1(X)d - 1$, a connected component S of $\mathbb{R}X$, and a real configuration $\underline{x}$ of $c_1(X)d - 1$ points in X containing exactly r points in S and $\frac{c_1(X)d-1-r}{2}$ pairs of complex conjugated points. When J is generic, Welschinger proved in [15] that the number of irreducible real rational J-holomorphic curves C, counted with multiplicity $(-1)^{m(C)}$, incident to $\underline{x}$ and realizing the class d is finite and depends only on d and r. This number is a *Welschinger invariant* of (X, ω, c), and we denote it by $W_{\mathbb{R}X,S}(d, r)$. We omit the reference to S when $S = \mathbb{R}X$, or to r when $r = c_1(X)d - 1$.

Suppose now that J is mildly non-generic as above, in particular the (-2)-curve E is real. Counting real rational J-holomorphic curves in X with multiplicity $(-1)^{m(C)}$ does not give a number depending only on d and r, since J is non-generic ([15], [7]).

Definition 2.1. *Let C be a nodal real rational J-holomorphic curve in X intersecting the (-2)-curve E transversally. We denote respectively by α and β the number of real and pairs of complex conjugated intersection points in $C \cap E$. For any integer $k \geq 0$, we define the two kth multiplicities of C as follows:*

$$\mu_k^+(C) = (-1)^{m(C)} \sum_{k=\alpha_k+2\beta_k} \binom{\alpha}{\alpha_k}\binom{\beta}{\beta_k}$$

and

$$\mu_k^-(C) = \begin{cases} (-1)^{m(C)+\beta}2^\beta & \text{if } \alpha = 0 \text{ and } k = \beta; \\ 0 & \text{otherwise.} \end{cases}$$

As above choose $d \in H_2(X,\mathbb{Z})$, an integer $0 \le r \le c_1(X)d - 1$, a connected component S of $\mathbb{R}X \setminus \mathbb{R}E$, and a generic real configuration $\underline{x}$ of $c_1(X)d-1$ points in X containing exactly r points in S and $\frac{c_1(X)d-1-r}{2}$ pairs of complex conjugated points. For each integer $k \ge 0$, we denote by $\mathcal{R}_k(d,\omega)$ the set of all irreducible rational real J-holomorphic curves in X passing through all points in $\underline{x}$ and realizing the class $d - kE$. The set $\mathcal{R}_k(d,\omega)$ is finite, and any curve in $\mathcal{R}_k(d,\omega)$ is nodal and intersects E transversally. Moreover $\mathcal{R}_k(d,\omega)$ is non-empty only for finitely many values of k. We define the two following numbers:

$$W^\pm_{\mathbb{R}X,S}(d,r) = \sum_{k\ge 0} \sum_{C\in\mathcal{R}_k(d,\omega)} \mu_k^\pm(C).$$

Let $(X^\#, \omega^\#, c^\#)$ be as above with $c^\# = c^\#_\pm$, and let $S^\#$ be the component of $\mathbb{R}X^\#$ containing the deformation of S. Note that the homology groups $H_2(X,\mathbb{Z})$ and $H_2(X^\#,\mathbb{Z})$ are canonically identified ([6]).

Theorem 2.2. *Under the above hypotheses, one has:*

(i) if $\chi(\mathbb{R}X^\#) = \chi(\mathbb{R}X)$, *then*

$$W_{\mathbb{R}X^\#,S^\#}(d,r) = W^+_{\mathbb{R}X,S}(d,r);$$

(ii) if $\chi(\mathbb{R}X^\#) = \chi(\mathbb{R}X) + 2$, *then*

$$W_{\mathbb{R}X^\#,S^\#}(d,r) = W^-_{\mathbb{R}X,S}(d,r).$$

As an immediate consequence of Theorem 2.2, the numbers $W^\pm_{\mathbb{R}X,S}(d,r)$ depend only on d and r. As mentioned in the introduction, part (i) in Theorem 2.2 is an immediate consequence of Abramovich-Bertram formula and was known before ([3], [4], [11], [13]).

3. Applications

Here we announce some consequences of Theorem 2.2, in particular when X is $\mathbb{C}P^2_6$, the complex projective plane $\mathbb{C}P^2$ blown up in 6 points.

3.1. **Computation for degree 6 curves with 6 fixed nodes.**

Let us also denote by $\widetilde{\mathbb{C}P^2_6}$ the projective plane $\mathbb{C}P^2$ blown up at 6 points lying on a smooth conic E. Here we enumerate real rational curves realizing twice the anti-canonical class $\delta = 2c_1(\mathbb{C}P^2_6)^\vee$ in $\widetilde{\mathbb{C}P^2_6}$ and $\mathbb{C}P^2_6$.

Given a real structure on $\widetilde{\mathbb{C}P^2_6}$, we denote by $\widetilde{\mathbb{R}P^2_6}$ its real part. Note that $\widetilde{\mathbb{R}P^2_6}$ is not necessarily $\mathbb{R}P^2$ blown up in 6 real points lying on a conic. Given a generic configuration $\underline{x}$ of $c_1(\mathbb{C}P^2_6)\delta - 1 = 5$ real points in $\widetilde{\mathbb{C}P^2_6}$, we set $n^\pm_{\chi(\widetilde{\mathbb{R}P^2_6})}(\delta - kE) := \sum_{C\in\mathcal{R}_k(\delta,\omega)} \mu_k^\pm(C)$.

Proposition 3.1. *For any choice of S, there exists a configuration of 5 real points in $\widetilde{\mathbb{C}P^2_6}$ such that:*

	n_{-5}^{+}	n_{-5}^{-}	n_{-3}^{+}	n_{-3}^{-}	n_{-1}^{+}	n_{-1}^{-}	n_{1}^{+}	n_{1}^{-}
δ	*522*	*522*	*236*	*236*	*78*	*78*	*0*	*0*
$\delta - E$	*236*	*0*	*140*	*0*	*76*	*0*	*36*	*0*
$\delta - 2E$	*1*	*0*	*1*	*0*	*1*	*0*	*1*	*0*

Corollary 3.2. *The surface $\mathbb{C}P_6^2$ has the following Welschinger invariants:*

$\chi(\mathbb{R}P_6^2)$	-5	-3	-1	1	3
$W_{\mathbb{R}P_6^2,S}(\delta)$	*1000*	*522*	*236*	*78*	*0*

The value $W_{\mathbb{R}P_6^2}(\delta)$ when $\chi(\mathbb{R}P_6^2) = -5$ has been first computed by the first author ([3], [4]). The numbers $W_{\mathbb{R}P_6^2}(\delta)$ when $\chi(\mathbb{R}P_6^2) = -3, -1, 1$, as well as $W_{\mathbb{R}P^2 \sqcup S^2, \mathbb{R}P^2}(\delta)$ have been first computed by Itenberg, Kharlamov and Shustin ([10]). The vanishing of $W_{\mathbb{R}P^2 \sqcup S^2, S}(\delta)$ is actually a general fact.

Proposition 3.3. *If (X, ω, c) is a real symplectic 4-manifold with disconnect real part, then for any $d \in H_2(X, \mathbb{Z})$, any $r \geq 2$, and any choice of S, one has*

$$W_{\mathbb{R}X,S}(d, r) = 0.$$

3.2. **Behavior of purely real Welschinger invariants with respect to Euler characteristic.** Given a real toric Del Pezzo surface X equipped with its tautological real toric structure and a class $d \in H_2(X, \mathbb{Z})$, one has ([8])

$$W_{\mathbb{R}X}(d) \geq W_{\mathbb{R}X}(d, c_1(X)d - 3).$$

Theorem 2.2 provides a natural generalization of this formula in the particular cases when X is $S^2 \times S^2$ or $\mathbb{C}P_6^2$.

Theorem 3.4. *Let (X_1, ω_1) and (X_2, ω_2) be two symplectic 4-manifolds deformation equivalent to either $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P_6^2$ equipped with their standard symplectic form. Choose a real structure c_1 on X_1, and a real structure c_2 on X_2. Then for any $d \in H_2(X, \mathbb{Z})$, one has*

$$W_{\mathbb{R}X_1,S_1}(d) \geq W_{\mathbb{R}X_2,S_2}(d) \quad \textit{if} \quad \chi(\mathbb{R}X_1) \leq \chi(\mathbb{R}X_2).$$

Note that Theorem 3.4 does not generalize immediately to any symplectic 4-manifold. Indeed, according to [2] one has $W_{\mathbb{R}P^2}(9, 2) < W_{\mathbb{R}P^2}(9, 0)$, i.e. Theorem 3.4 does not hold in the case of $\mathbb{C}P^2$ blown up in 26 points.

3.3. **Modified Welschinger invariants.** In the case when $\mathbb{R}X$ is not connected, one may slightly modify the definition of Welschinger invariants given in section 2. Namely, given S a connected component of $\mathbb{R}X$, the modified mass of a real rational curve C is defined as the number of solitary real nodes of C lying in S. Counting real curves with this sign produces a new invariant, denoted by $\widetilde{W}_{\mathbb{R}X,S}$.

Our method also allows us to compute these invariants in the case of $\mathbb{C}P_6^2$. In particular we have the following two propositions.

Proposition 3.5. $\widetilde{W}_{\mathbb{R}P^2 \sqcup S^2, \mathbb{R}P^2}(\delta) = 160$ *and* $\widetilde{W}_{\mathbb{R}P^2 \sqcup S^2, S^2}(\delta) = 96$.

The value of $\widetilde{W}_{\mathbb{R}P^2 \sqcup S^2, \mathbb{R}P^2}(\delta)$ has been first computed by Itenberg, Kharlamov and Shustin ([10]).

Proposition 3.6. *For any class $d \in H_2(\mathbb{C}P^2_6, \mathbb{Z})$, we have*

$$\widetilde{W}_{\mathbb{R}P^2 \sqcup S^2, \mathbb{R}P^2}(d) \geq \widetilde{W}_{\mathbb{R}P^2 \sqcup S^2, S^2}(d) \geq 0.$$

The positivity of $\widetilde{W}_{\mathbb{R}P^2 \sqcup S^2, \mathbb{R}P^2}(d)$ whenever d contains a real algebraic curve has first been established in [10].

3.4. **Relation to tropical Welschinger invariants of $\mathbb{F}_2$.** We end this note relating some tropical Welschinger invariants of $\mathbb{F}_2$ to genuine Welschinger invariants of the quadric ellipsoid Q. The only real homology classes of Q are multiple of the hyperplane section h. We say that a tropical curve in $\mathbb{R}^2$ is of class $aB + bF$ in $\mathbb{T}\mathbb{F}_2$ if its Newton polygon has vertices $(0,0)$, $(0,a)$, (a,b), and $(2a+b,0)$. We denote by $W_{\mathbb{T}\mathbb{F}_2}(dB)$ the irreducible tropical Welschinger invariant of $\mathbb{T}\mathbb{F}_2$ for curves of class dB ([9]).

Proposition 3.7. *For any positive integer d*

$$W_Q(dh) = W_{\mathbb{T}\mathbb{F}_2}(dB).$$

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