A NON-ALGEBRAIC PATCHWORK

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Abstract

Itenberg and Shustin's pseudoholomorphic curve patchworking is in principle more flexible than Viro's original algebraic one. It was natural to wonder if the former method allows one to construct non-algebraic objects. In this paper we construct the first examples of patchworked real pseudoholomorphic curves in Σ_n whose position with respect to the pencil of lines cannot be realised by any real algebraic curve of the same bidegree.

1   Introduction

Viro's patchworking has been since the seventies one of the most important and fruitful methods in topology of real algebraic varieties. It was applied in the proof of a lot of meaningful results in this field (e.g. [Vir84], [Vir89], [Shu99], [LdM],[Ite93], [Ite01], [Haa95],[Bru06], [Bih], [IV], [Ber06], [Mik05], and [Shu05]). Here we will only consider the case of curves in $\mathbb{C}P^2$ and in rational geometrically ruled surfaces Σ_n.

Viro Method allows to construct an algebraic curve A out of simpler curves A_i so that the topology of A can be deduced from the topology of the initial curves A_i. Namely one gets a curve with Newton polygon Δ out of curves whose Newton polygons are the 2-simplices of a subdivision σ of Δ, and one can see the curve A as a gluing of the curves A_i. Moreover, if all the curves A_i are real, so is the curve A. For a detailed account on the Viro Method we refer to [Vir84] and [Vir89] for example.

One of the hypotheses of the Viro Method is that σ should be convex (i.e. the 2-simplices of σ are the domains of linearity of a piecewise linear convex function). In the original Patchworking Theorem, one also requires the curves A_i to be totally nondegenerate. In particular, one can only glue nonsingular curves. Later, E. Shustin proved in [Shu98] that, under some numerical conditions depending on the types and number of the singularities, it is possible to patchwork singular curves keeping the singular points.

On the other hand, I. Itenberg and E. Shustin proved in [IS02] a pseudoholomorphic patchworking theorem: they showed that applying the Viro Patchworking with any subdivision (non necessarily convex) and with reduced curves A_i with arbitrary singularities, one can glue the A_i's, keeping singular points, to obtain a *real pseudoholomorphic curve*. More precisely, given some (maybe singular) curves A_i whose Newton polygons are the 2-simplices of a subdivision of the quadrangle with vertices $(0,0)$, $(0,k)$, (l,k) and $(l + nk,0)$, Itenberg and Shustin gave a way to construct a pseudoholomorphic curve C of bidegree (k,l) in the rational geometrically ruled surface Σ_n, whose position with respect to the pencil of lines can be deduced from the initial curves A_i. Isotopy types

realizable by (algebraic or pseudoholomorphic) curves obtained via a patchworking procedure are called *patchworked curves*.

Pseudoholomorphic curves were introduced by M. Gromov in [Gro85] to study symplectic 4-manifolds. A real pseudoholomorphic curve C on $\mathbb{C}P^2$ or Σ_n is an immersed Riemann surface which is a J-holomorphic curve in some tame almost complex structure J such that the exceptional section (in Σ_n with $n \geq 1$) is J-holomorphic, $conj(C) = C$, and $conj_* \circ J_p = -J_p \circ conj_*$ (where $conj$ is the standard complex conjugation and p is any point of C). It has been realized since then that real pseudoholomorphic curves share a lot of properties with real algebraic ones (see for example [Or99]). It is still unknown if there exist nonsingular real pseudoholomorphic curves in $\mathbb{C}P^2$ which are isotopic to no real algebraic curves of the same degree (this is the so called real symplectic isotopy problem). However, S. Orekov ([Or]) gave an example of a nonsingular real pseudoholomorphic curve in Σ_4 which is not istopic to any homologous real algebraic curve (two curves are said to be homologous if they realise the same homology class in $H_2(\Sigma_n; \mathbb{Z})$). Note that, not requiring the exceptional section in Σ_n to be J-holomorphic, J-Y. Welschinger constructed in [Wel02] examples of real pseudoholomorphic curves on Σ_n for $n \geq 2$ which are not isotopic to any real algebraic curve realizing the same homology class in $H_2(\Sigma_n; \mathbb{Z})$.

In the surfaces $\mathbb{R}\Sigma_n$, there is a natural pencil of lines $\mathcal{L}$, and one can study curves there up to fiberwise isotopy. Two curves C_1 and C_2 in $\mathbb{R}\Sigma_n$ are said to be $\mathcal{L}$-*isotopic* if there exists an isotopy $\phi(t, x)$ of $\mathbb{R}\Sigma_n$ mapping C_1 to C_2 such that for any $t \in [0; 1]$, for any $p \in C_1$ and for any fiber F of $\mathbb{R}\Sigma_n$, $\phi(t, F)$ is a fiber of $\mathbb{R}\Sigma_n$, and the intersection multiplicity of C_1 and F at p is the intersection multiplicity of $\phi(t, C_1)$ and $\phi(t, F)$ at $\phi(t, p)$.
A lot of examples are known of nonsingular real pseudoholomorphic curves in $\mathbb{R}\Sigma_n$ which are $\mathcal{L}$-isotopic to no homologous real algebraic curves (see for example [FLTO02], [OS02], [OS03], [Bru]). However, as far as we know none of those examples are constructed with the pseudoholomorphic patchworking of Itenberg and Shustin, and the question of the existence of a patchworked pseudoholomorphic curves with any kind of non-algebraic behaviour was open. The aim of this paper is to construct patchworked curves with such a non-algebraic behavior.

In [BB06] we proved that in the case of curves of bidegree $(3, 0)$ in Σ_n, the patchworked pseudoholomorphic curve is always isotopic to a real algebraic one in the same homology class.
In this paper we construct the first examples of patchworked real pseudoholomorphic curves in Σ_n whose position with respect to the pencil of lines cannot be realised by any homologous real algebraic curve.

A smooth curve C in $\mathbb{R}\Sigma_n$ is said to be $\mathcal{L}$-*nonsingular* if C intersects any fiber transversally, except for a finite number of fibers which have an ordinary tangency point with one of the branches of C, and intersect transversally the other branches of C.
A smooth curve C in $\mathbb{R}\Sigma_n$ which is $\mathcal{L}$-singular is called *smooth $\mathcal{L}$-singular*.

Theorem 1.1 *For any $d \geq 3$ there exists a smooth $\mathcal{L}$-singular real pseudoholomorphic patchworked curve of bidegree $(d, 0)$ in Σ_2 which is not $\mathcal{L}$-isotopic to any real algebraic curve in Σ_2 of the same bidegree.*

The pseudoholomorphic construction is done in Proposition 3.2, and the algebraic obstruction is proved in Proposition 3.4 where we show that no real algebraic curve of bidegree $(d, 0)$ can be positioned like the patchworked curve.

2 Rational geometrically ruled surfaces

In this section we fix our notations for the surfaces Σ_n. The nth *rational geometrically ruled surface*, denoted by Σ_n, is the surface obtained by taking four copies of $\mathbb{C}^2$ with coordinates (x, y), (x_2, y_2), (x_3, y_3) and (x_4, y_4), and by gluing them along $(\mathbb{C}^*)^2$ with the identifications $(x_2, y_2) = (1/x, y/x^n)$, $(x_3, y_3) = (x, 1/y)$ and $(x_4, y_4) = (1/x, x^n/y)$. Let us denote by E (resp. B and F) the algebraic curve in Σ_n defined by the equation $\{y_3 = 0\}$ (resp. $\{y = 0\}$ and $\{x = 0\}$). The coordinate system (x, y) is called *standard*. The projection $\pi : (x, y) \mapsto x$ on Σ_n defines a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$. The intersection numbers of B and F are respectively $B \circ B = n$, $F \circ F = 0$ and $B \circ F = 1$. The surface Σ_n has a natural real structure induced by the complex conjugation in $\mathbb{C}^2$, and the real part $\mathbb{R}\Sigma_n$ of Σ_n is a torus if n is even and a Klein bottle if n is odd. The restriction of π on $\mathbb{R}\Sigma_n$ defines a pencil of lines denoted by $\mathcal{L}$.

The group $H_2(\Sigma_n, \mathbb{Z})$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and is generated by the classes of B and F. Moreover, one has $E = B - nF$. An algebraic or pseudoholomorphic curve on Σ_n is said to be of *bidegree* (k, l) if it realizes the homology class $kB + lF$ in $H_2(\Sigma_n, \mathbb{Z})$. Its equation in $\Sigma_n \setminus E$ is

$$\sum_{i=0}^{k} a_{k-i}(X, Z) Y^i \qquad (*)$$

where $a_j(X, Z)$ is a homogeneous polynomial of degree $nj + l$.

3 Proof of the main Theorem

Our constructions use the Pseudoholomorphic Patchworking Theorem (see [IS02]). The fact that the curves constructed have a position with respect to $\mathcal{L}$ which is not realizable by an algebraic curve comes from a condition on the degree of a certain univariate polynomial.

Proposition 3.1 *Let $d \geq 3$ be a natural number and put $k = \left[\frac{d}{2}\right]$. Choose k real numbers $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k$ and define the curve C_{sing} by the equation*

$$\prod_{i=1}^{k} \left(Y^2 - \alpha_i X\right) \qquad \text{if } d \text{ is even}$$

$$Y \prod_{i=1}^{k} \left(Y^2 - \alpha_i X\right) \quad \text{if } d \text{ is odd.}$$

Then, for $\epsilon > 0$ small enough, the real algebraic curve $C = C_{sing} + \epsilon(Y^d + X - X^{d-1})$ in $\mathbb{C}P^2$ satisfies

- *C is nonsingular,*

- *at the point $[0 : 0 : 1]$ (resp. $[1 : 0 : 0]$), the curve C is locally given by the equation $Y^d + X = 0$ (resp. $Y^d - X^{d-1} = 0$),*

- *there exists a real number $a > 0$ such that the line with equation $X - aZ = 0$ intersects the curve C in d distinct real points.*

Proof. Straightforward. $\qquad\qquad\square$

Let us denote by $\widetilde{C}$ the curve defined by the equation $X^d C(X, \frac{Y}{X})$. Then, the Newton polygon of $\widetilde{C}$ is the triangle with vertices $(0, d)$, $(d + 1, 0)$ and $(2d - 1, 0)$ and the curve $\widetilde{C}$ is locally given by the equation $Y^d + X^{d+1} = 0$ (resp. $Y^d - X^{2d-1} = 0$) at the point $[0 : 0 : 1]$ (resp. $[1 : 0 : 0]$). Moreover,

there exists a real number $b > 0$ such that the line of equation $X - bZ = 0$ intersects the curve $\widetilde{C}$ in d distinct real points.

Define the polynomials $P_1 = Y^d + X + 1$, $P_2 = Y^d - X^{d-1} + X^d$, $P_3 = Y^d + X^d + X^{d+1}$ and $P_4 = Y^d - X^{2d-1} + X^{2d}$. The curves P_1 and P_3 (resp. P_2 and P_4) have a maximal tangency point with the line $X + Z = 0$ (resp. $X - Z = 0$).

One can now patchwork the polynomials C, $\widetilde{C}$, P_1, P_2, P_3, and P_4 (see Figure 1). According to
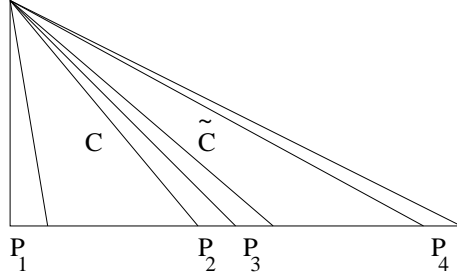

Figure 1:

Itenberg and Shustin's theorem (see [IS02]), one can glue pseudoholomorphically all the pieces of the patchwork keeping the tangency conditions with respect to $\mathcal{L}$. This proves the following proposition.

Proposition 3.2 *There exists a patchworked nonsingular real pseudoholomorphic curve Γ of bidegree $(d, 0)$ in Σ_2 and real numbers $x_1 < x_2 < 0 < z_1 < x_4 < z_2 < x_6$ such that*

- *Γ has a maximal tangency point with the fibers $X - x_i Z = 0$ with $i = 1, \ldots 4$,*

- *Γ has d distinct real intersection points with the fibers $X - z_i Z = 0$ with $i = 1, 2$.* □

Lemma 3.3 *Let $P(X, Y)$ be a real polynomial of the form*

$$Y^d + a_2(X)Y^{d-2} + a_3(X)Y^{d-3} + \ldots + a_d(X).$$

and x be a real number. If the univariate polynomial $Q_x(Y) = P(x, Y)$ has only real roots then $a_2(x) \leq 0$ and $a_2(x) = 0$ if and only if 0 is a root of order d.

Proof. Let y_i be the d roots of Q_x. The sum $\sum_{i=1}^{d} y_i$ is 0 and the second coefficient satisfies $a_2 = \sum_{i<j} y_i y_j$. Thus one has $\sum_{i<j} y_i y_j = -1/2 \sum y_i^2$ which proves the lemma. □

Using Lemma 3.3, we easily prove that there are no algebraic curves of the same bidegrees as curves in Proposition 3.2 having the same positions with respect to the line pencil $\mathcal{L}$.

Proposition 3.4 *There does not exist a nonsingular real algebraic curve Γ of bidegree $(d, 0)$ in Σ_2 such that there exists real numbers $x_1 < x_2 < 0 < z_1 < x_4 < z_2 < x_6$ satisfying*

- *Γ has a maximal tangency point with the fibers $X - x_i Z = 0$ with $i = 1, \ldots 4$,*

- *Γ has d distinct real intersection points with the fibers $X - z_i Z = 0$ with $i = 1, 2$.* □

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Proof. Suppose such a real algebraic curve Γ exists. Then, in an appropriate standard system of coordinates on Σ_2, the curve Γ has the following equation

$$Y^d + a_2(X)Y^{d-2} + a_3(X)Y^{d-3} + \ldots + a_d(X)$$

where $a_i(X)$ is real a polynomial of degree $2i$. According to Lemma 3.3, one has $a_2(z_i) < 0$, so the polynomial a_2 is not identically zero. According to Lemma 3.3, one has $a_2(x_i) = 0$, so the x_i's are exactly the simple roots of the polynomial a_2. However, as $z_1 < x_4 < z_2$ and $a_2(z_1)$ and $a_2(z_2)$ have the same sign, there should exist an extra root x_5 of a_2 in the interval $]z_1; z_2[$. Hence, a_2 would be a non-null polynomial of degree 4 with at least 5 roots, which is impossible. $\square$

4   Concluding remarks

1. The patchwork in Section 3 can be realized algebraically as soon as one keeps only any three out of the four maximal tangency points. Patchworking a curve with Newton polygon the triangle with vertices $(1,0), (0,d)$ and $(2d-1,0)$, one can keep two of them. Applying Shustin's theorems (see [Shu05], [Shu98]), one can keep the maximal tangency point coming either from the curve P_2 or from the curve P_3.

2. It is fairly easy to generalise the main theorem to other rationally ruled surfaces and to construct a lot of other examples. For instance in the same way one proves that there exists pseudoholomorphic patchworked curves of bidegree $(d,0)$ in Σ_n which are not $\mathcal{L}$-isotopic to any such real algebraic curve as soon as $n \geq 2$ and $d \geq 3$. One can also construct examples in Σ_1 of bidegree $(d,0)$ for $d \geq 5$.

3. This paper is part of a work in progress in which we investigate tangencies of curves with respect to a pencil of lines. For real algebraic curves, one can obtain restrictions valid in any degree by means of certain subresultants. We point out that most of these restrictions do not hold for real pseudoholomorphic curves. Our proof here relies on the simplest example of algebraic prohibitions obtained in this way.

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