SURFACES WITH MANY SOLITARY POINTS

ERWAN BRUGALLÉ AND OLIVER LABS

ABSTRACT. It is classically known that a real cubic surface in $\mathbb{R}P^3$ cannot have more than one solitary point (or $A_1^\bullet$-singularity, locally given by $x^2+y^2+z^2=0$) whereas it can have up to four nodes (or A_1^--singularities, locally given by $x^2+y^2-z^2=0$). We show that on any surface of degree $d \geq 3$ in $\mathbb{R}P^3$ the maximum possible number of solitary points is strictly smaller than the maximum possible number of nodes.

Conversely, we adapt a construction of Chmutov to obtain surfaces with many solitary points by using a refined version of Brusotti's Theorem. Combining lower and upper bounds, we deduce: $\frac{1}{4}d^3 + o(d^3) \leq \mu^3(A_1^\bullet, d) \leq \frac{5}{12}d^3 + o(d^3)$, where $\mu^3(A_1^\bullet, d)$ denotes the maximum possible number of solitary points on a real surface of degree d in $\mathbb{R}P^3$. Finally, we adapt this construction to get real algebraic surfaces in $\mathbb{R}P^3$ with many singular points of type $A_{2k-1}^\bullet$ for all $k \geq 1$.

INTRODUCTION

An *ordinary double point*, or A_1-singularity, of a hypersurface $f = 0$ in $\mathbb{R}P^n$ or $\mathbb{C}P^n$ is a non-degenerate singular point p of f; i.e. f and all its partial derivatives vanish at p, but the hessian matrix $H_f(p) = (\partial^2 f \,/\, \partial x_i \partial x_j(p))_{i,j=0\ldots n}$ is of rank n. In $\mathbb{R}P^3$, there are exactly two real types of ordinary double points: we call the ones which can be given locally by the affine equation $x^2+y^2-z^2=0$ *nodes* or A_1^-*-singularities*, and the others, locally given by $x^2+y^2+z^2=0$, *solitary ordinary double points*, $A_1^\bullet$*-singularities*, or *solitary points* for short.

[BLvS05] showed by construction that for large degree d the currently known maximum number of complex singularities on a surface of degree d in $\mathbb{C}P^3$ [Chm92] can also be achieved with a real surface with only real singularities. All real singularities appearing in their construction are nodes. In the present paper, we consider solitary points instead.

We denote the maximum possible number of complex A_1-singularities on a complex hypersurface of degree d in $\mathbb{C}P^3$ by $\mu^3(A_1, d)$, and similarly for the real A_1^-- and $A_1^\bullet$-singularities on real surfaces in $\mathbb{R}P^3$: $\mu^3(A_1^-, d)$, $\mu^3(A_1^\bullet, d)$.

Question 1. *It is clear that the maximum possible number of complex ordinary double points is at least as large as the corresponding real numbers:*

$$\mu^3(A_1^-, d),\ \mu^3(A_1^\bullet, d)\ \leq\ \mu^3(A_1, d).$$

Is any of these inequalities strict?

Question 2. *Classical results on cubic surfaces* [Sch63] *and quartic surfaces* [Roh13] *in* $\mathbb{R}P^3$ *show that we have:*

$$\mu^3(A_1^\bullet, 3) = 1 < 4 = \mu^3(A_1^-, 3) \quad \textit{and} \quad \mu^3(A_1^\bullet, 4) = 10 < 16 = \mu^3(A_1^-, 4).$$

degree d	1	2	3	4	5	6	7	8	large d
$\mu^3(A_1^\bullet, d) \geq$	0	1	1	10	12	29	45	81	$\approx \frac{1}{4}d^3$
$\mu^3(A_1^\bullet, d) \leq$	0	1	1	10	24	48	83	134	$\approx \frac{5}{12}d^3$
$\mu^3(A_1^-, d) \geq$	0	1	4	16	31	65	99	168	$\approx \frac{5}{12}d^3$
$\mu^3(A_1^-, d) \leq$	0	1	4	16	31	65	104	174	$\approx \frac{4}{9}d^3$

TABLE 1. An overview of the known bounds for the maximum possible number of both variants of the real ordinary double points on surfaces of degree d in $\mathbb{R}P^3$: solitary points and nodes.

These results suggest that it might be more difficult to have many solitary points on surfaces than to have many nodes. Is this true for all $d \geq 3$?

In this article, we answer those questions involving solitary points affirmatively in Theorem 13:

$$\text{If } d \geq 3 \text{ then } \mu^3(A_1^\bullet, d) \ < \ \mu^3(A_1^-, d),\ \mu^3(A_1, d).$$

However the case of real nodes remains open in general although it is clear that $\mu^3(A_1^-, d) \leq \mu^3(A_1, d)$ for all d. In fact, $\mu^3(A_1^-, d) = \mu^3(A_1, d)$ is only known for $d = 1, 2, \ldots, 6$.

The currently known lower bound for $\mu^3(A_1^\bullet, d)$ is still far from the best known upper bound. In the third section of this article, we improve the previously known maximum number of solitary points on a surface of degree d in $\mathbb{R}P^3$ by adapting a construction of Chmutov and by using Brusotti's Theorem. Altogether, we show for $d \in \mathbb{N}$ by combining lower bound (Theorem 14) and upper bound (Corollary 12):

$$\frac{1}{4}d^3 + o(d^3) \leq \mu^3(A_1^\bullet, d) \leq \frac{5}{12}d^3 + o(d^3).$$

Together with the known cases in low degree, we get table 1 which provides an overview of the known bounds for the maximum possible number of both variants of the real ordinary double points. In that table, the upper bounds for the case of A_1^--singularities are simply the complex ones most of which are due to Miyaoka [Miy84], the asymptotic lower bound was found in [BLvS05].

More generally, an A_j-singularity of a complex surface in $\mathbb{C}P^3$ is a singular point locally given by the equation $x^{j+1} + y^2 + z^2 = 0$. If $k \geq 2$ then there are three (two if $k = 1$) real types of A_{2k-1}-singularities, and we call the one given locally by the real equation $x^{2k} + y^2 + z^2 = 0$ an *$A_{2k-1}^\bullet$-point* . In section 4, we explain how to adapt our method to construct real surfaces in $\mathbb{R}P^3$ with many $A_{2k-1}^\bullet$-points. More precisely, we prove that (see Proposition 20) for $k, d \geq 1$:

$$\frac{1}{8k-4}d^3 + o(d^3) \leq \mu^3(A_{2k-1}^\bullet, d) \leq \frac{4k}{12k^2-3}d^3 + o(d^3).$$

The upper bound is again Miyaoka's bound on the number of complex A_{2k-1}-points of a complex surface of degree d in $\mathbb{C}P^3$.

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1. Plane Curves with Solitary Points

In our results on real surfaces with solitary points we will use some facts about real plane curves with solitary points. So, we give a brief overview about this classical subject. As in the case of A_1-singularities of surfaces mentioned in the introduction, there are exactly two real types of ordinary double points on a real plane curve, also denoted by $A_1^\bullet$ resp. A_1^-.

1.1. **Nodes.**

The value $\mu^2(A_1^-, d)$ of the maximum possible number of nodes on a real plane curve of degree d has been known for a long time:

$$\mu^2(A_1^-, d) = \frac{d(d-1)}{2}.$$

The upper bound is a consequence of the genus formula, and a generic configuration of d lines shows that this upper bound is sharp. The genus formula also shows that this bound can only be achieved with arrangements of d real lines no three of which meet in a point.

There is a classical theorem, the Brusotti Theorem, which shows that we can smooth each of the ordinary double points of a plane curve independently. Applied to the d generic lines in the plane mentioned above, we may deduce that for any integer r between 0 and $\frac{d(d-1)}{2}$, there is a real plane curve of degree d in $\mathbb{R}P^2$ with exactly r nodes as its only singularities.

Let us denote by $\mathcal{C}(d)$ (resp. $\mathbb{R}\mathcal{C}(d)$) the space of complex (resp. real) algebraic curves of degree d in $\mathbb{C}P^2$ (resp. $\mathbb{R}P^2$). These are projective spaces of dimension $\frac{d(d+3)}{2}$. Brusotti's result is the following:

Theorem 1 (Brusotti Theorem, usual formulation)**.** *Let C be a real algebraic curve of degree d in $\mathbb{R}P^2$ with ordinary double points as its only singularities. For any of these singularities, choose a local deformation. Then it is possible to vary the curve C in the space $\mathbb{R}\mathcal{C}(d)$ in such a way that all previously chosen deformations are realized.*

This is the form of the theorem which is usually given because it can be applied very easily. It is a straightforward corollary of the following result which will be more convenient for our purposes.

Theorem 2 (Brusotti Theorem, for a proof see e.g. [BR90])**.** *Let C be a complex algebraic curve of degree d in $\mathbb{C}P^2$ with ordinary double points $p_1, \dots, p_k$ as its only singularities. Then there exists a small neighborhood V_i of p_i in $\mathbb{C}P^2$ for each i, and a small neighborhood V of C in $\mathcal{C}(d)$ such that the analytic sets*

$$S_i = \{\widetilde{C} \in V \mid \widetilde{C} \text{ is non-singular except at some point in } V_i \text{ where it has an } A_1\}$$

are all non-singular and intersect transversely. Moreover, the tangent space of S_i at C is $\{\widetilde{C} \in \mathcal{C}(d) \mid p_i \in \widetilde{C}\}$.

1.2. **Solitary Points.** For solitary points, things are a bit more complicated, and the exact maximum number of solitary points is only known since the 80's.

Proposition 3. *Let $d \in \mathbb{N}$. Then:*

$$\mu^2(A_1^\bullet, d) \leq \frac{(d-1)(d-2)}{2} + 1.$$

Proof. According to Harnack's Theorem, a non-singular real algebraic curve of degree d in $\mathbb{R}P^2$ has at most $\frac{(d-1)(d-2)}{2} + 1$ connected components. Now the result follows from the Brusotti Theorem. □

In most cases, this upper bound can be refined using the Petrovskii inequality (see [Pet33] or [Vir84]):

Proposition 4. *If $2, 4 \neq d \in \mathbb{N}$ then:*

$$\mu^2(A_1^\bullet, d) \leq \frac{(d-1)(d-2)}{2}.$$

Proof. This bound is trivial for curves of odd degree, as one component of the curve is not contractible in $\mathbb{R}P^2$. The Petrovskii inequality for plane curves implies that if a curve of degree $d = 2k$ has $\frac{(d-1)(d-2)}{2} + 1$ ovals, then at least one of them contains another oval if $k \geq 3$. □

The union of two complex conjugated lines is a real conic with one $A_1^\bullet$-point. If $P_1(x, y) = 0$ and $P_2(x, y) = 0$ are real equations of two real conics intersecting in four real points, then the real quartic with equation $P_1^2(x, y) + P_2^2(x, y) = 0$ has four $A_1^\bullet$-points. Hence, one has $\mu^2(A_1^\bullet, 2) = 1$ and $\mu^2(A_1^\bullet, 4) = 4$ (i.e. Proposition 3 is sharp in degree 2 and 4). The proof that the upper bound given in Proposition 4 is sharp for any other degree has first been given by Viro in the 80's. As Viro's original proof was not available to us, we sketch Kenyon and Okounkov's [KO06] here:

Theorem 5 (Viro, see [Vir83], [KO06], or see [Shu93] for a proof of a more general case)**.** *If $d \neq 2, 4$ then:*

$$\mu^2(A_1^\bullet, d) = \frac{(d-1)(d-2)}{2}.$$

Proof. Let ε be a primitive d^{th} root of unity, and define the polynomial $\widetilde{P}_d(x, y) = \prod_{i,j=1}^{d}(\varepsilon^i x + \varepsilon^j y + 1)$. Then, $\widetilde{P}_d(x, y) = P_d(x^d, y^d)$ where $P_d(x, y)$ is a real polynomial of degree d, and the curve with equation $P_d(x, y) = 0$ has exactly $\frac{(d-1)(d-2)}{2}$ real solitary non-degenerate double points. □

As in the case of nodes, it follows from Brusotti's Theorem that for any integer r between 0 and $\frac{(d-1)(d-2)}{2}$, there exists a real algebraic plane curve of degree d in $\mathbb{R}P^2$ with exactly r solitary nodes as its only singularities.

For what follows, we need to introduce a distinction among solitary points of a real algebraic curve $P(x, y) = 0$ of even degree in $\mathbb{C}^2$: those who are local minima of the function $(x, y) \mapsto P(x, y)$ and those who are local maxima.

Proposition 6 ([KO06],[Mik00])**.** *Let $d \geq 2$ be even and let $P_d(x, y) = 0$ be the polynomial constructed in the proof of Proposition 5. Then $\frac{3d(d-2)}{8}$ solitary points of the curve $P(x, y) = 0$ are local maxima of the polynomial $P(x, y)$ and the $\frac{(d-2)(d-4)}{8}$*

other solitary points of the curve $P(x, y) = 0$ *are local minima of the polynomial* $P(x, y)$*.*

Proof. Kenyon's and Okounkov's polynomial $P_d(x, y)$ defines a Harnack curve, and Mikhalkin showed that these curves have the desired property. □

2. Surfaces in $\mathbb{R}P^3$: Upper Bounds

In order to prove the upper bounds mentioned in the introduction, we need — in analogy to the Brusotti Theorem in the case of plane curves — a result about smoothings of algebraic varieties.

By a *smoothing* of a singular (real) algebraic hypersurface X of degree d in $\mathbb{C}P^n$, we mean a small perturbation of the coefficients of X such that the result is a non-singular (real) algebraic hypersurface of degree d in $\mathbb{C}P^n$. The Coste-Hironaka Theorem now says that one can always smooth a real projective hypersurface in such a way that no connected component disappears into the complex world:

Theorem 7 (Coste-Hironaka, [Cos92])**.** *Let* X *be a singular real algebraic hypersurface in* $\mathbb{R}P^n$*. Then there is a smoothing* $\widetilde{X}$ *of* X *such that*

$$b_0(X) \leq b_0(\widetilde{X}),$$

where b_0 *denotes the* 0^{th} *Betti number, i.e. the number of connected components.*

Remark 8. *In the special case of hypersurfaces with only solitary (ordinary double!) points as singularities, it is easy to prove this result using the construction given in the paper* [Cos92]*: indeed, let* $P(X_0, \ldots, X_n) = 0$ *be the equation of such a hypersurface in* $\mathbb{R}P^n$ *which does not have a singularity in the point* $(1 : 0 : \cdots : 0)$ *(which we may assume after a suitable change of coordinates). Then define*

$$\widetilde{P}(X_0, \ldots, X_n) := P(X_0, \ldots, X_n) + \sum_{i=1}^{n} \varepsilon_i X_i \frac{\partial P}{\partial X_i}(X_0, \ldots, X_n).$$

Each singular point p *of* $P = 0$ *will still be a point on* $\widetilde{P} = 0$*. Moreover, a short computation shows that there are* ε_i *small enough such that* $\widetilde{P} = 0$ *is non-singular in* p *because of the hessian criterion for* A_1*-singularities. But this means that if the* ε_i *are small enough then near each solitary point* p*, the hypersurface* $P = 0$ *is smoothed into a small connected component of* $\widetilde{P} = 0$ *homeomorphic to an* n*-sphere and containing* p*.*

As we know the homology of projective non-singular complex algebraic hypersurfaces, the Coste-Hironaka Theorem 7 combined with Smith Theory (see [Bre72]) implies the following corollary:

Corollary 9 (Coste-Hironaka, [Cos92])**.** *Let* X *be a (possibly singular) real algebraic hypersurface of degree* d *in* $\mathbb{R}P^n$*. Then*

$$b_0(X) \leq \frac{1}{2}\left(\frac{(d+1)^{n+1} - (-1)^{n+1}}{d} + n - (-1)^n\right).$$

In the case of projective surfaces in $\mathbb{R}P^3$ one can improve the upper bound on the number of connected components thanks to the Petrovskii-Oleinik inequality (see, e.g., [DK00]):

Corollary 10. *Let S be a (possibly singular) real algebraic surface of degree d in $\mathbb{R}P^3$. Then*

$$b_0(S) \leq \frac{5d^3 - 18d^2 + 25d}{12}.$$

Remark 11. *Note that starting from degree 5, the maximal possible value of $b_0(S)$ when S is a real algebraic surface of degree d is still unknown.*

Applying Corollary 10 in the special case of real surfaces with solitary double points in $\mathbb{R}P^3$, we get:

Corollary 12. *For $d \in \mathbb{N}$, we have:*

$$\mu^3(A_1^{\bullet}, d) \leq \begin{cases} \left\lfloor \frac{5d^3-18d^2+25d}{12} \right\rfloor, & d \text{ even}, \\ \left\lfloor \frac{5d^3-18d^2+25d}{12} \right\rfloor - 1, & d \text{ odd}. \end{cases}$$

Proof. In odd degree, we can subtract one because in that case at least one of the connected components from Corollary 10 is not homeomorphic to a sphere. □

Comparing this upper bound with the lower bound obtained in the case of nodes (see Theorem 2 in [BLvS05] for a detailed formula) which is given by

$$\mu^3(A_1^-, d) \geq \begin{matrix} \frac{5}{12}d^3 - \frac{13}{12}d^2 + o(d^2), & d \text{ even}, \\ \frac{5}{12}d^3 - \frac{14}{12}d^2 + o(d^2), & d \text{ odd}, \end{matrix}$$

we may deduce that one cannot reach the maximum number of nodes with surfaces having only solitary points:

Theorem 13. *The maximum possible number of solitary points on a surface of degree $d, d \geq 3$, in $\mathbb{R}P^3$ is strictly smaller than the maximum possible number of nodes:*

$$\mu^3(A_1^{\bullet}, d) \ < \ \mu^3(A_1^-, d), \ \mu^3(A_1, d).$$

We already mentioned in the introduction that this result is not very surprising because it has been known for degree 3 and 4 for a long time. However, notice that the corresponding statement in the case of plane curves does not hold: the maximum number of nodes on an irreducible curve of degree d equals the maximum number of solitary points on an irreducible curve of degree d: in both cases, it is the genus of a smooth plane curve of degree d, as mentioned earlier.

3. Surfaces in $\mathbb{R}P^3$: Lower Bounds by Constructions

In the preceding section we have shown that the maximum possible number of solitary points on a surface in $\mathbb{R}P^3$ is less than the corresponding number of nodes. Here we improve the currently known maximum number of solitary points. Indeed, in this section we show:

Theorem 14. *Let $d \in \mathbb{N}$. Then:*

$$\mu^3(A_1^{\bullet}, d) \geq \frac{(d-2)(2d^2-3d+4)}{8} \qquad \text{if } d \text{ is even},$$
$$\mu^3(A_1^{\bullet}, d) \geq \frac{(d-1)^2(d-2)}{4} \qquad \text{if } d \text{ is odd}.$$

We prove Theorem 14 in section 3.3. It is based on Chmutov's method to construct singular complex surfaces. We thus explain this method first. Then we discuss how to adapt it to obtain real algebraic surfaces with solitary nodes; finally, we show in section 3.4 that our result is asymptotically the best that one can achieve using Chmutov's method.

3.1. **Known Constructions.** Notice that the previously best known lower bound for the maximum possible number $\mu^3(A_1^\bullet, d)$ of solitary points on a surface in $\mathbb{R}P^3$ is far below $\frac{1}{4}d^3$.

Not many constructions are known. Certainly the most sophisticated one is Shustin's variant of Viro's patchworking method for the singular case. This method yields the optimal result in the case of plane curves, but already for complex surfaces in $\mathbb{C}P^3$ it only yields $\mu^3(A_1, d) \geq \frac{1}{6}d^3 + o(d^3)$ (see [SW04]).

There is another construction which is natural to consider: we take a polynomial $P_d(x, y)$ of degree d in two variables and we set

$$f(x, y, z) = P_d(x, y) + g(z),$$

where $g(z)$ is a polynomial function of degree d in one variable z with the maximum possible number $\lceil \frac{d}{2} \rceil$ of local maxima z_i with value $g(z_i) = 0$. An *even solitary point* of an affine plane curve given by the equation $P(x, y) = 0$ is a solitary point (x, y) of the curve $P(x, y) = 0$ which is a local minimum of the polynomial $P(x, y)$, i.e. locally at p the graph of $P(x, y)$ looks like $z = x^2 + y^2$. We denote by $\mathrm{es}(P_d)$ the number of even solitary points of the curve $P_d(x, y) = 0$. With these preliminaries, it is clear that the surface $f(x, y, z) = 0$ has $\lceil d/2 \rceil \cdot \mathrm{es}(P_d)$ solitary points: for each even solitary point (a, b) of the affine plane curve $P_d(x, y) = 0$, we thus get a point (a, b, z_i) of $f(x, y, z) = 0$ which is locally of the form $x^2 + y^2 + z^2$. However, it is well known that for a curve of degree d, one has (see [Vir84])

$$\mathrm{es}(P_d) \leq \frac{7}{16}d^2 + o(d^2),$$

so one cannot expect to construct in this way surfaces of degree d with more than $\frac{7}{32}d^3 + o(d^3)$ solitary points. Combining this method with Chmutov's method we improve the leading coefficient $\frac{7}{32}$.

3.2. **Chmutov's method.** We describe briefly how Chmutov constructed surfaces with many (complex) ordinary double points in the 90's [Chm92]. It is similar to the idea mentioned in the previous paragraph. Despite its simplicity, Chmutov's surfaces still yield the best known lower bound for the maximum number of ordinary double points on a complex surface of degree $d \geq 13$. The best known lower bound in the case of real nodes (A_1^--singularities) which we mentioned above and which equals the current lower bound in the complex case is an adaption of Chmutov's construction to real nodes [BLvS05]. So, it is quite natural to try to adapt the method to solitary points. However, we will see that this process is not completely straightforward, and we will need a refined version of Brusotti's Theorem to make it work.

3.2.1. *Chmutov's Constructions.* Let $T_d(z) \in \mathbb{R}[z]$ be the Tchebychev polynomial of degree d with $\lceil \frac{d-1}{2} \rceil$ extremal points with value -1 and $\lfloor \frac{d-1}{2} \rfloor$ with value $+1$. This can either be defined recursively by $T_0(z) := 1,\ T_1(z) := z,\ T_d(z) := 2{\cdot}z{\cdot}T_{d-1}(z) - T_{d-2}(z)$ for $d \geq 2$, or implicitly by $T_d(\cos(z)) = \cos(dz)$. In [Chm92], Chmutov used the Tchebychev polynomials to construct surfaces in $\mathbb{C}P^3$ with $\approx \frac{5}{12}d^3$ (complex) nodes using the so-called *folding polynomials* $F_d^{A_2}(x,y) \in \mathbb{R}[x,y]$ associated to the root-system A_2:

$$\mathrm{Chm}_d^{A_2}(x,y,z) := F_d^{A_2}(x,y) + \frac{1}{2}(T_d(z)+1).$$

The polynomials $F_d^{A_2}(x,y)$ have critical points with only three different critical values: 0, -1, and 8. The surface $\mathrm{Chm}_d^{A_2}(x,y,z)=0$ is singular exactly at those points at which the critical values of $F_d^{A_2}(x,y)$ and $\frac{1}{2}(T_d(z)+1)$ sum up to zero (i.e., either both are 0 or the first one is -1 and the second one is $+1$).

3.2.2. *Adaption to Real Nodes.* In [BLvS05], this construction was modified to yield real surfaces $\mathrm{Chm}_{\mathbb{R},d}^{A_2}(x,y,z)=0$ with real nodes as singularities by using the so-called *real folding polynomials*

$$F_{\mathbb{R},d}^{A_2}(x,y) := F_d^{A_2}(x+iy,\ x-iy),$$

where i is the imaginary number. It is not difficult to see that the singularities of the surface $\mathrm{Chm}_{\mathbb{R},d}^{A_2}(x,y,z)=0$ are indeed nodes, i.e. of type A_1^-, by using the fact that the plane curve $F_{\mathbb{R},d}^{A_2}(x,y)=0$ is actually a product of d real lines no three of which meet in a common point.

3.2.3. *Adaption to Solitary Points.* From the explanations in the previous paragraphs it is clear how to adapt Chmutov's construction to yield solitary points: we need to show the existence of a real polynomial $f(x,y)$ with many local minima with value $+1$ and local maxima with value -1. More precisely:

Proposition 15. *Let $f(x,y)$ be a real polynomial of even (resp. odd) degree d with α local minima with value $+1$, and β local maxima with value -1. Then the affine surface defined by $f(x,y)-T_d(z)=0$ has $\frac{1}{2}(\alpha\cdot d+\beta\cdot(d-2))$ (resp. $\frac{1}{2}(\alpha+\beta)\cdot(d-1)$) solitary points. The corresponding projective surface in $\mathbb{R}P^3$ has at most $O(d^2)$ additional singularities.*

Notice that we cannot use a product of real lines such as $F_{\mathbb{R},d}^{A_2}(x,y)$ as the polynomial $f(x,y)$ in order to obtain many solitary points because it has the wrong critical values: the minima have critical value -1 and the maxima $+1$.

3.3. **Proof of the Lower Bound of Theorem 14.** We are now ready to prove Theorem 14 on the lower bound for $\mu^3(A_1^\bullet, d)$. According to Chmutov's construction and in particular Proposition 15, we have to construct polynomials in 2 variables whose graphs have many minima with value $+1$ and many maxima with value -1. The existence of such polynomials is established by Proposition 16 below applied to polynomials constructed in Propositions 5 and 6. This completes the proof of Theorem 14. □

Proposition 16. *Let $d \in \mathbb{N}$ and $P(x,y)$ be a real polynomial of degree d with α (resp. β) local minima (resp. local maxima) with value 0. Then, there exists a real polynomial in two variables of degree d with α local minima with value $+1$ and β local maxima with value -1.*

Proof. We start with the following observation: if $f(x,y)$ is a polynomial of degree d, then the graph of f, defined by the equation $f(x,y)-z=0$, is a special line in the space $\mathcal{C}(d)$ of plane curves of degree d. Indeed, if $f(x,y)=\sum a_{i,j}x^iy^j$, then the section of the graph of f by the hyperplane $z=t$ is given by the equation

$\sum a_{i,j}x^iy^j - t = 0$. If $(a_{0,0} : a_{1,0} : a_{0,1} : \cdots : a_{0,d})$ are the coordinates in the space $\mathcal{C}(d)$, then the graph of f can be parameterized by the line $t \mapsto (a_{0,0} - t : a_{1,0} : a_{0,1} : \cdots : a_{0,d})$. For $t = \infty$, this line passes through the point $(1 : 0 : 0 : \cdots : 0)$ which represents the multiple line z^d. Conversely, any line in the space of plane curves of degree d passing through the point $(1 : 0 : 0 : \cdots : 0)$ admits a parameterization of the form $t \mapsto (a_{0,0} - t : a_{1,0} : a_{0,1} : \cdots : a_{0,d})$ which defines a polynomial $f(x, y) = \sum a_{i,j}x^iy^j$ of degree d.

Let us go back to the polynomial $P(x, y)$ of the Theorem. By assumption, the curve defined by P has $\alpha + \beta$ solitary points. Now we show that we can perturb the polynomial $P(x, y)$ in such a way that all local minima (resp. maxima) stay on the same level a (resp. b) with $a > b$ (see Figure 1).

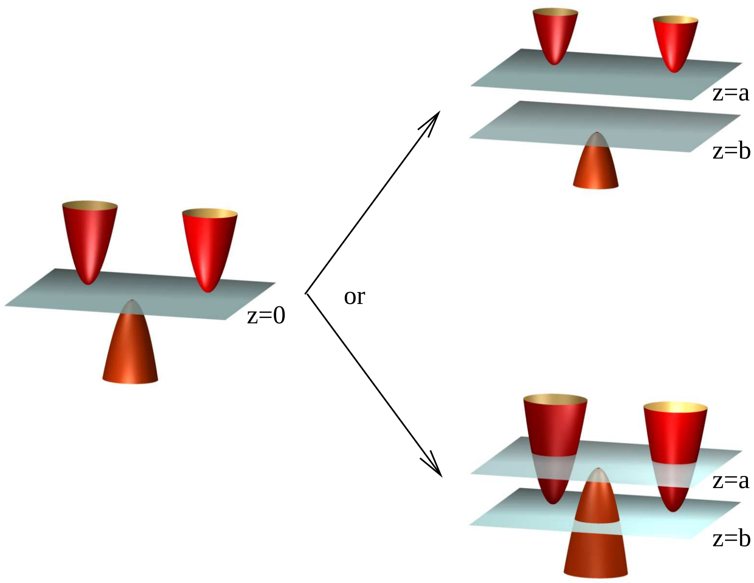

FIGURE 1. Two ways to perturb $P(x, y)$.

For any solitary point p of the curve $P(x, y) = 0$, we choose a small neighborhood $V(p)$ of p in $\mathbb{R}P^2$ such that $V(p) \cap V(q) = \emptyset$ if $q \neq p$ is another solitary point. We denote by $M(P)$ (resp. $m(P)$) the set of solitary points of the curve $P(x, y) = 0$ corresponding to local maxima (resp. minima) of $P(x, y)$. Moreover, we denote by $\Sigma_{M(P)}$ (resp. $\Sigma_{m(P)}$) the stratum of real algebraic plane curves in $\mathcal{C}(d)$ in a small neighborhood of $P(x, y) = 0$ with one solitary point in $V(p)$ for any $p \in M(P)$ (resp. $m(P)$). Then, according to Brusotti's Theorem, $\Sigma_{M(P)}$ and $\Sigma_{m(P)}$ are smooth and intersect transversely at the curve $P(x, y) = 0$. Moreover, we have:

$$\operatorname{codim}(\Sigma_{M(P)}) = \beta, \quad \operatorname{codim}(\Sigma_{m(P)}) = \alpha$$

and

$$\operatorname{codim}(\Sigma_{M(P)} \cap \Sigma_{m(P)}) = \beta + \alpha \leq \frac{(d-1)(d-2)}{2} = \frac{d(d+3)}{2} - (3d - 1).$$

One can suppose that $\alpha > 0$ and $\beta > 0$ otherwise the proposition is trivial. We denote by L the line in the space $\mathcal{C}(d)$ passing through the curve $P(x,y) = 0$ and z^d. By a simple dimension computation, we prove that we can perturb L to a line $\widetilde{L}$ still passing through $z^d = 0$ and intersecting the stratum $\Sigma_{M(P)}$ and $\Sigma_{m(P)}$ one after the other:

Define the projection

$$\begin{array}{cccc} \pi : & \mathbb{R}P^{\frac{d(d+3)}{2}} & \longrightarrow & \mathbb{R}P^{\frac{d(d+3)}{2}-1} \\ & (a_{0,0} : a_{1,0} : a_{0,1} : \cdots) & \longmapsto & (a_{1,0} : a_{0,1} : \cdots). \end{array}$$

None of the tangent spaces of $\Sigma_{M(P)}$ and $\Sigma_{m(P)}$ contains the point $z^d = 0$, so $\pi\big(\Sigma_{M(P)}\big)$ and $\pi\big(\Sigma_{m(P)}\big)$ are non-singular and intersect transversely. Hence, we have:

$$\operatorname{codim}(\pi\big(\Sigma_{M(P)}\big)) = \beta - 1, \quad \operatorname{codim}(\pi\big(\Sigma_{m(P)}\big)) = \alpha - 1,$$
$$\operatorname{codim}(\pi\big(\Sigma_{M(P)} \cap \Sigma_{m(P)}\big)) = \beta + \alpha - 1$$

and

$$\operatorname{codim}(\pi\big(\Sigma_{M(P)}\big) \cap \pi\big(\Sigma_{m(P)}\big)) = \beta + \alpha - 2.$$

So, we have one degree of freedom to move from $\pi(P(x,y) = 0)$ out of $\pi\big(\Sigma_{M(P)} \cap \Sigma_{m(P)}\big)$ staying in $\pi\big(\Sigma_{M(P)}\big) \cap \pi\big(\Sigma_{m(P)}\big)$ which means exactly that we can perturb L to a line $\widetilde{L}$ still passing through $z^d = 0$ and intersecting the stratum $\Sigma_{M(P)}$ and $\Sigma_{m(P)}$ one after the other.

As $\pi\big(\Sigma_{M(P)}\big) \cap \pi\big(\Sigma_{m(P)}\big) \setminus \pi\big(\Sigma_{M(P)} \cap \Sigma_{m(P)}\big)$ has two connected components, we have two possible choices to perturb L. One will correspond to move up (resp. down) the local maxima (resp. minima) and the other will correspond to move up (resp. down) the local minima (resp. maxima), see Figure 1. Choosing the latter possibility, we prove the proposition. □

The proposition can be interpreted as a refined version of Brusotti's Theorem in a special case. Indeed, it does not only show that we can perturb each solitary point of a real plane curve $P(x,y) = 0$ into one of the two topological possibilities, but it proves that we can in addition put all solitary points which are deformed in the same topological way on the same level of $P(x,y)$, i.e. transform the points into extremal points of the graph of $P(x,y)$ with the same value.

3.4. **Optimality of our Construction.** We now show that using Chmutov's method it is asymptotically not possible to improve our lower bound obtained in Theorem 14. Let us denote by $\mu_{Ch}(d)$ the maximal possible number of solitary points of a real algebraic surface of degree d in $\mathbb{R}P^3$ constructed using Chmutov's method.

Proposition 17. *Let $d \in \mathbb{N}$. Then:*

$$\mu_{Ch}(d) = \frac{1}{4}d^3 + o(d^3).$$

Proof. The result is an immediate corollary of Theorem 14, Proposition 15 and of the following Proposition 18. □

Let us denote by $\mu_{extr}(d)$ the maximum possible number of local extrema of a real polynomial $f(x,y)$ of degree d. We believe that the bound we establish now is known, but as we did not find a reference for it, we include a proof here:

Proposition 18. *With the notation of the preceding proof, we have:*

$$\mu_{extr}(d) \leq \frac{1}{2}d^2 + o(d^2).$$

Proof. Denote by h the height function $(x, y, z) \mapsto z$ defined on $\mathbb{R}^3$. Let $f(x, y)$ be a real polynomial of degree d and denote by $\nu_0(f)$ (resp. $\nu_1(f)$) the number of local extrema (resp. hyperbolic critical points) of f. Consider a very large ball B in $\mathbb{R}^2$ containing all critical points of f, and consider $D(f)$ the intersection of the graph of f with the cylinder with base B. Then, one can glue in $\mathbb{R}^3$ a disk to $D(f)$ along its border $\partial D(f)$ adding a number of critical points for h which is at most linear in d. Then, we obtain a sphere S^2 and h defines a Morse function on it. Hence, we have $\nu_0(f) - \nu_1(f) \leq 2 + ad$ with a some integer number.

On the other hand, the number of real critical points of f is not more than its number of complex critical points, which is equal to $(d-1)^2$. Taking all this together, we get: $\mu_{extr}(d) \leq \frac{1}{2}d^2 + o(d^2)$. □

4. Higher singularities

Proposition 15 can also be applied to construct real algebraic surfaces in $\mathbb{R}P^3$ with many $A^{\bullet}_{2k-1}$ singularities. The method is exactly the same as in section 3.2.3, but instead of Tchebychev polynomials, we use polynomials with very degenerate critical points of critical values ± 1. The existence of such polynomials is guaranteed by applying the real version of Dessins d'Enfants (e.g. see [Bru06]) to the construction in [Lab06b].

Lemma 19. *Let $d, k \geq 1$. Then there is a real polynomial $T_d^{2k}(z)$ of degree d with $\left[\frac{d-1}{4k-2}\right]$ local maxima (resp. minima) which are critical points of multiplicity $2k-1$ (resp. non-degenerate critical points) and with value $+1$ (resp. -1).*

Proposition 20. *Let $k, d \geq 1$. We have:*

$$\frac{1}{8k-4}d^3 + o(d^3) \leq \mu^3(A^{\bullet}_{2k-1}, d) \leq \frac{4k}{12k^2-3}d^3 + o(d^3).$$

Proof. The upper bound is Miyaoka's bound. Let $f_d(x, y)$ be a real polynomial of degree d with α local minima with value 1, with β local maxima with value -1, and such that $\alpha + \beta = \frac{(d-1)(d-2)}{2}$. According to Theorem 5 and Proposition 16, such a polynomial exists. The lower bound in the theorem is given by considering the surface with equation $f_d(x, y) - T_d^{2k}(z) = 0$. □

Remark 21. *Using the method "$P_d(x, y) + g(z) = 0$" described in section 3.1, one could expect to obtain better lower bounds for $\mu^3(A^{\bullet}_{2k-1}, d)$ as soon as $k \geq 2$. In this case, $P_d(x, y) = 0$ should be a plane curve with many even $A^{\bullet}_{2k-1}$-points. However, up to our knowledge, the currently known constructions give only*

$$\mu^2(A^{\bullet}_{2k-1}, d) \geq \frac{1}{4k}d^2 \qquad (1)$$

which provide lower bounds a bit worse than ours for $\mu^3(A^{\bullet}_{2k-1}, d)$. The lower bound (1) can be obtained by considering the polynomials $T_d(x) - \widetilde{T}_d^{2k}(y)$ where $T_d(x)$ is the Tchebychev polynomial of degree d and $\widetilde{T}_d^{2k}(y)$ is a polynomial of degree d which has $\lfloor \frac{d}{2k} \rfloor$ local minima which are critical points of multiplicity $2k-1$ and with value $+1$. The existence of the polynomials $\widetilde{T}_d^{2k}(y)$ can be proved with the same technique as in Lemma 19.

In [Wes03, Proposition 3.5]*, Westenberger claims that* $\mu^2(A^\bullet_{2k-1}, d) \geq \frac{1}{4k-2}d^2$*. However, his proof of this proposition uses* [Wes03, Lemma 3.1] *which is wrong for solitary points. Indeed, this lemma states that there exists a real algebraic curve with Newton polygon the quadrangle with vertices* $(0,1)$*,* $(0,2)$*,* $(1,0)$ *and* $(2k-1,1)$*, and with one* $A^\bullet_{2k-1}$ *point. However, such a curve cannot exist due to the following proposition. The case of* $A^\bullet_1$*-singularities is easy to verify by hand; for the general statement, we need some more work:*

Proposition 22. *For any* $k \geq 1$*, no real algebraic curve with Newton polygon the quadrangle with vertices* $(0,1)$*,* $(0,2)$*,* $(1,0)$ *and* $(2k-1,1)$ *can have an* $A^\bullet_{2k-1}$ *point.*

Proof. For brevity, we will use the notations, definitions and basic results of [Bru06, section 4]. Suppose that there exists a curve C contradicting the proposition. Without loss of generality, we can assume that the equation of C is $y^2+P(x)y+x=0$, where $P(x)$ is a real univariate polynomial in x of degree $2k-1$. The discriminant of C seen as a polynomial in y is $R(x) = P^2(x)-4x$, and it is clear that the topology of C can be recovered out of the root scheme realized by the polynomials $P^2(x)$, $Q(x) = -4x$, and $R(x)$. One sees that $R(x) > 0$ for $x \leq 0$, and since C has an $A^\bullet_{2k-1}$ point, $R(x)$ must have a root of order $2k$ close to which $R(x)$ is non positive. It follows that the polynomials $P^2(x)$, $Q(x)$ and $R(x)$ realize the root scheme

$$\big((p,2b),(q,1),(r,a_1),(p,2b_1),(r,a_2),(p,2b_2),\ldots,(r,a_i),(p,2b_i),(r,2k),\\ (p,2b_{i+1}),(r,a_{i+1}),(p,2b_k),(r,a_k)\big)$$

where i, k, a_j, b and b_j are some non negative integers, and $a_1 > 0$. It is not hard to see from the real rational graphs (or Dessins d'Enfants) that this is equivalent to the existence of three real polynomials $\widetilde{P}^2(x)$, $\widetilde{Q}(x)$ and $\widetilde{R}(x)$ of degree $4k-2$ and which realize the root scheme

$$\big((r,1),(p,2b),(r,a_1-1),(p,2b_1),(r,a_2),(p,2b_2),\ldots,(r,a_i),(p,2b_i),(r,2k),\\ (p,2b_{i+1}),(r,a_{i+1}),(p,2b_k),(r,a_k)\big).$$

But then, $\widetilde{Q}(x) = -\beta^2$ with β a nonzero real number, and $\widetilde{R}(x) = \widetilde{P}(x)^2 - \beta^2 = (\widetilde{P}(x)-\beta)(\widetilde{P}(x)+\beta)$. Now, the polynomials $\widetilde{P}(x)-\beta$ and $\widetilde{P}(x)+\beta$ are relatively prime and of degree $2k-1$, so $\widetilde{Q}(x)$ cannot have a root of order $2k$. □

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Univ. Paris 6, 175 rue du Chevaleret, 75 013 Paris, France
E-mail address: brugalle@math.jussieu.fr

Universität des Saarlandes, Mathematik und Informatik, Gebäude 2.4, D-66123 Saarbrücken, Germany
E-mail address: Labs@math.uni-sb.de, mail@OliverLabs.net