N° d'Ordre : 3133

T H È S E

Présentée

DEVANT L'UNIVERSITÉ DE RENNES I

pour obtenir

le grade de DOCTEUR DE L'UNIVERSITÉ DE RENNES I

Mention Mathématiques et Applications

par

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U.F.R. de Mathématiques

TITRE DE LA THÈSE :

Courbes algébriques réelles et courbes pseudoholomorphes réelles dans les surfaces réglées.

Soutenue le 10 décembre 2004 devant la Commission d'Examen

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Remerciements

Cette thèse est l'aboutissement de trois ans de travail. Durant cette période, de nombreuses personnes m'ont, de près ou de loin, aidé, soutenu et encouragé. Je profite de cette page à l'origine blanche pour les en remercier.

Je tiens tout d'abord à remercier chaleureusement Ilia Itenberg pour sa disponibilité et son soutien constant au cours de ces trois années. Il aura toujours répondu patiemment à mes nombreuses sollicitations mathématiques et angoisses administratives. Je lui dois cette fascination pour le 16$^{\text{ème}}$ problème de Hilbert et en discuter avec lui aura été une source continuelle de (re)motivation et d'inspiration. Je tiens à le remercier aussi, ainsi que sa famille, pour son accueil lors de mes visites à Strasbourg.

Ma thèse doit aussi beaucoup à Stepan Orevkov. Il m'aura donné de nombreuses explications sur son travail, et ce avec patience. Mes discussions avec lui auront joué un rôle prépondérant dans l'élaboration de cette thèse. De plus, je n'oublierai jamais l'accueil extraordinaire de la famille Orevkov lors de mon séjour en Russie, ainsi que les visites guidées de Stepan à Moscou, Zagorsk (Sergueïev Possad) et St Pétersbourg.

Je remercie également Stepan Orevkov et Jean Jacques Risler d'avoir accepté d'être rapporteurs ainsi que Michel Coste, Louis Mahé et Adam Parusinski de faire partie de mon jury.

La forme finale de cette thèse comporterait beaucoup plus de fautes (de français, d'anglais et de maths) sans les relectures attentives d'Assia (la plus jolie), d'Ilia, de Michel, de Jean Jacques, de Stepan, de Nico, de D'Jo et de Nenelle. Leurs remarques judicieuses m'auront permis, je l'espère, de rendre ce manuscrit beaucoup plus clair et compréhensible.

Je remercie aussi Vincent qui s'est courageusement chargé de toutes les formalités administratives de la soutenance.

J'ai grandement apprécié l'univers mathématique au sein duquel j'ai évolué pendant trois ans. L'équipe de géométrie algébrique réelle de Rennes tout d'abord, son activité, son ambiance détendue, la gentillesse et la simplicité de ses membres. L'IRMAR ensuite, où je remercie spécialement Claude Boschet, Danielle Lanneau et Karine Falc'hon pour leur gentillesse et leur efficacité. Le réseau européen RAAG enfin, dont j'ai eu la chance de faire partie dès le début de ma thèse. J'ai ainsi eu l'opportunité de rencontrer et de discuter avec de nombreuses personnes ayant les mêmes centres d'intérêts mathématiques, et souvent bien plus expérimentées. J'adresse un grand merci aux responsables de RAAG, et plus particulièrement à Michel Coste, responsable de l'équipe française qui a toujours facilité les déplacements des jeunes, et à Francesca Acquistapace pour son aide lors de l'organisation des rencontres jeunes chercheurs du réseau en Espagne. Mais surtout, dans ce monde, évoluent aussi Benito, Fred, Lucia, Luis, Nico, Olive, Solen et Vincent (bis) avec qui vivre et faire des maths aux quatre coins de la planète est un immense plaisir, et Bert, Damien, Élise, Glenn, Guillaume, Guillaume (bis), Gwéno, Mariana et Vincent, aux côtés desquels il fait bon travailler, mais également passer de bonnes soirées. Une petite pensée particulière pour Gwéno, le manitou informatique le plus gentil et le plus humain que je connaisse.

Cependant, tout cela serait resté sans saveur sans le soutien et les encouragements constants de ma famille, Nard&Jo et Jagwé, et la présence de Ben, Cécile, David, Greg, Guy'C, L'ami Cause, Magali, Raph, Seal et Stéphane qui m'auront bien souvent changé les idées.

J'espère que tout ceux que j'ai oubliés voudront bien me pardonner...

Mais celle que je tiens le plus à remercier, et que j'espère remercier encore à l'occasion de mon premier poste permanent, de mon habilitation, de mon passage au rang de Professeur et de mon accession à la Présidence de la République, c'est Assia, sans qui le reste ne serait que roupie de sansonnet.

Contents

Chapter 1

Introduction

1.1 Origines de l'étude topologique des variétés algébriques réelles

L'étude systématique de la topologie de variétés algébriques réelles commence en 1876 avec les travaux d'A. Harnack lorsqu'il pose et résout la question suivante dans [Har76].

Problème 1 *Quel est le nombre maximal de composantes connexes de la partie réelle d'une courbe algébrique réelle non singulière de degré d dans le plan projectif?*

On rappelle qu'une courbe algébrique réelle dans $\mathbb{R}P^2$ est un polynôme réel homogène a trois variables considéré a multiplication par une constante réelle non nulle près. Par abus de langage, on emploiera aussi la dénomination "courbe algébrique réelle" pour désigner l'ensemble des zéros dans $\mathbb{C}P^2$ d'un polynôme réel homogène a trois variables. L'ensemble des zéros réels sera lui appelé la *partie réelle* de la courbe. Par exemple, la partie réelle de la courbe définie par le polynôme $X^2 + Y^2 - Z^2$ est dans le plan affine $\{Z = 1\}$ le cercle de centre $(0, 0)$ et de rayon 1.
Pour une courbe algébrique réelle C, on notera $\mathbb{R}C$ sa partie réelle. Une courbe algébrique réelle C est dite *non singulière* si il n'existe aucun point de $\mathbb{C}P^2$ où les polynômes $\frac{\partial C}{\partial X}$, $\frac{\partial C}{\partial Y}$, $\frac{\partial C}{\partial Z}$ s'annulent simultanément.

En d'autres termes, Harnack soulevait et résolvait le problème de la classification topologique de la partie réelle des courbes algébriques réelles non singulières dans $\mathbb{R}P^2$ pour un degré fixé. La réponse est plutôt simple : il n'y a qu'une seule obstruction.

Théorème (Harnack, [Har76]) *Une courbe algébrique réelle non singulière de degré d dans $\mathbb{R}P^2$ a au plus $c(d) = \frac{(d-1)(d-2)}{2} + 1$ composantes connexes.*
De plus, pour tout nombre l entre 0 et $c(d)$ si d est pair, ou entre 1 et $c(d)$ si d est impair, il existe une courbe algébrique réelle non singulière de degré d dans $\mathbb{R}P^2$ avec l composantes connexes.

Une courbe qui possède le nombre maximal de composantes connexes autorisé par le Théorème de Harnack est appelée une M-courbe, ou une courbe maximale.
F. Klein et D. Hilbert prolongèrent le travail de Harnack en étudiant le problème 1 reposé pour des surfaces, ou en étudiant non seulement le nombre de composantes connexes d'une courbe algébrique réelle mais aussi leur position dans le plan projectif. Ces deux généralisations mènent aux deux problèmes fondamentaux suivants de la topologie des variétés algébriques réelles.

Problème 2 *Quels sont, à homéomorphisme près, les ensembles réalisés par les hypersurfaces algébriques réelles non singulières de degré d dans* $\mathbb{R}P^n$ *?*

Problème 3 *Quelles sont, à isotopie près, les ensembles réalisés par les hypersurfaces algébriques réelles non singulières de degré d dans* $\mathbb{R}P^n$ *?*

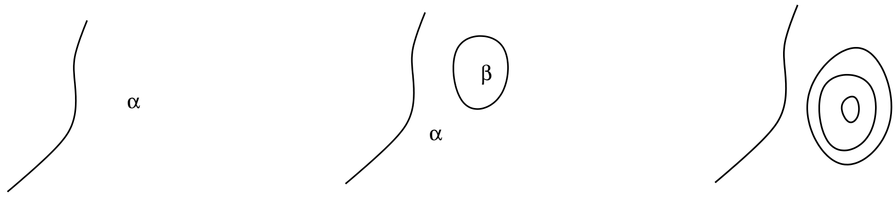
Une reformulation naïve du problème 3 est la suivante : comment peuvent être positionnées les composantes connexes d'une hypersurface algébrique réelle dans $\mathbb{R}P^n$? Assez rapidement, le problème 3 est résolu pour les courbes jusqu'en degré 5 et pour les surfaces jusqu'en degré 3.
En 1900, Hilbert énonce sa célèbre liste des 23 problèmes mathématiques pour le $XX^{\text{ème}}$ siècle (voir [Hil01]). Il reformule le problème 3 dans la première partie de son $16^{\text{ème}}$ problème, en mettant l'accent sur les courbes de degré 6 et les surfaces de degré 4.
La première moitié du $XX^{\text{ème}}$ siècle est marquée par les travaux en ce sens de L. Brusotti, A. Wiman et I. G. Petrovsky (voir [Vir89]), mais ce n'est qu'en 1969 que D. A. Gudkov parvient à obtenir une classification à isotopie de $\mathbb{R}P^2$ près des courbes non singulières projectives de degré 6 (voir [Gud69]). Les années 70 verront se réaliser d'importants progrès avec les apports conséquents de V. I. Arnold, V. A. Rokhlin, V. M. Kharlamov et O. Ya. Viro (voir par exemple [Wil78], [Vir89] ou [DK00]). En particulier Kharlamov termine les classifications correspondantes aux problèmes 2 et 3 pour les surfaces de degré 4 (voir [Kha78]). A la fin des années 70, Viro invente une méthode puissante pour construire des hypersurfaces algébriques dans $\mathbb{R}P^n$ (et plus généralement dans toutes les variétés toriques) et obtient ainsi dans [Vir84a] la classification à isotopie de $\mathbb{R}P^2$ près des courbes non singulières projectives de degré 7.
Le problème 3 est toujours ouvert pour les courbes de degré 8. Quant aux dimensions supérieures, peu de réponses on été apportées au problème 2. La méthode des tresses inventée par S. Yu. Orevkov à la fin des années 90 a fourni un moyen très efficace pour étudier la position d'une courbe par rapport à un pinceau de droites. Cette méthode à été un des outils essentiels dans la réduction du nombre de classes d'isotopie dont la réalisation par une courbe algébrique réelle maximale de degré 8 reste inconnue.
On peut bien sûr étudier d'autres classifications plus ou moins fines. Par exemple, si l'on note $\mathbb{R}V_d^n$ l'ensemble des hypersurfaces algébriques réelles de degré d dans $\mathbb{R}P^n$ et $\mathbb{R}\Delta_d^n$ le sous-ensemble de $\mathbb{R}V_d^n$ constitué des hypersurfaces singulières, le problème de classification à isotopie rigide près s'énonce comme suit.

Problème 4 *Classifier les composantes connexes de* $\mathbb{R}V_d^n \setminus \mathbb{R}\Delta_d^n$.

Autrement dit, on cherche à savoir si deux hypersurfaces non singulières de même degré et dimension peuvent être reliées par une isotopie de $\mathbb{R}P^n$ pour laquelle toutes les hypersurfaces intermédiaires durant l'isotopie sont algébriques et non singulières. Cette question à été rapidement résolue pour les courbes jusqu'en degré 4 et pour les surfaces jusqu'en degré 3. Le cas des courbes de degré 5 (Kharlamov, [Kha81]) et 6 (V. V. Nikulin, [Nik79]) et des surfaces de degré 4 (Kharlamov, [Kha84]) ne sont résolus qu'après les années 70.

1.2 Courbes algébriques réelles

Nous présentons ici quelques problèmes relatifs aux courbes qui ont motivé notre travail. Cette liste ne saurait en aucun cas être exhaustive. D'autres problématiques intéressantes sont mentionnées dans les articles [Wil78], [Vir89] et [DK00], ainsi que les généralisations aux dimensions supérieures des notions exposées ici pour les courbes.
Nous commencerons par rappeler quelques définitions et résultats de classification pour les courbes algébriques réelles dans $\mathbb{R}P^2$, puis nous introduirons les deux problèmes centraux de cette thèse.



$0 \leq \alpha \leq 15$ $\qquad$ $0 \leq \alpha \leq 13,\ 1 \leq \beta \leq 13,$ $\alpha + \beta \leq 14$

Figure 1.1:

1.2.1 Généralités

Comme $H_1(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, un cercle topologique peut être plongé de deux façons dans $\mathbb{R}P^2$: soit il réalise la classe nulle, soit il réalise la classe non triviale. Dans le deuxième cas, ce cercle est non contractile dans $\mathbb{R}P^2$ et est appelé une *pseudo-droite*. On remarque que le complémentaire d'une pseudo-droite dans le plan projectif est connexe. Si le cercle réalise la classe triviale, il est appelé *ovale*. Un ovale divise $\mathbb{R}P^2$ en deux composantes connexes, dont l'une est homéomorphe à un disque (l'*intérieur* de l'ovale) et l'autre à un ruban de Möbius (l'*extérieur* de l'ovale).
Une collection de cercles disjoints dans $\mathbb{R}P^2$ est appelée un schéma réel. On dira qu'un schéma réel est réalisable par une courbe algébrique réelle non singulière de degré d si il est isotope à la partie réelle d'une telle courbe.
Les classifications des schémas réels réalisés par les courbes algébriques réelles non singulières dans $\mathbb{R}P^2$ jusqu'en degré 5 sont connues dès le $XIX^{\text{ème}}$ siècle. Celles des courbes de degré 6 et 7 sont obtenues respectivement par Gudkov (voir [Gud69]) en 1969 et Viro (voir [Vir84a]) à la fin des années 70. On rappelle ici la classification des courbes algébriques réelles planes de degré 7.

Théorème (Viro, [Vir84a]) *Les schémas réels réalisables par les courbes algébriques réelles non singulières de degré 7 sont exactement ceux représentés sur la Figure 1.1 (les nombres représentent autant d'ovales, tous les uns en dehors des autres).*

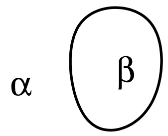
La classification des M-courbes de degré 8 n'est toujours pas achevée. Cependant, il ne reste qu'un petit nombre de schémas réels dont la réalisation est inconnue. Le théorème suivant rassemble les connaissances actuelles sur ce sujet. Il est le résultat des travaux de B. Chevallier (voir [Che02]), T. Fiedler (voir [Fie83]), A. B. Korchagin (voir [Kor89]), S. Yu. Orevkov (voir [Ore02a]), E. Shustin (voir [Shu87], [Shu88] et [Shu91]) et O. Ya. Viro (voir [Vir80], [Vir89], [Vir84b] et [Vira]).

Théorème *Les schémas réels réalisés par les courbes algébriques réelles maximales non singulières de degré 8 sont contenus dans ceux représentés sur les Figures 1.2 et1.3. De plus, tous les schémas réels représentés sur la figure 1.2 sont réalisables par les courbes algébriques réelles non singulières de degré 8.*

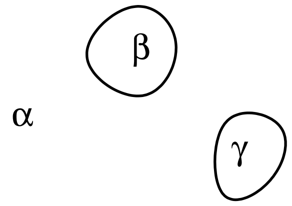
On ignore toujours actuellement si les schémas réels représentés sur la Figure 1.3 sont réalisables par les courbes algébriques réelles non singulière de degré 8 dans $\mathbb{R}P^2$.

1.2.2 Problème d'isotopie symplectique réelle

Dans les années 80, Viro constate que la majorité des restrictions connues sur la topologie des courbes algébriques réelles sont obtenues par des moyens purement topologiques, et s'appliquent

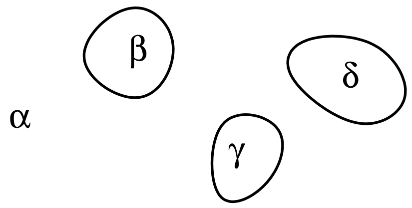


$\alpha \equiv 2\ [4],\ \alpha + \beta = 21$



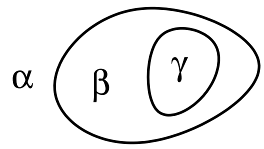
$\alpha \equiv 1\ [4],\ \alpha + \beta + \gamma = 20,\ 1 \leq \beta \leq \gamma$

si $\alpha = 1,\ \beta \neq 1, 3, 4, 6, 7, 9$



$\alpha \equiv 0\ [4],\ \alpha + \beta + \gamma + \delta = 19,$
$1 \leq \beta \leq \gamma \leq \delta$ et $\beta,\ \gamma,\ \delta$ impairs

si $\alpha = 0$ et $\beta = 1,\ \gamma \neq 3, 5, 9$
si $\alpha = 0,\ \beta \neq 3$
si $\alpha = 4$ et $\beta = 3,\ \gamma \neq 3$



$\alpha \geq 1,\ \beta \equiv 2\ [4],\ \gamma \geq 1,\ \alpha + \beta + \gamma = 20$

si $\beta = 2,\ \alpha \neq 4, 14$

Figure 1.2:

donc à une classe plus vaste d'objets. Il nomme ces objets les *courbes flexibles* (voir [Vir84c] pour une définition précise) et pose la question suivante.

Problème 5 *Existe-t-il une courbe flexible dans $\mathbb{R}P^2$ réalisant un schéma réel réalisable par aucune courbe algébrique réelle non singulière de même degré?*

Autrement dit, est ce que des méthodes purement topologiques permettent de résoudre le $16^{\text{ème}}$ problème de Hilbert, ou est-il nécessaire de trouver des obstructions algébriques?
Il apparaît dans les années 90 avec les travaux d'Orevkov qu'il est opportun de recentrer le problème 5 sur une sous-classe des courbes flexibles : les *courbes pseudoholomorphes réelles.* En effet, ces objets vérifient le Théorème de Bézout, fondamental pour les courbes algébriques, ce qui n'est à priori pas le cas des courbes flexibles. Les courbes pseudoholomorphes, introduites par M. Gromov dans [Gro85] pour étudier les 4-variétés symplectiques, sont l'analogue des courbes algébriques mais pour une structure presque complexe de $\mathbb{C}P^2$ au lieu de la structure complexe standard (on demande de plus que la structure presque complexe soit *calibrée* par une forme symplectique, voir la section 2.4 pour des définitions précises). Le problème 5 se reformule alors de la manière suivante.

Problème 6 *Existe-t-il une courbe pseudoholomorphe réelle non singulière dans $\mathbb{C}P^2$ réalisant un schéma réel réalisable par aucune courbe algébrique réelle non singulière de même degré?*

On peut en fait voir cette question comme une version réelle du *problème d'isotopie symplectique* dans $\mathbb{C}P^2$. Ce problème est directement issu de l'étude des 4-variétés symplectiques et peut s'énoncer comme suit (voir par exemple [She]).

$\alpha = 1, \beta = 1, 4, 7, 9$ $\beta = 2, \alpha = 4, 14$

Figure 1.3:

Problème 7 *Toute courbe symplectique non singulière de $\mathbb{C}P^2$ est-elle isotope à une courbe algébrique non singulière de même degré?*

A priori, on peut considérer tout type d'isotopie (continue, lisse ou symplectique) car la réponse est inconnue dans tous les cas. Cette question a cependant été résolue de façon affirmative pour des degrés inférieurs ou égaux à 6 par V. Shevchischin dans [She].
On sait que toute surface symplectique non singulière de $\mathbb{C}P^2$ est une courbe J-holomorphe pour une certaine structure presque complexe J calibrée par la forme symplectique, et que cette structure peut être choisie réelle si la surface est réelle. Ainsi, le problème 6 peut se reformuler de la manière suivante, qui est bien l'analogue réel du problème 7.

Problème 8 *Existe-t-il une courbe symplectique réelle non singulière dans $\mathbb{C}P^2$ réalisant un schéma réel réalisable par aucune courbe algébrique réelle non singulière de même degré?*

Pour l'instant, cette question est toujours un problème ouvert et semble plutôt difficile. Toutes les classifications connues coïncident pour les deux classes d'objets. On notera que mêmes les démonstrations sont les identiques... En revanche, en généralisant un peu ce problème, des contre-exemples non triviaux ont été exhibés par S. Fiedler-Le Touzé, Orevkov et Shustin. Ainsi, on connaît maintenant des courbes pseudoholomorphes réelles réductibles dans $\mathbb{R}P^2$ (resp. non singulières dans les surfaces rationnelles géométriquement réglées) dont la partie réelle réalise une classe d'isotopie (resp. isotopie par rapport au pinceau de droites) qui n'est réalisée par aucune courbe algébrique réelle ayant la même classe d'homologie. On peut voir à ce sujet [FLTO02], [Ore], [OS02] et [OS].
La méthode des tresses est un outil très puissant pour interdire ou construire des courbes pseudoholomorphes réelles. Ainsi, il est plus facile de travailler avec ces dernières qu'avec les courbes algébriques réelles. Par exemple, la classification des M-courbes pseudoholomorphes réelles non singulières de degré 8 dans $\mathbb{R}P^2$ à été achevée par Orevkov, alors que l'analogue algébrique n'est toujours pas connu.

Théorème (Orevkov,[Ore02a]) *Les schémas réels réalisés par les courbes pseudoholomorphes réelles maximales non singulières de degré 8 sont exactement ceux représentés sur les Figures 1.2 et 1.3.*

Dans cette thèse, nous ne considérons que des structures presque complexes dans les surfaces rationnelles géométriquement réglées pour lesquelles l'éventuel diviseur exceptionnel de la surface est pseudoholomorphe (voir section 2.4). En ne tenant pas compte de cette condition, J. Y. Welschinger a construit dans [Wel02] une famille infinie de courbes symplectiques dans Σ_n avec $n \geq 2$ qui réalisent des classes d'isotopies qu'aucune courbe algébrique réelle non singulière ayant la même classe d'homologie ne réalise.

1.2.3 Ovales pairs et conjecture de Ragsdale

Un ovale d'une courbe algébrique réelle non singulière de degré pair dans $\mathbb{R}P^2$ est dit *pair* (resp. *impair*) si il est contenu dans un nombre pair (resp. impair) d'ovales. Le nombre maximal d'ovales pairs que peut contenir une courbe de degré $2k$ est important, car il est relié aux nombres de Betti maximaux des certaines surfaces algébriques réelles (voir la section 4.5). On note p le nombre d'ovales pairs d'une courbe, et n son nombre d'ovales impairs.

Problème 9 *Quel est la valeur maximale de p et n pour une courbe de degré $2k$?*

En 1906, V. Ragsdale proposa dans [Rag06] une série de conjectures sur la topologie des courbes algébriques réelles dont la plus célèbre est la suivante.

Conjecture (Ragsdale, [Rag06]) *Pour toute courbe de degré $2k$, on a*

$$p \leq \frac{3k(k-1)}{2} + 1 \ et \ n \leq \frac{3k(k-1)}{2}.$$

On notera $R(k) = \frac{3k(k-1)}{2} + 1$. Dans les années 30, Petrovsky démontra dans [Pet33] des inégalités moins fortes, et proposa quelques conjectures similaires à celles de Ragsdale. Il parait cependant assez clair que Petrovsky n'était pas familier avec le travail de Ragsdale.

Théorème (Petrovsky, [Pet33]) *Pour toute courbe de degré $2k$, on a*

$$p - n \leq \frac{3k(k-1)}{2} + 1 \ et \ n - p \leq \frac{3k(k-1)}{2}.$$

En combinant ces inégalités avec le Théorème de Harnack, on obtient immédiatement une borne supérieure pour p et n.

Proposition *Pour toute courbe de degré $2k$, on a*

$$p \leq \frac{7k^2}{4} - \frac{9k}{4} + \frac{3}{2} \ et \ n \leq \frac{7k^2}{4} - \frac{9k}{4} + 1.$$

Il est alors tout à fait naturel de se demander si ces bornes sont atteintes, au moins asymptotiquement. Le terme asymptotiquement est expliqué dans l'énoncé du problème suivant.

Problème 10 *Existe-t-il une famille de courbes algébriques réelles non singulières de degré $2k$ dans $\mathbb{R}P^2$ avec $\frac{7}{4}k^2 + o(k^2)$ ovales pairs?*
Existe-t-il une famille de courbes algébriques réelles non singulières de degré $2k$ dans $\mathbb{R}P^2$ avec $\frac{7}{4}k^2 + o(k^2)$ ovales impairs?

Au début des années 80, Viro réfuta la conjecture de Ragsdale en construisant des courbes avec $n = R(k)$ (ce n'étaient en revanche pas des contre-exemples à la conjecture de Petrovsky). En 1993, I. Itenberg construisit en utilisant la T-construction des contre-exemples de degré $2k$ ayant $R(k) + \frac{k^2}{8} + O(k)$. Ce résultat fur amélioré par B. Haas dans [Haa95] puis par Itenberg dans [Ite01] et avant les résultats de cette thèse, les meilleurs résultats était les suivants.

Théorème (Itenberg, [Ite01]) *Il existe une famille de courbes algébriques réelles non singulières de degré $2k$ dans $\mathbb{R}P^2$ avec $\frac{81}{48}k^2 + O(k)$ ovales pairs.*
Il existe une famille de courbes algébriques réelles non singulières de degré $2k$ dans $\mathbb{R}P^2$ avec $\frac{81}{48}k^2 + O(k)$ ovales impairs.

A priori les problèmes 9 et 10 se posent aussi pour des courbes pseudoholomorphes. En cherchant à améliorer les résultats d'Itenberg, F. Santos construit dans [San] des T-courbes ayant un grand nombre d'ovales pairs. Malheureusement, les triangulations utilisées ne sont pas convexes. Mais d'après [IS02], si les courbes de Santos ne sont pas algébriques, elles sont au moins pseudoholomorphes.

Théorème (Santos, [San]) *Il existe une famille de courbes pseudoholomorphes réelles non singulières de degré $2k$ dans $\mathbb{R}P^2$ telle que*

$$\lim_{k\to\infty} \frac{p}{k^2} = \frac{17}{10}.$$

Il existe une famille de courbes pseudoholomorphes réelles non singulières de degré $2k$ dans $\mathbb{R}P^2$ telle que

$$\lim_{k\to\infty} \frac{n}{k^2} = \frac{17}{10}.$$

Il est intéressant de constater qu'aucun contre-exemple à la conjecture de Ragsdale n'est connu parmi les M-courbes. De plus, Haas a démontré dans [Haa] que tout contre-exemple maximal hypothétique T-construit est un "petit" contre-exemple.

Théorème (Haas, [Haa]) *Soit C une M-courbe pseudoholomorphe réelle non singulière de degré $2k$ obtenue par T-construction. Alors*

$$p \leq R(k) + 1 \text{ et } n \leq R(k) + 4.$$

1.3 Résultats

Nous énonçons ici les principaux résultats obtenus dans cette thèse.

1.3.1 Graphes rationnels sur $\mathbb{C}P^1$

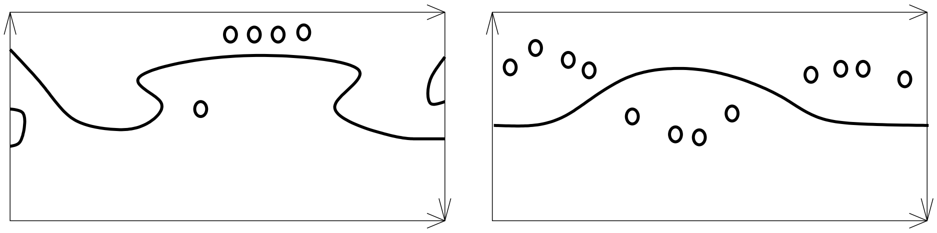
S'inspirant de travaux antérieurs de A. Zvonkin, Orevkov a proposé dans [Ore03] une nouvelle méthode de construction de courbes algébriques réelles trigonales dans les surfaces réglées Σ_n. L'idée est de reformuler la question de l'existence d'une telle courbe réalisant un certain $\mathcal{L}$-*schéma* par celle de l'existence d'un certain graphe (appelé *graphe trigonal réel* dans cette thèse) dans $\mathbb{C}P^1$. L'intérêt de ces graphes est que leur partie réelle peut se lire sur le $\mathcal{L}$-schéma.
En fait, cette méthode se généralise immédiatement pour étudier l'existence de deux polynômes à une variable $P(X)$ et $Q(X)$ de degré n tels que les racines réelles de $P(X)$, $Q(X)$ et $P(X)+Q(X)$ aient une certaine disposition sur $\mathbb{R}$. Nous formulons donc d'abord cette généralisation et introduisons ainsi les *graphes rationnels réels*.
Nous nous concentrons ensuite sur l'étude des graphes trigonaux réels. Nous donnons un algorithme efficace pour décider si, étant donner le *graphe réel* d'un $\mathcal{L}$-schéma, ce graphe est la partie réelle d'un graphe trigonal réel ou non. Nous introduisons pour ce faire le semi groupe des *peignes*.
Les résultats de ce chapitre sont utilisés au chapitre 4 pour démontrer la prochaine proposition et au chapitre 5 pour démontrer entre autre les deux suivantes.

Proposition *Pour tout entier $n \in \mathbb{N}^*$, il existe 3 polynômes $a_1(X)$, $a_2(X)$ et $b(X)$ de degré n tels que*

- *toutes les racines de a_1, a_2, b et $a_1b + a_2$ soient réelles,*
- *toutes les racines de a_2 et $a_1b + a_2$ soient plus petites que les racines de b.*

Proposition *Il n'existe pas de courbe algébrique réelle trigonale dans Σ_3 réalisant le $\mathcal{L}$-schéma représenté sur la Figure 1.4a).*

Proposition *Il n'existe pas de courbe algébrique réelle trigonale dans Σ_5 réalisant le $\mathcal{L}$-schéma représenté sur la Figure 1.4b).*



a) b)

Figure 1.4:

1.3.2 Construction d'une famille de courbes avec le nombre asymptotiquement maximal d'ovales pairs

Nous construisons dans cette thèse une famille de courbes qui donnent une réponse affirmative au problème 10.

Théorème *Il existe une famille de courbes algébriques réelles non singulières de degré $2k$ dans $\mathbb{R}P^2$ avec $\frac{7}{4}k^2 + o(k^2)$ ovales pairs.*
Il existe une famille de courbes algébriques réelles non singulières de degré $2k$ dans $\mathbb{R}P^2$ avec $\frac{7}{4}k^2 + o(k^2)$ ovales impairs.

Toutes les familles de courbes, algébriques ou pseudoholomorphes, avec un grand nombre d'ovales pairs construites avant cette thèse étaient des familles de T-courbes. Il nous a semblé que la T-construction est plutôt "rigide" et que la méthode de Viro dans toute sa généralité offrait de bien meilleures possibilités pour construire des courbes algébriques réelles. Nous avons ainsi repris les constructions d'Itenberg et de Santos en essayant d'optimiser la densité d'ovales pairs des courbes construites. Pour cela, nous recollons des courbes dont le polygone de Newton n'est plus un triangle mais un hexagone. Pour construire ces courbes auxiliaires, nous utilisons la méthode des graphes rationnels réels sur $\mathbb{C}P^1$.

1.3.3 Courbes réelles symétriques de degré 7 dans $\mathbb{R}P^2$

Le problème d'isotopie symplectique réelle dans $\mathbb{R}P^2$ semble vraiment difficile. Afin de trouver des différences entre les courbes algébriques et les courbes pseudoholomorphes, une idée est d'étudier des courbes "plus simples". Une première voie est de regarder les courbes singulières, et des différences ont été trouvées par Fiedler-Le Touzé, Orevkov et Shustin. Une autre approche est de considérer des courbes qui admettent d'autres symétries que la seule invariance par la conjugaison complexe. C'est la voie que nous empruntons dans ce chapitre où nous étudions les *courbes symétriques* de degré 7. Une courbe dans $\mathbb{R}P^2$ est dite symétrique si elle admet une symétrie par rapport à une droite dans une certaine carte affine de $\mathbb{R}P^2$.

La classification des courbes symétriques dans $\mathbb{R}P^2$ était connue jusqu'en degré 6, le degré 6 ayant été traité de manière simple par I. Itenberg et V. Itenberg dans [II01]. Encore une fois, les classifications algébriques et pseudoholomorphes coïncident pour ces degrés. D'un autre coté, Orevkov et Shustin ont obtenu dans [OS02] une classification schémas réels réalisés par les M-courbes symétriques réelles de degré 8, et il est apparu alors que les deux classifications n'étaient pas les mêmes.
Ainsi il semblait naturel de traiter le cas du degré 7.
Le premier travail à été d'obtenir les classifications à isotopie près, ce qui ne fait apparaître aucune différence.

Théorème *Les schémas réels suivants ne sont pas réalisables par une courbe pseudoholomorphe réelle non singulière symétrique de degré 7 dans* $\mathbb{R}P^2$ *:*

- $\langle J \amalg (14-\alpha) \amalg 1\langle\alpha\rangle\rangle$ *avec* $\alpha = 6, 7, 8, 9,$
- $\langle J \amalg (13-\alpha) \amalg 1\langle\alpha\rangle\rangle$ *avec* $\alpha = 6, 7, 9.$

De plus, tout schéma réel réalisable par les courbes algébriques réelles non singulières de degré 7 dans $\mathbb{R}P^2$ *non mentionné dans la liste ci-dessus est réalisable par une telle courbe symétrique.*

Remarque. Nous utilisons ici les notations standards, introduites dans [Vir84c] pour les schémas réels[1].
Il nous a semblé alors opportun de raffiner ce résultat en étudiant, parmi ces schémas réels, lesquels sont réalisés par des courbes *séparantes* et lesquels sont réalisés par des courbes *non séparantes.* Nous avons ainsi exhibé deux schémas réels qui sont réalisables par des courbes pseudoholomorphes séparantes symétriques de degré 7 mais pas par de telles courbes algébriques.

Théorème (Classification pseudoholomorphe) *Les schémas réels*

$$\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle \text{ avec } \alpha = 2, 6 \text{ et } \alpha + \beta = 12,\ \alpha = \beta = 4,$$

ne sont pas réalisables par une courbe pseudoholomorphe réelle non singulière séparante symétrique de degré 7 dans $\mathbb{R}P^2$.
De plus, tout schéma réel réalisable par les courbes pseudoholomorphes réelles non singulières séparantes de degré 7 dans $\mathbb{R}P^2$ *non mentionné dans la liste ci-dessus et le théorème précédent est réalisable par une telle courbe symétrique.*
Tout schéma réel réalisable par les courbes pseudoholomorphes réelles non singulières non séparantes de degré 7 dans $\mathbb{R}P^2$ *non mentionné dans le théorème précédent est réalisable par une telle courbe symétrique.*

Théorème (Classification algébrique) *Les schémas réels*

$$\langle J \amalg 8 \amalg 1\langle 4\rangle\rangle \text{ et } \langle J \amalg 4 \amalg 1\langle 8\rangle\rangle$$

ne sont pas réalisables par une courbe algébrique réelle non singulière séparante symétrique de degré 7 dans $\mathbb{R}P^2$.
Tout autre schéma réel réalisable par une courbe pseudoholomorphe réelle non singulière séparante symétrique de degré 7 dans $\mathbb{R}P^2$ *est réalisable par une telle courbe algébrique.*
Tout schéma réel réalisable par une courbe pseudoholomorphe réelle non singulière non séparante symétrique de degré 7 dans $\mathbb{R}P^2$ *est réalisable par une telle courbe algébrique.*

[1] par exemple, $\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle$ signifie une pseudo-droite, un ovale avec β ovales à l'intérieur et α ovales à l'extérieur.

Les résultats sur les courbes pseudoholomorphes sont obtenus essentiellement grâce à la méthode des tresses. Pour les obstructions algébriques, nous utilisons la méthode des peignes et la résolvante cubique d'une courbe trigonale introduite par Orevkov. Pour les constructions algébriques, nous utilisons essentiellement des transformations birationnelles de $\mathbb{C}P^2$, la méthode de Viro et le théorème de perturbation de courbes singulières de Shustin.

Chapter 2

Preliminaries

2.1 Viro's method

The Viro method turned out to be one of the most powerful method to construct real algebraic hypersurfaces with prescribed topology in toric varieties. O. Ya. Viro invented it in the late 70's and since then, this method has been used to obtain many important results in topology of real algebraic varieties. Here are some examples : classification of the real schemes realized by nonsingular curves of degree 7 in $\mathbb{R}P^2$ (Viro [Vir84a]), smoothing of curves with complicated singularities (Viro, [Vir89], E. Shustin, [Shu99]), curvature of plane algebraic curves (L. Lopez de Medrano, [LdM]), existence of projective M-hypersurfaces of any degree in any dimension (I. Itenberg and O. Ya. Viro, [IV]), existence of asymptotically maximal families of hypersurfaces in any toric variety (B. Bertrand, [Ber]), and construction of projective hypersurfaces with big Betti numbers (F. Bihan, [Bih]).
We follow here the presentation of the Viro method exposed in [IS03]. The interested reader can also refer to [Vir84a], [Vir89], [Virb] and [Ris92].

2.1.1 Integer convex polyhedron, moment map and their complexification

Here we recall some definitions we need to state the Viro Theorem.

Definition 2.1 *Let*

$$F = \sum_{(i_1,\dots,i_n)\in\mathbb{N}^n} a_{i_1,\dots,i_n} X_1^{i_1} \dots X_n^{i_n}$$

be a polynomial in $\mathbb{C}[X_1,\dots,X_n]$.
The Newton polyhedron of F*, denoted by* $\Delta(F)$*, is the convex hull in* $\mathbb{R}^n$ *of the set*

$$\{(i_1,\dots,i_n) : a_{i_1,\dots,i_n} \neq 0\}.$$

If δ *is a face of* $\Delta(F)$*, the* δ*-truncation of* F*, denoted by* F^δ*, is the polynomial defined by*

$$\sum_{(i_1,\dots,i_n)\in\delta} a_{i_1,\dots,i_n} X_1^{i_1} \dots X_n^{i_n}.$$

The polynomial F *is said to be non-degenerate if* F *and* F^δ *have a nonsingular zero set in* $(\mathbb{C}^*)^n$ *for all proper faces* δ *of* $\Delta(F)$.

Definition 2.2 *An integer convex polyhedron in $\mathbb{R}^n$ is the convex hull of a finite subset $\mathcal{A}$ of $\mathbb{Z}^n$*

As usual, put $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{R}_+^* = \{x \in \mathbb{R} : x > 0\}$, and $\mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\}$.
Given an integer convex polyhedron Δ in $\mathbb{R}^n$ with interior $I(\Delta)$ and vertices $V(\Delta)$, one can define the well known *moment map* as follows :

$$\begin{array}{cccc} \mu_\Delta : & (\mathbb{R}_+^*)^n & \to & I(\Delta) \\ & (x_1, \ldots, x_n) & \mapsto & \frac{\sum_{(i_1,\ldots,i_n)\in V(\Delta)} x_1^{i_1} \ldots x_n^{i_n} . (i_1,\ldots,i_n)}{\sum_{(i_1,\ldots,i_n)\in V(\Delta)} x_1^{i_1} \ldots x_n^{i_n}} \end{array} .$$

If $dim(\Delta) = n$, then μ_Δ is a diffeomorphism. From now on, let us suppose that $\Delta \subset (\mathbb{R}_+)^n$.
Let us consider the diffeomorphism

$$\begin{array}{cccc} \phi : & (\mathbb{C}^*)^n & \to & (\mathbb{R}_+^*)^n \times (S^1)^n \\ & (z_1, \ldots, z_n) & \mapsto & \Big((|z_1|, \ldots, |z_n|), \Big(\frac{z_1}{|z_1|}, \ldots, \frac{z_n}{|z_n|} \Big) \Big) \end{array} .$$

The inverse of ϕ naturally extends to a surjection $\theta : (\mathbb{R}_+)^n \times (S^1)^n \to \mathbb{C}^n$. Given any subset E of $(\mathbb{R}_+)^n$, we denote by $\mathbb{C}E$ the subset $\theta(E \times (S^1)^n)$ of $\mathbb{C}^n$.

Definition 2.3 *The set $\mathbb{C}\Delta$ is called the complexification of Δ.*
The map

$$\mathbb{C}\mu_\Delta : (\mathbb{C}^*)^n \xrightarrow[\phi]{} (\mathbb{R}_+^*)^n \times (S^1)^n \xrightarrow[(\mu_\Delta, Id)]{} I(\Delta) \times (S^1)^n \xrightarrow[\theta]{} \mathbb{C}I(\Delta)$$

is called the complexification of the moment map μ_Δ.

Proposition 2.4 *The real part $\mathbb{R}\Delta$ of $\mathbb{C}\Delta$ is the union of Δ with all its symmetric copies with respect to the coordinates hyperplanes.*

Proposition 2.5 *The map $\mathbb{C}\mu_\Delta$ is surjective and commutes with the complex conjugation. It is a diffeomorphism when the dimension of Δ is equal to n. The real part of $\mathbb{C}I(\Delta)$ is the image of $(\mathbb{R}^*)^n$ by $\mathbb{C}\mu_\Delta$.*

2.1.2 Toric varieties

We recall now briefly some facts about *toric varieties*. More details can be found, for example, in [Ful93].
Let $\Delta \subset (\mathbb{R}_+)^n$ be an integer convex polyhedron and put $N = Card(\Delta \cap \mathbb{Z}^n)$.

Definition 2.6 *The toric variety associated to Δ and denoted by $Tor_\mathbb{C}(\Delta)$ is the Zariski closure of the set*

$$\{[z_1^{i_1} \ldots z_n^{i_n}]_{(i_1,\ldots,i_n)\in\Delta\cap\mathbb{Z}^n} : (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n\} \subset \mathbb{C}P^{N-1}.$$

This is a complex algebraic variety which has the same dimension as Δ.

Define a map $\nu_\Delta : \mathbb{C}(\Delta) \to Tor_\mathbb{C}(\Delta)$ in the following way : given δ a face of Δ or Δ itself and $\boldsymbol{z} \in I(\mathbb{C}\delta)$ such that $\boldsymbol{z} = \mathbb{C}\mu_\delta(w_1, \ldots, w_n)$, then

$$\nu_\Delta(\boldsymbol{z}) = (a_{i_1,\ldots,i_n})_{(i_1,\ldots,i_n)\in\Delta\cap\mathbb{Z}^n} \text{ where}$$
$$a_{i_1,\ldots,i_n} = w_1^{i_1} \ldots w_n^{i_n} \text{ if } (i_1, \ldots, i_n) \in \delta \text{ and } a_{i_1,\ldots,i_n} = 0 \text{ otherwise.}$$

Proposition 2.7 *The map ν_Δ is equivariant, continuous and surjective. Moreover, if the dimension of Δ is equal to n, then $\nu_{\Delta|I(\mathbb{C}\Delta)}$ is a diffeomorphism on its image.*

Definition 2.8 *An isotopy in $Tor_\mathbb{C}(\Delta)$ is said to be tame if for any face δ of Δ, the restriction of this isotopy to $Tor_\mathbb{C}(\delta)$ is an isotopy in $Tor_\mathbb{C}(\delta)$.*
Such an isotopy is said to be equivariant if it commutes with the complex conjugation.

2.1.3 Real and complex charts of a polynomial

These notions are among the fundamental ingredients of the Viro method.
Let $F = \sum_{(i_1,\dots,i_n)} a_{i_1,\dots,i_n} X_1^{i_1} \dots X_n^{i_n}$ be a polynomial in $\mathbb{C}[X_1,\dots,X_n]$ and $\Delta \subset \mathbb{R}_+^n$ be an integer convex polyhedron.

Definition 2.9 *The complex chart of the polynomial F in Δ, denoted by $\mathbb{C}Ch_\Delta(F)$, is the closure of the set $\mathbb{C}\mu_\Delta(\{F=0\} \cap (\mathbb{C}^*)^n) \in \mathbb{C}\Delta$.*
If F is a real polynomial, then its real chart in Δ, denoted by $\mathbb{R}Ch_\Delta(F)$, is defined as $\mathbb{C}Ch_\Delta(F) \cap \mathbb{R}\Delta$.
If $\Delta = \Delta(F)$, we simply denote the complex (resp., real) chart of F in $\Delta(F)$ by $\mathbb{C}Ch(F)$ (resp., $\mathbb{R}Ch(F)$).

Proposition 2.10 *For any face δ of Δ, one has $\mathbb{C}Ch(F) \cap \mathbb{C}\delta = \mathbb{C}Ch(F^\delta)$.*
If F is real, then $\mathbb{C}Ch(F)$ is invariant with respect to the action of complex conjugation.

2.1.4 Viro's Theorem

Let us fix k polynomials $F_1,\dots,F_k$ in $\mathbb{C}[X_1,\dots,X_n]$, and let $\Delta \subset (\mathbb{R}_+)^n$ be an integer convex polyhedron of dimension n such that

- $\Delta = \Delta(F_1) \cup \dots \cup \Delta(F_k)$,
- for all r and s, the intersection $\Delta(F_r) \cap \Delta(F_s)$ is either empty or a face δ of $\Delta(F_r)$ and $\Delta(F_s)$. Moreover, in the latter case, we have $F_r^\delta = F_s^\delta$.

Definition 2.11 *The set*

$$T\mathbb{C}Ch(F_1,\dots,F_n) = \nu_\Delta\left(\bigcup_{r=1}^{k} \mathbb{C}Ch(F_r)\right) \in Tor_{\mathbb{C}}(\Delta)$$

is called a C-hypersurface in $Tor_{\mathbb{C}}(\Delta)$.

Proposition 2.12 *If all the polynomials F_r are real, then $T\mathbb{C}Ch(F_1,\dots,F_n)$ is invariant with respect to the action of complex conjugation.*

Definition 2.13 *A subdivision $\Delta = \Delta_1 \cup \dots \cup \Delta_k$ of an integer convex polyhedron Δ is said to be convex if there exists a piecewise-linear convex function from Δ to $\mathbb{R}$ whose domains of linearity are the polyhedrons $\Delta_1,\dots,\Delta_k$.*

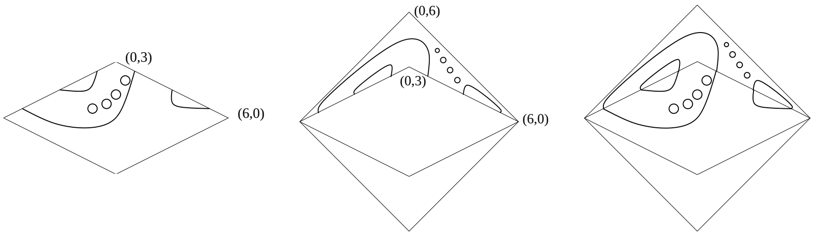
Let us write $F_r = \sum_{(i,j)\in\Delta(F_r)} a_{i,j} X^i Y^j$ for all $r \in \{1,\dots,k\}$.

Theorem 2.14 (Viro, [Vir84a]) *Suppose that all the polynomials F_r are non-degenerate and that the subdivision $\Delta = \Delta(F_1) \cup \dots \cup \Delta(F_k)$ is convex. Let us consider a piecewise-linear convex function μ certifying the convexity of the subdivision of Δ and define*

$$F_{t,\mu} = \sum_{(i,j)\in\Delta} a_{i,j} t^{\mu(i,j)} X^i Y^j.$$

Then, the hypersurface defined by $F_{t,\mu}$ is tame isotopic to the C-hypersurface $T\mathbb{C}Ch(F_1,\dots,F_n)$ in $Tor_{\mathbb{C}}(\Delta)$ for $t>0$ small enough.
Moreover, if all the F_r are real, this tame isotopy can be made equivariant.

One usually says that the hypersurface given by $F_{t,\mu}$ in $Tor_{\mathbb{C}}(\Delta)$ is obtained by gluing the charts of the F_r. The polynomial $F_{t,\mu}$ for t small enough is called a Viro polynomial of the gluing.
For example, Viro constructed in [Vir84a], non-degenerate polynomials whose real charts are depicted on Figures 2.1a) and b) with arbitrary truncation on the segment $[(0,3);(6,0)]$. Gluing two of them by the Viro Theorem as depicted on Figure 2.1c), one obtains the so called Gudkov curve in $\mathbb{R}P^2$.



a) b) c)

Figure 2.1:

A very famous particular case of the Viro Theorem is when all the $\Delta(F_i)$ are simplices and the only non-zero coefficients of the F_i are the vertices of $\Delta(F_i)$. This is the so called *T-construction* (see [Ite01], [KI96], [LdM], [IV], [Ber] for examples of applications of the T-construction).

2.1.5 Remarks on the development of the Viro method

Since the 70's, the Viro Theorem has been improved in order to show that it is still valid under weaker hypothesis. As we won't use these improvements in this text, we just mention some of them.

2.1.5.1 Gluing of singular varieties

One of the conditions to apply the Viro Theorem is to glue polynomials which are nonsingular in $(\mathbb{C}^*)^n$ as well as their truncation on the faces of their Newton polyhedron. In [Shu98] and [Shu05], Shustin showed that, under some conditions on singularities, Theorem 2.14 remains true even if the F_i or their truncations have singularities in $(\mathbb{C}^*)^n$. One can construct singular hypersurfaces in this way, and Shustin's conditions can easily be checked in the case of curves. In particular, using these improvements, Shustin constructed in [Shu98] real algebraic curves in $\mathbb{R}P^2$ with many cusps and proved in [Shu05] the tropical correspondence theorem in a way different from that of G. Mikhalkin (see [Mik]).

2.1.5.2 Using nonconvexe subdivisions

What can one say about the C-hypersurface $T\mathbb{C}Ch(F_1,\ldots,F_n)$ in $Tor_{\mathbb{C}}(\Delta)$ if the subdivision of Δ is nonconvexe? In the case of curves in $\mathbb{C}P^2$ or the rational geometrically rules surfaces, Itenberg and Shustin proved in [IS02] that one obtains pseudoholomorphic curves. Their gluing Theorem also works with singular varieties, without the conditions given in [Shu98] for algebraic curves. In this way, they constructed real pseudoholomorphic curves in $\mathbb{C}P^2$ with a collection of singular points that no known algebraic curve has.

2.2 Perturbing singular curves

For the definition of the type of a singular point, one can refer to [AGZV85] or [Vir89].
Almost all known isotopy types realized by nonsingular real algebraic curves in toric surfaces have been constructed in the following way : start with a curve having isolated singularities and perturb them. The word "perturb" means here "move a little this curve inside a given linear system". Since L. Brusotti's work (see [Bru21]), it is known that if a curve has only nondegenerate double points, one can perturb any of them independently from the others. All the isotopy types realized by curves of degree less or equal than 6 in $\mathbb{R}P^2$ can be obtained using Brusotti's Theorem. However, this kind of constructions starts to be very complicated with the degree 6, and no one managed to achieved the classification for the degree 7 perturbing curves with only nondegenerate double points.
The Viro method has provided a way to perturb more complicated singularities, called *Newton nondegenerate*. Perturbing curves of degree 7 with two singular points of type J_{10} (i.e. three branches with second order tangency), Viro achieved the classification of real schemes realizable by nonsingular real algebraic curves of degree 7 in [Vir84a]. Several nonsingular real algebraic curves of degree 8 were constructed by perturbing other singularities (see [Vir89] for more details). However, the realizability of some real schemes by curves of degree 8 remained unknown and it appeared quite clear that one should deal with more complicated singularities than the Newton nondegenerate ones. Indeed, new curves of degree 8 were constructed by perturbing *generalized Newton nondegenerate* singular points (see [Shu87] and [Che02]).
Modifying the original Viro method, Shustin in [Shu99] gave a method to perturb curves in toric surfaces with arbitrary many generalized Newton nondegenerate singular points arbitrary placed in the surface. Using Shustin's results, S. Yu. Orevkov constructed several reducible curves of degree 7 (see [Ore98a], [Ore98b] and [Ore] for examples).

2.2.1 Perturbing one Newton nondegenerate singular point

Here, we follow [Vir89] and [Virb].
Let C be a nondegenerate polynomial in $\mathbb{C}[X,Y]$. For a face δ of $\Delta(C)$, we denote by $K(\delta)$ the convex hull of $\delta \cup \{0\}$.

Definition 2.15 *The Newton diagram $\Gamma(C)$ of C at the origin is the union of the faces δ of $\Delta(C)$ such that $K(\delta)$ has dimension 2 and $\Delta(C) \cap K(\delta) = \delta$.*

We will denote by $K(\Gamma(C))$ the union of the closures of $K(\delta)$ taken over all faces δ of $\Gamma(C)$.
Let us suppose that the curve C in $\mathbb{C}^2$ has an isolated singularity at $(0,0)$.

Definition 2.16 *We say that the origin is a Newton nondegenerate singular point of C if*

- *the Newton diagram $\Gamma(C)$ of C touches the axes $\{X=0\}$ and $\{Y=0\}$,*
- *for every face δ of $\Gamma(C)$, the polynomial C^δ is nondegenerate.*

The Viro method can be seen as a way to perturb such a singular point of a curve. Consider a nondegenerate algebraic curve C in $\mathbb{C}^2$ with an isolated Newton nondegenerate singular point at the origin, and assume that the curve is smooth everywhere else. That means in particular that the point $(0,0)$ does not belong to $\Delta(C)$. Now, using the Viro Theorem, glue to C some curves whose Newton polygons are contained in $K(\Gamma(C))$. We choose the function μ mentioned in the Viro Theorem to be null on $\Delta(C)$ (it is always possible) and we denote by $\widetilde{C}_{t,\mu}$ the Viro polynomials of the gluing. Then, as $\mu_{|\Delta(C)} = 0$, $\widetilde{C}_{t,\mu}$ tends to C as t tends to 0. Thus, the curves $\widetilde{C}_{t,\mu}$ can be viewed as perturbations of C.

For example, in the gluing depicted on Figure 2.1, let us choose the piecewise-linear convex function satisfying

$$\mu(0,0) = 1, \mu(0,3) = \mu(6,0) = \mu(0,6) = 0.$$

Then the curves $\widetilde{C}_{t,\mu}$ for $t > 0$ can be seen as smoothings of the singular point of type J_{10} of the curve $\widetilde{C}_{0,\mu}$ as depicted on Figure 2.2.

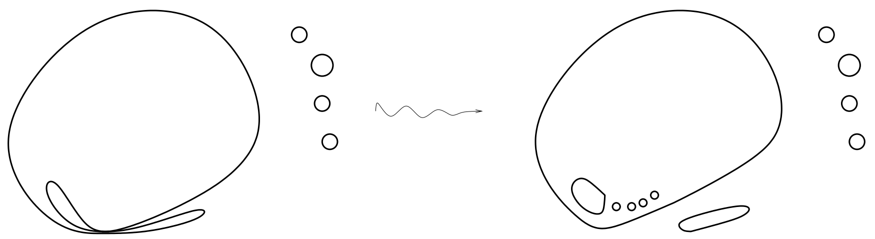


Figure 2.2:

Remark. One can use the Viro method as explained above to perturb any singular algebraic curve in $\mathbb{C}P^2$ whose singularities lie among the points $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$ and are Newton nondegenerate.

2.2.2 Perturbing many generalized Newton nondegenerate singular points

Here, we follow [Shu98] and [Shu99].

Definition 2.17 *Let C be a curve in a nonsingular algebraic surface S. A point z of C is said to be a generalized Newton nondegenerate (briefly, GNND) singular point if there exists a neighborhood U of z in S, a neighborhood V of $(0,0)$ in $\mathbb{C}^2$ and a biholomorphism $\phi : U \to V$ such that $\phi(z)$ is a Newton nondegenerate singular point of $\phi(C \cap U)$.*
An equation of $\phi(C \cap U)$ is called a representative of the GNND singular point z of C.
If all the objects are real, ϕ can be made equivariant.

In [Shu99], Shustin gave conditions to perturb algebraic hypersurfaces (not only curves) with many GNND singular points. The idea is to perturb one of the representative of each singular point using the Viro method as explained in the previous section, and to check if all the perturbations can be done independently on C. Shustin's conditions are quite technical, so we restrict ourselves here to give all the corresponding results in the case of curves, when the singular points are either smoothed or kept. All the criterions given in [Shu99] are formulated for curves in $\mathbb{C}P^2$. However, results from [Shu98] allow one to formulate those criterions for singular curves on other toric surfaces and to obtain perturbations preserving Newton polygon.
Given a curve C in an algebraic surface nonsingular S, we denote by $Sing(C)$ the set of its singular points.

2.2.2.1 Deformations of singular points

Definition 2.18 *Let C_1 (resp., C_2) be an algebraic curve in a nonsingular algebraic surface S and z_1 (resp., z_2) a point of C_1 (resp., C_2). The points z_1 and z_2 are said to be topologically equivalent if there exists a neighborhood U_1 (resp., U_2) of z_1 (resp., z_2) in S and an homeomorphism of U_1 to U_2 that takes $C_1 \cap U_1$ to $C_2 \cap U_2$.*

Let C be an algebraic curve in S and z a singular point of C. Following J. Milnor (see [Mil68]), consider a sufficiently small neighborhood B of z in S diffeomorphic to the 4-ball B^4 such that

- $C_0 = C \cap B$ is a compact variety with an isolated singular point z and boundary $\partial C_0 \subset \partial B$,
- C_0 is transversal to ∂B along ∂C_0 and homeomorphic to a cone over ∂C_0.

Put $D_\epsilon = \{t \in \mathbb{C} : |t| < \epsilon\}$ and $\pi : B \times D_\epsilon \to D_\epsilon$ the projection.

Definition 2.19 *A (one parametric) deformation of a singular point z of C is an analytic hypersurface $W \subset B \times D_\epsilon$ such that*

- $(\pi_{|W})^{-1}(0) = C_0$,
- *for t in D_ϵ, $C_t = (\pi_{|W})^{-1}(t)$ is a compact variety with isolated singularities in $Int(B)$, whose boundary ∂C_t is contained in ∂B, and C_t is transversal to ∂B along ∂C_t.*

Moreover, we say that the deformation is topologically compatible if for any t_1 and t_2 in D_ϵ, there exists a bijection between $Sing(C_{t_1})$ and $Sing(C_{t_2})$ such that the corresponding points in $Sing(C_{t_1})$ and $Sing(C_{t_2})$ are topologically equivalent.
If z, C and W are real, then we take only real nonnegative values of the parameter t and consider $W \cap (B \times [0;\epsilon])$.
If all the W_t are nonsingular for $t \neq 0$, we speak about a (one parametric) smoothing of z.

Now define *models for smoothing a GNND singular point.* Let z be a GNND singular point of C, and F one of its representatives. Suppose we are given k polynomials $F_1, \dots, F_k$ in $\mathbb{C}[X,Y]$ such that

- $\Delta(F_1), \dots, \Delta(F_k)$ form a subdivision of $K(\Gamma(F))$,
- the polynomials $F, F_1, \dots, F_k$ verify the hypothesis of the Viro Theorem.

Put $\Phi = \bigcup_{\alpha=1}^{k} \mathbb{C}Ch(F_\alpha) \subset \mathbb{C}K(\Gamma(F)$.

Definition 2.20 *The pair $(\mathbb{C}K(\Gamma(F)), \Phi)$ is called a model for a topologically compatible smoothing W of $z \in C$ if for any $t \neq 0$ there exists a homeomorphism $(B, C_t) \to (\mathbb{C}K(\Gamma(F)), \Phi)$.*
If all the objects are real, the homeomorphism must be taken equivariant.

Remark. One can also define models for a topologically compatible deformation of a GNND singular point (see [Shu99]).

2.2.2.2 A topological invariant of curve singularities

Let C be an algebraic curve in an algebraic smooth surface S, z an isolated singular point of C, T its embedded resolution tree, C_q the strict transform of C corresponding to an infinitely near point $q \in T$, and E_q the reduced exceptional divisor (see [Sha77]).
We will say that a point q of T is *terminal* if it is nonsingular both for C_q and E_q. We remove from T all the points following terminal points.

Definition 2.21 (see [Shu99]) *Define $b(C,z)$ and $b(C,Q)$ as follows, where Q is a local branch of C at z :*

- *if z is of type A_{2k-1}, $k \geq 1$, then C has two nonsingular branches Q_1, Q_2 at z and put $b(C,Q_1) = b(C,Q_2) = k-1$,*

- *if z is of type A_{2k}, $k \geq 1$, then C has one branch Q at z and put $b(C,Q) = 2k-1$,*
- *if z is not of type A_k, let Q be a local branch of C at z. Let $b'(C,Q)$ be the sum of the multiplicities of the strict transform Q_q of Q at nonterminal points $q \in T$, and $b''(C,Q)$ be $Q_q \circ E_q - 1$, where q is the terminal point of Q; finally, put $b(C,Q) = b'(C,Q) + b''(C,Q)$,*
- *$b(C,z)$ is the sum of $b(C,Q)$ over all branches of C centered at z.*

2.2.2.3 Perturbing curves with GNND singular points

Let C be an algebraic curve in a projective algebraic nonsingular toric surface S.

Theorem 2.22 (Shustin,[Shu98], [Shu99]) *Suppose that the set $Sing(C)$ splits into two disjoint subsets S and S' such that the singular points $z \in S'$ are GNND. Suppose also that for any point of S', we are given a smoothing model $(\mathbb{C}K(\Gamma(F_z)), \Phi_z)$ and that*

- *if C is irreducible,*
$$\sum_{z \in Sing(C)} b(C,z) < Card(\partial\Delta(C) \cap \mathbb{Z}^2),$$
- *if C splits into irreducible components $C_1, \dots, C_r$,*
$$\sum_{z \in Sing(C)} \sum_{Q_z^{(i)}} b(C, Q_z^{(i)}) < Card(\partial\Delta(C_i) \cap \mathbb{Z}^2) \text{ for all } i \in \{1, \dots, r\}$$

  where $Q_z^{(i)}$ is a local branch of C_i at z.

Then there exists a family of curves C_t with Newton polygon $\Delta(C)$ with $|t| < \epsilon$ such that $C_0 = C$ and that the family C_t realizes

- *for any $z \in S$, a topological equisingular deformation of $z \in C$,*
- *for any $z \in S'$, a topological smoothing of $z \in C$ with the model $(\mathbb{C}K(\Gamma(F_z)), \Phi_z)$.*

If all the data are real, then there exists an equivariant family of curves C_t with Newton polygon $\Delta(C)$ with $t \in [0; \epsilon[$ with the above properties.

An example of application of Shustin's Theorem is another proof of a result of B. Chevallier in [Che02] on the smoothing of four conics that have a single point in common at which they intersect with multiplicity 4. Using this method, Chevallier has constructed four M-curves of degree 8 whose realizability was previously unknown. One of these smoothing is depicted on Figure 2.3.

2.3 Rational geometrically ruled surfaces

A *rational geometrically ruled surface* is a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$ that is to say a complex surface S equipped with a fibration $\pi : S \to \mathbb{C}P^1$ with fiber $\mathbb{C}P^1$. The two simplest examples are $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2$ blown up in a point. In the latter case, the fibration is given by the extension to the blowing up of the projection from p to a line which does not pass through p. If S is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, let us denote by E any line $\mathbb{C}P^1 \times \{x\}$. Otherwise, there exists a unique nonsingular algebraic section E on S, called the exceptional section, with a negative auto-intersection. The classification of rational geometrically ruled surfaces up to biholomorphism is well known (see [Bea83]).

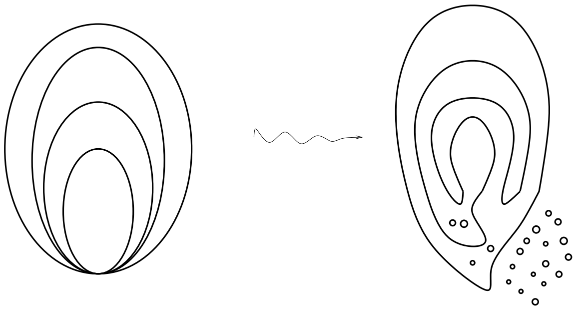


Figure 2.3:

Theorem 2.23 *Two rational geometrically ruled surfaces are biholomorphic if and only if their exceptional sections have the same auto-intersection.*

Definition 2.24 *If the auto-intersection of E is equal to $-n$, then the surface S is called the n^{th} rational geometrically ruled surface and is denoted by Σ_n.*

One can note that from a smooth point of view, the situation is completely different, and there are only two rational geometrically ruled surfaces, namely Σ_0 and Σ_1 (see [MS98]).

Proposition 2.25 *The surfaces Σ_{n_1} and Σ_{n_2} are diffeomorphic if and only if n_1 and n_2 have the same parity.*

The surface Σ_n can be obtained by taking four copies of $\mathbb{C}^2$ with coordinates (X_1, Y_1), (X_2, Y_2), (X_3, Y_3) and (X_4, Y_4), and by gluing them along $(\mathbb{C}^*)^2$ with the identifications

$$(X_2, Y_2) = (1/X_1, Y_1/X_1^n),\ (X_3, Y_3) = (X_1, 1/Y_1) \text{ and } (X_4, Y_4) = (1/X_1, X_1^n/Y_1).$$

The coordinate system (X_1, Y_1) is said to be *standard*.
For Σ_n the fibration is given by $\pi : (X, Y) \mapsto X$.
If $n \geq 1$, the exceptional section is given by the equation $\{Y_3 = 0\}$. Let us denote by B (resp., F) the curve given by the equation $\{Y_1 = 0\}$ (resp., $\{X_1 = 0\}$). We have $B \circ B = n$, $E \circ E = -n$, $F \circ B = 1$ and $F \circ F = 0$.
The surface Σ_n is also a projective toric surface defined by the trapeze with the vertices $(0,0)$, $(0,1)$, $(1,1)$ and $(1+n, 0)$. Hence, one can use the Viro Theorem to construct (real) algebraic curves in Σ_n. The surface Σ_n has a natural real structure induced by the complex conjugation in $\mathbb{C}^2$.

Proposition 2.26 *The real part of Σ_n is a torus if n is even and a Klein bottle if n is odd.*

We will depict the real part of Σ_n as a quadrangle whose opposite sides are identified in a suitable way. Moreover, the two horizontal sides will represent the exceptional divisor.
The group $H_1(\Sigma_n, \mathbb{Z})$ is isomorphic to $\mathbb{Z}\times\mathbb{Z}$ and is generated by the classes of B and F. Moreover, one has $E = B - nF$. An algebraic curve on Σ_n is said to be of *bidegree* (k, l) if it realizes the homology class $kB + lF$ in $H_1(\Sigma_n, \mathbb{Z})$. The Newton polygon of such a curve in a standard coordinate system lies inside the trapeze defined by the vertices $(0,0)$, $(0,k)$, (l,k), and $(nk + l, 0)$.

Definition 2.27 *A curve of bidegree $(1,0)$ is called a base.*
A curve of bidegree $(3,0)$ is called a trigonal curve.

One can construct the surface Σ_{n+1} starting with Σ_n : blow up a point of the exceptional section and blow down the strict transform of the fiber.
In the rational geometrically ruled surfaces, we have a natural pencil of lines (i.e. a smooth $\mathbb{C}P^1$-bundle over $\mathbb{R}P^1$), which will be denoted by $\mathcal{L}$, given by the real lines of the fibration. So, it is natural to study not only isotopy types of real algebraic curves in Σ_n, but also *isotopy types with respect to* $\mathcal{L}$. Two curves will be said to be isotopic with respect to $\mathcal{L}$ if there exists an isotopy of Σ_n which brings the first curve to the second one, and which transforms each line of $\mathcal{L}$ in another line of $\mathcal{L}$.

Definition 2.28 *An arrangement of circles A, which may be nodal, in Σ_n up to isotopy of $\Sigma_n \setminus E$ which respects the pencil of lines $\mathcal{L}$ is called an $\mathcal{L}$-scheme.*
An $\mathcal{L}$-scheme is realizable by a real algebraic (or pseudoholomorphic, see section2.4) curve of bidegree (k,l) in Σ_n if there exists such a curve whose real part is isotopic with respect to $\mathcal{L}$ to A in $\mathbb{R}\Sigma_n$.
A trigonal $\mathcal{L}$-scheme is an $\mathcal{L}$-scheme which intersects each fiber in 1 or 3 points and which does not intersect the exceptional section.

Proposition 2.29 *There exists a unique base of Σ_n which passes through $n+1$ given generic points in Σ_n.*

Proof. The equation for such a base in a standard coordinate system is $Y+\sum_{i=0}^{n} a_iX^i$, where a_i are some complex numbers. The condition to pass through $n+1$ points imposes $n+1$ linear equations on the a_i. If the points are generic, then the obtained system of linear equations has a unique solution. □

2.4 Pseudoholomorphic curves

Analyzing restrictions on the topology of real algebraic curves in $\mathbb{R}P^2$, Viro noticed that almost all of them have a topological origin. He introduced in [Vir84c] *flexible curves* which are smooth surfaces embedded in $\mathbb{C}P^2$, invariant by the complex conjugation and which satisfy all the topological restrictions on plane real algebraic curves.
Let C_1 and C_2 two smooth oriented 2-submanifold of an oriented smooth 4-manifold X, let x be a isolated common point of C_1 and C_2, and let (v_1, v_2) (resp., (v_3, v_4)) be a positive basis of T_xC_1 (resp., T_xC_2).

Definition 2.30 *The intersection index of C_1 and C_2 at x is defined as follows*

- *if C_1 and C_2 intersect transversally at x, the intersection index is 1 (resp., -1) if (v_1, v_2, v_3, v_4) is a positive (resp., negative) basis of T_xX,*
- *if C_1 and C_2 do not intersect transversally at x, let U be a small ball in X centered on x such that x is the only intersection point of C_1 and C_2 in U. Perform a smooth isotopy of C_1 in U such that all the intersection points in U of the obtained curve and C_2 are transverse. The sum of the index of all these intersection points is the intersection index of C_1 and C_2 at x.*

If the intersection index at x is positive (resp., negative), the point x is said to be a positive (resp., negative) intersection point of C_1 and C_2.

An intersection point of two flexible curves is not necessarily positive. In particular, the Bézout theorem is not true for those curves. Later, Orevkov considered in [Ore99] a subclass of flexible curves, *pseudoholomorphic curves.* These objects, introduced by M. Gromov in [Gro85] to study

symplectic 4-manifolds, verify the Bézout theorem (if they are J-holomorphic with respect to the same almost complex structure J). Moreover, they are much easier to handle. Indeed, there are methods to construct pseudoholomorphic curves which are not necessarily algebraic (braid theoretical method of Orevkov, see section 2.7, T-construction using nonconvexe triangulations, see [IS02]). For example, the classification up to isotopy of real pseudoholomorphic M-curves of degree 8 in $\mathbb{R}P^2$ was achieved by Orevkov [Ore02a]. However, it remains 6 real schemes for which it is unknown whether they are realizable by algebraic M-curves of degree 8 in $\mathbb{R}P^2$ or not.
More references and material about pseudoholomorphic curves can be found in [MS98].

Definition 2.31 *Let X be a differential manifold. A symplectic form on X is 2-form on X such that*

- *ω is closed,*
- *for all x in X, ω is nondegenerate on T_xX (i.e. $\forall v \in T_xX, \omega(u,v) = 0 \Rightarrow u = 0$).*

A pair (X, ω), where ω is a symplectic form on X, is called a symplectic manifold.

A straightforward consequence of this definition is that a symplectic manifold must have an even dimension.

Definition 2.32 *An almost complex structure J on X is a smooth family $(J_p)_{p\in X}$ of linear maps $J_p : T_pX \to T_pX$ such that $J_p^2 = -Id$.*
We say that J is tame with respect to ω if

$$\forall p \in X, \forall v \in T_pX, v \neq 0, \omega(v, J_p(v)) > 0.$$

An example of almost complex structure on a symplectic manifold is $\mathbb{C}P^n$ with its usual complex structure and the Fubini-Study form (see [MS98]).
From now on, let (X, ω) be a symplectic 4-manifold, and J a tame almost complex structure on (X, ω).

Definition 2.33 *A Riemann surface A immersed in X is called a J-holomorphic curve if*

$$\forall p \in A, \forall v \in T_pA, J_p(v) \in T_pA.$$

The following proposition ensures that the Bézout theorem is true for J-holomorphic curves.

Proposition 2.34 (McDuff, [McD94]) *All the intersection points of two J-holomorphic curves are positive.*

Now, we suppose that $X = \mathbb{C}P^2$ or Σ_n. Then we have the standard complex conjugation on X, denoted by $conj$.

Definition 2.35 *The almost complex structure J is real if $conj_* \circ J_p = -J_p \circ conj_*$.*

Definition 2.36 *A Riemann surface A is called a real pseudo-holomorphic curve in X if*

- *A is J-holomorphic for some real almost complex structure J on X,*
- *If $X = \Sigma_n$ with $n \geq 1$, the exceptional divisor of X is J-holomorphic,*
- *$conj(A) = A$.*

As an example, a real algebraic curve in $\mathbb{C}P^2$ is a real pseudo-holomorphic curve for the standard complex structure and the Fubini-Study form. If the curve C is immersed in $\mathbb{C}P^2$ (resp., Σ_n) and realizes the homology class $d[\mathbb{C}P^1]$ in $H_2(\mathbb{C}P^2, \mathbb{Z})$ (resp., $kB + lF$ in $H_2(\Sigma_n, \mathbb{Z})$), then C is said to be of degree d (resp., of bidegree (k, l)). All intersections of two J-holomorphic curves are positive, so the Bézout theorem is still true for two J-holomorphic curves.

Proposition 2.37 (Gromov, [Gro85]) *There exists a unique J-line passing through 2 given points in $\mathbb{C}P^2$.*
There exists a unique J-conic passing through 5 given generic points in $\mathbb{C}P^2$.
Moreover, if J and the configurations of points are real, then so are the J-line and the J-conic.

Remark. As soon as the degree is greater than 3, the uniqueness is no more true. This is a direction to find some differences between algebraic and pseudo-holomorphic curves (see [FLTO02], where the authors use pencils of cubics).

Corollary 2.38 *Suppose that $n = 1, 2$ or 3. Then, there exists a unique J-holomorphic base of Σ_n which passes through $n + 1$ given generic points in Σ_n.*

2.5 Real Curves

A real algebraic curve is a compact algebraic curve C defined over $\mathbb{C}$ equipped with an antiholomorphic involution c. The set of fixed points of c is denoted by $\mathbb{R}C$ and is called *the real part* of the curve C. In this section, the expression *real curve* means *real algebraic curve* or *real pseudoholomorphic curve*. The latter case implies that the curve is supposed to be embedded in a tame almost complex symplectic variety of dimension 4.
A more detailed exposition can be found in the surveys [Wil78], [Vir84c], and [DK00].

2.5.1 General facts

The following proposition was first proved by A. Harnack in the case of plane projective real algebraic curves. F. Klein has reformulated and proved it in a more general setting.

Proposition 2.39 (Klein,[Kle22]) *Let S be a smooth compact connected surface of genus g, equipped with a smooth involution c which has no isolated fixed points. Then, the set of fixed points of c has at most $g + 1$ connected components.*

Hence, if we know the genus of a nonsingular real curve, we also know the maximal number of connected components of its real part. For curves in toric surfaces, this is given by the next proposition.

Proposition 2.40 (see, for example, [Ful93]) *Let C be a nonsingular algebraic curve with Newton polygon Δ in a toric surface. Then the genus of C is equal to the number of integer points in the interior of Δ.*

Remark. If the toric surface is $\mathbb{C}P^2$ or Σ_n, then Proposition 2.40 is also valid for pseudoholomorphic curves.

Definition 2.41 *A nonsingular real curve C of genus g such that $\mathbb{R}C$ has $g + 1 - i$ connected components is called an $(M - i)$-curve.*
If $i = 0$, the curve is simply called an M-curve or a maximal curve.

A nonsingular real curve C is a 2-dimensional object and $C \setminus \mathbb{R}C$ is either connected or it has two connected components. In the former case, we say that $\mathbb{R}C$ is a *non dividing curve*, or of type II, and in the latter case, we say that $\mathbb{R}C$ is a *dividing curve*, or of type I.

Lemma 2.42 (Klein,[Kle22]) *An M-curve is of type I.*

Proposition 2.43 (Klein,[Kle22]) *If an $(M-i)$-curve is of type I, then i is even.*

2.5.2 Real curves in the projective plane

Historically, the first result in topology of real algebraic curves in the projective plane was obtained by Harnack. One can see it as a consequence of Propositions 2.39 and 2.40.

Theorem 2.44 (Harnack, [Har76]) *A nonsingular real algebraic curve in $\mathbb{C}P^2$ of degree d has at most $\frac{(d-1)(d-2)}{2}+1$ connected components.*

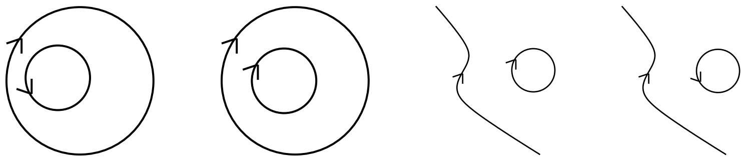
As $H_1(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, a connected component of a nonsingular curve in $\mathbb{R}P^2$ can be situated in two topologically distinct ways : either it realizes the 0 homology class, or it realizes the 1 homology class. In the former case, the component does not disconnect $\mathbb{R}P^2$ and is called a *pseudo-line* of the curve, and in the former case, it is called an *oval*.
All the connected components of a curve of even degree are ovals, and a curve of odd degree has exactly one pseudo-line. The complement of an oval is formed by two connected components, one of which is homeomorphic to a disk (called the *interior* of the oval) and the other to a Möbius strip (called the *exterior* of the oval).

Definition 2.45 *An oval of a nonsingular real plane curve is said to be empty (resp., non-empty) if it contains no other (resp., some other) oval of the curve.*

Two ovals in $\mathbb{R}P^2$ are said to constitute an *injective pair* if one of them is enclosed by the other. A set of ovals, each pair of which is injective is called a *nest*. The number of ovals in a nest is called the *depth* of the nest. The following lemma is a direct consequence of the Bézout theorem.

Lemma 2.46 *A nest of a nonsingular real pseudoholomorphic curve of degree m in $\mathbb{R}P^2$ has a depth less or equal to $[\frac{m}{2}]$. A nest of depth $[\frac{m}{2}]$ is called a maximal nest.*

Let C be a nonsingular dividing pseudoholomorphic curve in $\mathbb{R}P^2$. Then the two halves of $C \setminus \mathbb{R}C$ induce two opposite orientations on $\mathbb{R}C$ which are called *complex orientations* of the curve. An injective pair of ovals of $\mathbb{R}C$ is said to be *positive* if the orientations of the two ovals are induced by one of the orientations of the annulus in $\mathbb{R}P^2$ bounded by the two ovals (see figure 2.4a)), and *negative* otherwise (see figure 2.4b)). Let us denote by Π_+ (resp., Π_-) the number of positive (resp., negative) injective pairs of ovals of the curve. If the degree of C is odd, one can also speak about positive and negative ovals. Let us consider one oval of a dividing curve of odd degree. If the integral homology classes realized by the pseudo-line of the curve and the ovals in the Möbius strip defined by the exterior of this oval have the same sign, we say that the oval is *negative* (see figure 2.4c)), and *positive* (see figure 2.4d)) otherwise. Let us denote by Λ_+ (resp., Λ_-) the number of positive (resp., negative) of ovals of the curve.
Complex orientations were introduced by V. A. Rokhlin in [Rok74] and turned out to be an efficient tool in the study of plane curves. The first application was the following proposition.



a) b) c) d)

Figure 2.4:

Proposition 2.47 (Rokhlin-Mischachev's orientation formula, [Rok74], [Mis75]) *If C is a dividing nonsingular pseudoholomorphic curve of degree d in $\mathbb{R}P^2$ with l ovals, then*

$$2(\Pi_+ - \Pi_-) = l - k^2 \text{ if } d = 2k$$

$$\Lambda_+ - \Lambda_- + 2(\Pi_+ - \Pi_-) = l - k(k+1) \text{ if } d = 2k+1$$

In [Rok74], Rokhlin proved the orientation formula for curves of even degree. Later, N. M. Mischachev proved in [Mis75] the part relative to curves of odd degree.

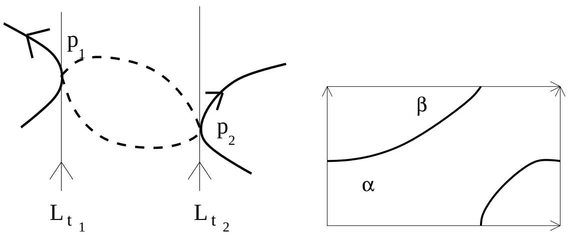
2.5.3 Real curves in Σ_n

Taking into account the arrangement of a curve in Σ_n with respect to the pencil of lines can be useful to extract some information about this curve. For example, the following theorem, due to T. Fiedler, permits to determine the complex orientations of a dividing curve in some cases (see [Fie83], [Vir84c], and [Tri01]). One can define complex orientations of a real dividing curve in Σ_n as for real dividing curve in $\mathbb{R}P^2$.

Theorem 2.48 (Fiedler's orientations alternating rule, [Fie83]) *Let C be a real curve of type I in Σ_n and $(L_t)_{t\in[t_1;t_2]}$ a portion of the pencil $\mathcal{L}$ such that (see figure 2.5):*

- *the line L_{t_i} is tangent to $\mathbb{R}C$ in p_i, $i = 1, 2$,*
- *there exists a connected component of $(\bigcup_{t\in[t_1;t_2]} L_t) \cap C$, invariant by complex conjugation whose real part is made of p_1 and p_2.*
- *the lines of $(\mathbb{R}L_t)_{t\in[t_1;t_2]}$ are compatibly oriented.*

Then, if the orientations of $\mathbb{R}C$ and $\mathbb{R}L_{t_1}$ are compatible in p_1, so are those of $\mathbb{R}C$ and $\mathbb{R}L_{t_2}$ in p_2.



a) b)

Figure 2.5:

Remark. Blowing down the exceptional divisor on Σ_1, we can apply Theorem 2.48 for a dividing curve and a pencil of lines in $\mathbb{C}P^2$.

The following lemma deals with real curves of bidegree $(3, 1)$ in Σ_2, and will be useful in Chapter 5.

Lemma 2.49 *Let C be a curve of bidegree $(3,1)$ on Σ_2, whose real part is depicted in Figure 2.5b). We orient $\mathcal{L}$, the pencil of lines of Σ_2, from the left to the right. Then, either the lines of $\mathcal{L}$ meet first all the ovals β and then all the ovals α or the converse.*
Moreover, if $\alpha + \beta \geq 3$, then, the lines of $\mathcal{L}$ meet first all the ovals β and then all the ovals α.

Proof. Suppose there exist 3 ovals contradicting the lemma. Choose a point inside each of these ovals. Then the base passing through the chosen points intersects the curve in 9 points, and we get a contradiction with the Bézout theorem. □

2.5.4 Real curves of degree 7 in $\mathbb{R}P^2$

We recall here some known classification results about curves of degree 7. The rigid isotopic classification for this degree seems far to be achieved. We also give some results we will use in Chapter 5.
First, Lemma 2.46 implies that a pseudoholomorphic curve of degree 7 in $\mathbb{R}P^2$ has at most one nest of depth greater or equal than 2. Moreover, this nest has depth 2 if the curve has at least 2 empty ovals.

Definition 2.50 *For a nonsingular real pseudoholomorphic curve of degree 7 in $\mathbb{R}P^2$ with a nest and with at least 2 empty ovals, the ovals lying inside the non-empty oval are called the inner ovals while those lying outside are called the outer ovals.*

Given a curve of odd degree, one can define a notion of convexity in $\mathbb{R}P^2$: the segment defined by two points a and b is the connected component of the line $(a,b) \setminus \{a,b\}$ which has an even number of intersection points with the pseudo-line of the curve.

Lemma 2.51 *If a real pseudoholomorphic curve of degree 7 with at least 6 ovals has a nest, then one can choose a point in each inner oval such that these points are the vertices of a convex polygon in $\mathbb{R}P^2$. Moreover, if a line L passes through two outer ovals O_1 and O_2 of the curve and separates the inner ovals in two non-empty groups, then O_1 and O_2 does not intersect the same connected component of $L \setminus (Int(O) \cup \mathcal{J})$, where O is the non-empty oval.*

Proof. Suppose that there exist four empty ovals contradicting the lemma as depicted in Figure 2.6a) and b). Then a conic passing through these ovals and another one intersects the curve in at least 16 points, and we get a contradiction with the Bézout theorem. □

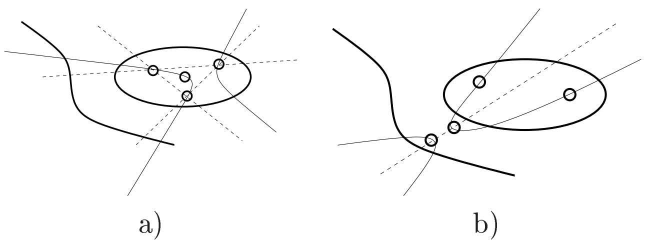


Figure 2.6:

Definition 2.52 *An arrangement of disjoint circles in $\mathbb{R}P^2$ up to isotopy is called a real scheme. A real scheme A is realizable by a real algebraic (or pseudoholomorphic) curve of degree d in $\mathbb{C}P^2$ if there exists such a curve whose real part is isotopic A in $\mathbb{R}P^2$.*

The classification of real schemes realizable by nonsingular plane curves of degree 7 has been obtained by Viro in the late 70's (see [Vir84a]). The notations used to encode real schemes are the usual ones proposed in [Vir84c][1].

Theorem 2.53 (Viro, [Vir84a]) *Any nonsingular real pseudoholomorphic curve of degree* 7 *in* $\mathbb{R}P^2$ *has one of the following real schemes :*

- $\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle$ *with* $\alpha+\beta \leq 14$, $0 \leq \alpha \leq 13$, $1 \leq \beta \leq 13$,
- $\langle J \amalg \alpha\rangle$ *with* $0 \leq \alpha \leq 15$,
- $\langle J \amalg 1\langle 1\langle 1\rangle\rangle\rangle$.

Moreover, any of these 121 real schemes is realizable by nonsingular real algebraic curves of degree 7 *in* $\mathbb{R}P^2$.

The second natural question is to determine real schemes that are realizable by dividing and non dividing curves. The case of $(M-2)$-curves was done by S. Fiedler-Le Touzé (see [FLT97]). Surprisingly, we did not succeed to find the complete statements in the literature, despite they are quite easy to obtain. The proof of these prohibitions is a straightforward application of the Bézout theorem, the Fiedler orientation alternating rule (see section 2.3) and the Rokhlin-Mischachev orientation formula. All the constructions are performed in [Sou01] and [FLT97].

Proposition 2.54 *Any nonsingular dividing real pseudoholomorphic curve of degree* 7 *in* $\mathbb{R}P^2$ *has one of the following real schemes :*

- $\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle$ *with* $\alpha+\beta \leq 14$, $\alpha+\beta = 0$ *mod* 2, $0 \leq \alpha \leq 13$, $1 \leq \beta \leq 13$, *if* $\alpha = 0$ *then* $\beta \neq 2, 6, 8$ *and if* $\alpha = 1$ *then* $\beta \geq 5$,
- $\langle J \amalg \alpha\rangle$ *with* $6 \leq \alpha \leq 15$, $\alpha = 1$ *mod* 2,
- $\langle J \amalg 1 \amalg 1\langle 1\langle 1\rangle\rangle\rangle$.

Moreover, any of these real schemes is realizable by nonsingular dividing real algebraic curves of degree 7 *in* $\mathbb{R}P^2$.

Proposition 2.55 *Any nonsingular non-dividing real pseudoholomorphic curve of degree* 7 *in* $\mathbb{R}P^2$ *has one of the following real schemes :*

- $\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle$ *with* $\alpha+\beta \leq 13$, $0 \leq \alpha \leq 12$, $1 \leq \beta \leq 13$,
- $\langle J \amalg \alpha\rangle$ *with* $0 \leq \alpha \leq 14$.

Moreover, any of these real schemes is realizable by nonsingular non-dividing real algebraic curves of degree 7 *in* $\mathbb{R}P^2$.

Definition 2.56 *An arrangement of disjoint oriented circles in* $\mathbb{R}P^2$ *up to isotopy and up to reversing all the orientations is called a complex scheme.*
A complex scheme A *is realizable by a real algebraic (or pseudoholomorphic) dividing curve of degree* d *in* $\mathbb{C}P^2$ *if there exists such a curve whose real part equipped with one of its complex orientations is isotopic* A *in* $\mathbb{R}P^2$.

[1] for example, $\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle$ means a pseudo, an oval with β empty ovals in its interior and α empty ovals in its exterior.

The next theorem we state is the classification of complex schemes realized by M-curves of degree 7 in $\mathbb{R}P^2$. We will not use it in this thesis. However, as we will give the classification of complex schemes realized by plane symmetric M-curves of degree 7 in section 5, we give here the classification of complex schemes of M-curves of degree 7 in $\mathbb{R}P^2$ in order to compare the two classifications. Theorem 2.57 has been finally proved by V. Florens in [Flo], concluding the previous works of T. Fiedler (see [Fie83]), S. Fiedler-Le Touzé (see [FLT]) and S. Yu. Orevkov (see [Ore00],[Ore01a],[Ore03] and [Ore01b]).
The notations used to encode complex schemes are the usual ones proposed in [Vir84c].

Theorem 2.57 (Fiedler, Fiedler-Le Touzé, Florens, Orevkov) *Any nonsingular pseudoholomorphic M-curve of degree 7 in $\mathbb{R}P^2$ has one of the following complex schemes :*

- $\langle J \amalg 9_+ \amalg 6_- \rangle_I$
- $\langle J \amalg (7-k)_+ \amalg (6-k)_- \amalg 1_-\langle (k+1)_+ \amalg k_- \rangle \rangle_I$ *with* $0 \leq k \leq 4$ *or* $k=6$,
- $\langle J \amalg (7-k)_+ \amalg (6-k)_- \amalg 1_+\langle k_+ \amalg (k+1)_- \rangle \rangle_I$ *with* $0 \leq k \leq 6$,
- $\langle J \amalg (5-k)_+ \amalg (7-k)_- \amalg 1_-\langle (k+2)_+ \amalg k_- \rangle \rangle_I$ *with* $0 \leq k \leq 5$.

Moreover, any of these complex schemes is realizable by nonsingular real algebraic M-curves of degree 7 in $\mathbb{R}P^2$.

One can note that in these three classifications, there are no differences between the algebraic and the pseudoholomorphic classifications. Historically, the first differences appeared studying reducible curves. Pseudoholomorphic arrangements of a line and a sextic which cannot be algebraic were exhibited by Fiedler-Le Touzé, Orevkov and Shustin in [FLTO02], [OS02], and [OS]; Orevkov found in [Ore] a pseudoholomorphic algebraically unrealizable arrangement of a cubic and a quartic.

2.6 Cubic resolvent of a real algebraic curve of bidegree $(4,0)$ in Σ_n

These objects have been introduced by Orevkov and are inspired by the well known cubic resolvent of a polynomial of degree 4. The reader who would like to have more details about this material can refer to [OS].
Suppose we are given a real *algebraic* curve C of bidegree $(4,0)$ in Σ_n realizing an $\mathcal{L}$-scheme A. The cubic resolvent gives a way to associate to C a real trigonal curve C' on Σ_{2n}. Moreover the $\mathcal{L}$-scheme realized by C' can be extracted from A. As there exist powerful tools to study real algebraic trigonal curves in rational geometrically ruled surfaces (see Chapter 3), this method can be used to prohibit *algebraically* some $\mathcal{L}$-schemes of bidegree $(4,0)$ in Σ_n.
In what follows, n is a positive integer and

$$P(X,Y) = Y^4 + b_2(X)Y^2 + b_3(X)Y + b_4(X)$$

is a real polynomial, where $b_j(X)$ is a real polynomial of degree jn in X. By a suitable change of coordinates in Σ_n, each curve of bidegree $(4,0)$ in Σ_n can be put into this form. Let us denote by $Y_1(X)$, $Y_2(X)$, $Y_3(X)$ and $Y_4(X)$ the roots of $P(X,Y)$ when one specializes the variable X, and put

$$Z_1(X) = (Y_1(X)+Y_2(X))(Y_3(X)+Y_4(X)) = -(Y_1(X)+Y_2(X))^2,$$

$$Z_2(X) = (Y_1(X)+Y_3(X))(Y_2(X)+Y_4(X)) = -(Y_1(X)+Y_3(X))^2,$$

$$Z_3(X) = (Y_1(X)+Y_4(X))(Y_2(X)+Y_3(X)) = -(Y_1(X)+Y_4(X))^2.$$

One can check that

$$Z_1(X) - Z_2(X) = (Y_1(X) - Y_4(X))(Y_3(X) - Y_2(X)),$$

$$Z_1(X) - Z_3(X) = (Y_4(X) - Y_2(X))(Y_1(X) - Y_3(X)),$$

$$Z_2(X) - Z_3(X) = (Y_4(X) - Y_3(X))(Y_1(X) - Y_2(X)).$$

Definition 2.58 *The cubic resolvent of P is the trigonal curve in Σ_{2n} defined in a standard coordinate system by the polynomial*

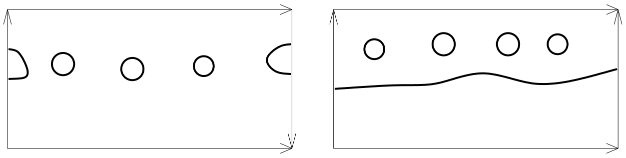
$$R = (Z - Z_1(X))(Z - Z_2(X))(Z - Z_3(X)) = Z^3 - 2b_2(X)Z^2 + (b_2^2(X) - 4b_4(X))Z + b_3^2(X).$$

The following lemma allows one to find the topology of the cubic resolvent of a curve. The lemma is a direct corollary of the above identity.

Lemma 2.59 (Orevkov, [OS]) *One has*

- *If $Y_1(X)$, $Y_2(X)$, $Y_3(X)$ and $Y_4(X)$ are real, then*
  1. *if $Y_1(X) < Y_2(X) < Y_3(X) < Y_4(X)$ then $Z_1(X) < Z_2(X) < Z_3(X) \leq 0$,*
  2. *if $Y_1(X) = Y_2(X) < Y_3(X) < Y_4(X)$ then $Z_1(X) < Z_2(X) = Z_3(X) < 0$,*
  3. *if $Y_1(X) < Y_2(X) = Y_3(X) < Y_4(X)$ then $Z_1(X) = Z_2(X) < Z_3(X) < 0$,*
  4. *if $Y_1(X) < Y_2(X) < Y_3(X) = Y_4(X)$ then $Z_1(X) < Z_2(X) = Z_3(X) < 0$.*
- *If $Y_1(X)$ and $Y_2(X)$ are real but $Y_3(X)$ and $Y_4(X)$ are not, then*
  1. *If $Y_1(X) \neq Y_2(X)$ then $Z_2(X)$ and $Z_3(X)$ are non-real,*
  2. *If $Y_1(X) = Y_2(X)$ then $Z_2(X) = Z_3(X) > 0$.*
- *If $Y_1(X)$, $Y_2(X) = \overline{Y_1(X)}$, $Y_3(X)$ and $Y_4(X) = \overline{Y_3(X)}$ are all distinct and non-real. Let $Im(Y_1(X) > 0$ and $Im(Y_3(X) > 0$. Then $Z_1(X) < 0 \leq Z_3(X) < Z_2(X)$. Moreover, $Z_3(X) = 0$ if and only if either $Im(Y_1(X)) = Im(Y_3(X))$ or $Re(Y_1(X)) = Re(Y_3(X))$.*

For example, let us consider a curve of bidegree $(4, 0)$ in Σ_1 realizing the $\mathcal{L}$-scheme shown on Figure 2.7a). Then, according to Lemma 2.59, the cubic resolvent of this curve realizes in Σ_2 the $\mathcal{L}$-scheme shown on Figure 2.7b).



a) b)

Figure 2.7:

Lemma 2.60 *If $P(x_0, Y)$ has a triple root, then so has $R(x_0, Z)$.*

Proof. Straightforward. ∎

2.7 Braid theoretical method

This method, invented by Orevkov in the 90's, provides a powerful tool to study real pseudoholomorphic curves in rational geometrically ruled surfaces. The idea is to replace the study of pseudoholomorphic realizability of an $\mathcal{L}$-scheme by the study of some braids. The corresponding problem in the braid group is called the *quasipositivity problem.* Using this method, Orevkov has achieved the classification of real pseudoholomorphic affine sextics (see [Ore99]) and pseudoholomorphic M-curves of degree 8 (see [Ore02a]).

2.7.1 Basic knot and braid theory

We just define here notions we will use later. For detailed proofs or more material about knots and braids, one can refer to [Lic97], [PS97] or [Ore99].
In this section, we fix an orientation of S^3.

2.7.1.1 Links and Alexander polynomial

Definition 2.61 *An oriented link with l components in the 3-sphere S^3 is a smooth embedding of a disjoint union of l oriented circles in S^3. A link with 1 component is called a knot.*
Two links L_1 and L_2 are said to be isotopic if there exists an isotopy of S^3 which brings L_1 to L_2.

Let L be a link in S^3 and p a point of $S^3 \setminus L$. Denote by π the projection $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$. Let us choose an orientation preserving diffeomorphism $\phi : S^3 \setminus \{p\} \to \mathbb{R}^3$ such that $\pi \circ \phi_{|L}$ is an embedding, the set $X = \{x \in \mathbb{R}^2 \mid Card((\pi \circ \phi_{|L})^{-1}(x)) > 1\}$ is finite and each element of X has exactly 2 preimages by $\pi \circ \phi_{|L}$. The point of X are called the *crossings* of L with respect to ϕ or, when ϕ is not specified, the crossings of X. Thanks to the natural order of $\mathbb{R}$, one can speak about the *over* and the *under* branch at each crossing of X. The image of $\phi_{|L}$ in $\mathbb{R}^2$ together with the over and under information is called a *link diagram of X.*
For example, (a link diagram of) the well known *trefoil knot* is depicted on Figure 2.8a).

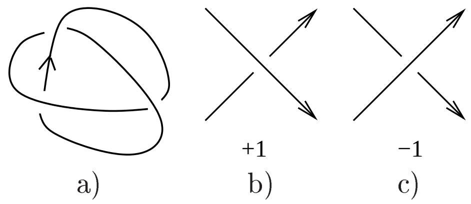


Figure 2.8:

One of the goals of knot theory is to associate isotopy invariants to links, that is to associate "quantities" to links in such a way that these "quantities" are equal for two isotopic links. The *Alexander polynomial* constitutes such a link invariant, and we will use it intensively. This polynomial is an isotopy invariant, up to multiplication by $\pm t^{\pm 1}$. In order to define it, we first have to introduce *Seifert surfaces.*

Definition 2.62 *A Seifert surface of a link L in S^3 is a connected compact oriented surface in S^3 having L as its oriented boundary.*

Let us fix a link L and a Seifert surface F for L. For an element x of $H_1(F, \mathbb{Z})$, we will denote by x^+ the result of a small shift of a cycle representing x in S^3 along a positive normal vector field to F.

Definition 2.63 *Let L_1 and L_2 be two oriented links in S^3. Label each crossing of L_1 and L_2 as depicted on Figure 2.8a) and b). The linking number of L_1 and L_2, denoted by $lk(L_1, L_2)$, is defined as the half sum of the signs of all the crossings of L_1 and L_2.*

Proposition 2.64 *The linking number of K_1 and K_2 is an isotopy invariant of the link $K_1 \sqcup K_2$.*

Definition 2.65 *The bilinear form* $\begin{array}{ccc} H_1(F,\mathbb{Z}) \times H_1(F,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ (x,y) & \longmapsto & \mathrm{lk}(x^+, y) \end{array}$ *is called a Seifert form of L.*
A Seifert matrix is the Gramm matrix of a Seifert form in some basis of $H_1(F,\mathbb{Z})$.

The following proposition can be proved using the infinite cyclic covering of L (see [Lic97]). We denote the transposed of a matrix G by G^τ.

Proposition 2.66 (Alexander) *Let G be a Seifert matrix of the link L. Then, the polynomial $\det(G - tG^\tau)$, considered up to multiplication by $\pm t^{\pm 1}$, depends only on the isotopy class of L.*

Definition 2.67 *The polynomial $\det(G - tG^\tau)$, considered up to multiplication by $\pm t^{\pm 1}$, is called the Alexander polynomial of L.*

Definition 2.68 *The value of the Alexander polynomial of L at -1 is called the determinant of L and denoted by $\det(L)$.*

It is clear from Proposition 2.66 that $|\det(L)|$ is an isotopy invariant of L.

2.7.1.2 Braids

Now we introduce *braids*. One of the interests of braids is that they form a natural group which has a nice presentation. This means in particular that most of the computations in the braid group can be performed by a computer. Alexander showed that each link can be put into the form of a *closed braid*, hence studying a link (for example, computing its Alexander polynomial) is sometimes easier using its closed braid form.

Definition 2.69 *A braid of m strings is the graph of a smooth m-valued function $f : [0;1] \to \mathbb{C}$ such that*

- *for each $t \in [0;1]$, the m values of $f(t)$ are pairwise distinct,*
- *the real parts of the m values of $f(0)$ and $f(1)$ are pairwise distinct.*

Two braids b_1 and b_2 are said to be isotopic if there exists an isotopy of $[0;1] \times \mathbb{C}$ which brings b_1 to b_2 and such that the image of b_1 during the isotopy remains a braid.
The set of all isotopy classes of braids of m strings is denoted by B_m.

We still denote by b the isotopy class of the braid b and we still call braids the elements of B_m.
The projection we use to depict braids is $(t,z) \mapsto (t, Re(z))$. An example of a braid is depicted on Figure 2.9a). We will denote this braid by σ_i.
There is a natural way to multiply two braids b_1 and b_2. choose a smooth m-valued function f (resp., g) : $[0;1] \to \mathbb{C}$ whose graph represents b_1 (resp., b_2) in B_m and such that $f(1) = g(0)$. Define $h : [0;1] \to \mathbb{C}$ by $h(t) = f(2t)$ if $t \in [0; \frac{t}{2}]$ and $h(t) = g(2t-1)$ if $t \in [\frac{t}{2}; 1]$. The braid $b_1 b_2$ in B_m is defined as the graph of h. One can see easily that the result does not depend on the choice of the representatives. For example, σ_i^2 is depicted on Figure 2.9b).
The set B_m equipped with this multiplication is a group.

Theorem 2.70 (Artin) *The group B_m is isomorphic to the finitely generated group*

$$\langle \sigma_1, ..., \sigma_{m-1} | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \text{ and } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1\rangle.$$

Definition 2.71 *The braids σ_i are called the standard generators.*

A corollary of Theorem 2.70 is the fact that for any presentation $\prod_{j=1}^{n} \sigma_{i_j}^{e_j}$ of a braid b in B_m, the integer $\sum_{j=1}^{n} e_j$ does only depend on b (that is, does not depend on the presentation).

Definition 2.72 *The latter integer, denoted by $e(b)$, is called the exponent sum of the braid b.*

A natural way to associate a link to a braid is to join its two ends. The result is called a closed braid.

Definition 2.73 *Consider the standard embedding of the solid torus $\phi : ([0;1] \times \mathbb{C})/_{(0,z)\sim(1,z)}$ into S^3. For any braid b in B_m, there exists a representative of b, whose image by ϕ is a link in S^3. This link is called the closure of b.*

So one can define the Alexander polynomial and the determinant of a braid : it is those of its closure. In [Ore99], Orevkov gives an efficient algorithm to compute a Seifert matrix of a closed braid. Thus, one can also easily compute the Alexander polynomial of a braid. We do not give here this algorithm, but all the computations of Alexander polynomials of braids we made in the following chapters are done using this algorithm.
Some braids in B_m are of special interest. We have already seen the standard generators. Here is another one.

Definition 2.74 *The Garside element of B_m is defined by*

$$\Delta_m = (\sigma_1 \dots \sigma_{m-1})(\sigma_1 \dots \sigma_{m-2}\sigma_1) \dots (\sigma_1\sigma_2)\sigma_1.$$

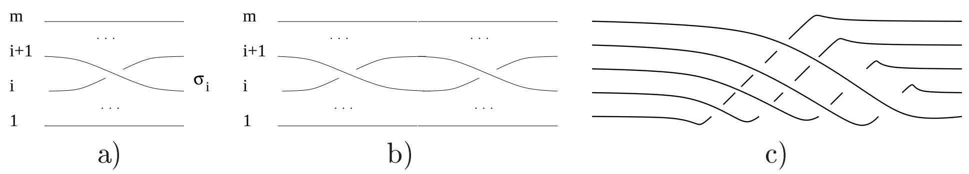


Figure 2.9:

For example, Δ_5 is depicted on Figure 2.9c).

The braid group being finitely generated, a natural problem, the so called *word problem* arises: given two elements of the free group generated by $\sigma_1, \dots, \sigma_{m-1}$, does they represent the same element in B_m?
F. Garside solved in [Gar69] the word problem in the braid group. He found a way to associate to any braid a *normal form.* He gave an algorithm to rewrite any braid in such a way that two braids represent the same element in B_m if and only if this rewriting algorithm gives the same result for both. The result of the rewriting algorithm is called the *Garside normal form.*
We do not define here the Garside normal form, the interested reader can refer to [Gar69] or [Jac90]. We will only use it in sections 5.4.4 and 5.5 to decide whether a braid is trivial or not. Hence we just give the Garside normal form of the trivial braid.

Lemma 2.75 *A braid is trivial if and only if its Garside normal form is equal to 1.*

The complexity of the initial algorithm proposed by Garside is exponential with respect to n, so we will use the algorithm given by A. Jacquemard in [Jac90] which is polynomial in n.

Lastly, we define a fundamental notion for the Orevkov method, the notion of quasipositivity for a braid. This notion was introduced by L. Rudolph in [Rud83] in the study of complex algebraic curves.

Definition 2.76 *A braid is said to be quasipositive if it can be represented in B_m as*

$$\prod_j w_j \sigma_{i_j} w_j^{-1}.$$

2.7.2 Orevkov's method

From now on, we fix a base B of Σ_n, a fiber l_∞ of $\mathcal{L}$, and an $\mathcal{L}-$ *scheme* A in Σ_n, which may be nodal. Suppose moreover that all the intersections of A and E are real and are not a tangency point of A with the pencil of lines, and that A intersects each fiber in m or $m-2$ real points (counted with multiplicities) and l_∞ in m distinct real points. Choose a standard coordinate system on Σ_n such that l_∞ has equation $\{x_2 = 0\}$ (see section 2.3) and a trivialization of the $\mathbb{C}P^1$-bundle over $B \setminus (B \cap l_\infty)$.
Now, we describe how to encode such an $\mathcal{L}$-scheme A. Examine the real part of the fibration from $x_1 = -\infty$ to $x_1 = +\infty$. Each time the pencil of real lines has a tangency point with A or meets a double point of A or an intersection point of A with E, denote by k the number of real intersection points of the fiber and A below this point and do the following :

- if the pencil of real lines has a tangency point with A, write $\supset_k$ if A intersects a fiber in m real points before the tangency point, and $\subset_k$ otherwise,
- if the pencil of real lines meets a double point of A, write $\bullet_k$ if the tangents are non-real and $\times_k$ otherwise,
- if A intersects E, write $/$ if the branch of A passing through E lies in the region $\{y > 0\}$ just before the infinite point, and $\backslash$ otherwise.

Replace all the occurrences $\subset_k$ $\supset_k$ and $\bullet_k$ by o_k. Thus, we have now a coding $s_1 \ldots s_r$ of the $\mathcal{L}-$ *scheme* A. In order to obtain a braid from this coding, perform the following substitutions :

- replace each $\times_j$ which appears between $\subset_s$ and $\supset_t$ by σ_j^{-1},
- replace each $\backslash_1$ which appears between $\subset_s$ and $\supset_t$ by $\sigma_1\sigma_2 \ldots \sigma_{m-1}$,
- replace each $/_m$ which appears between $\subset_s$ and $\supset_t$ by $\sigma_{m-1}\sigma_{m-2} \ldots \sigma_1$,
- replace each subword $\supset_s \times_{i_{1,1}} \ldots \times_{i_{r_1,1}} ?_1 \times_{i_{1,2}} \ldots \times_{i_{r_2,2}} ?_2 \ldots \times_{i_{1,p}} \ldots \times_{i_{r_p,p}} \subset_t$ with $?_i = \backslash$ or $/$ by $\sigma_s^{-1}\tau_{s,m-1}\sigma_{i_{1,1}}^{-1} \ldots \sigma_{i_{r_1,1}}^{-1} v_1 \sigma_{i_{1,2}}^{-1} \ldots \sigma_{i_{r_2,2}}^{-1} v_2 \ldots v_p \sigma_{i_{1,p}}^{-1} \ldots \sigma_{i_{r_p,p}}^{-1} \tau_{m-1,t}$,
  where

$$v_l = \begin{cases} \sigma_1\sigma_2\dots\sigma_{m-3}\sigma_{m-2}^2 & \text{if } ?_l = \backslash, \\ \sigma_{m-2}^{-1}\sigma_{m-1}^2\sigma_{m-2}\dots\sigma_1 & \text{if } ?_l = /, \end{cases}$$

$$\tau_{s,t} = \begin{cases} (\sigma_{s+1}^{-1}\sigma_s)(\sigma_{s+2}^{-1}\sigma_{s+1})\dots(\sigma_t^{-1}\sigma_{t-1}) & \text{if } t > s, \\ (\sigma_{s-1}^{-1}\sigma_s)(\sigma_{s-2}^{-1}\sigma_{s-1})\dots(\sigma_t^{-1}\sigma_{t+1}) & \text{if } t < s, \\ 1 & \text{if } t = s. \end{cases}$$

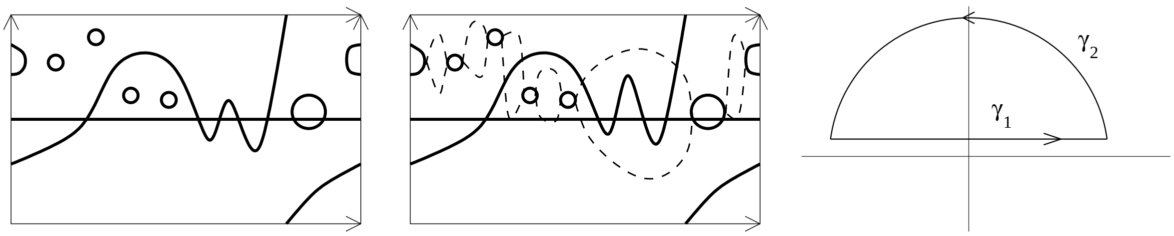
Then we obtain a braid $b_{\mathbb{R}}$.
Remark. The symbols / and \ do not appear in [Ore99]. However, studying the perturbations of A at these points as in [Ore99] with the local model $z = constant$ for the pencil of lines, and $z = \frac{1}{w}$ for A, one sees immediatly that the obtained braid is the one described above.

Then we obtain a braid $b_{\mathbb{R}}$.

Definition 2.77 *The braid associated to the $\mathcal{L}-$ scheme A, denoted b_A, is the braid $b_{\mathbb{R}}\Delta_m^n$ (Δ_m is the Garside element of B_m).*

For example, the coding and the braid corresponding to the real $\mathcal{L}-$ *scheme* in Σ_2 depicted in Figure 2.10a) are, respectively,

$$\supset_3 o_3^2 \times_1 o_2^2 \times_1^4 /_2 \subset_3 \times_2^2 \supset_3\subset_3 \quad \text{and } \sigma_3^{-3}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_2^{-2}\sigma_3^{-1}\sigma_2\sigma_1^{-4}\sigma_2^{-1}\sigma_3^2\sigma_2\sigma_1\sigma_2^{-2}\sigma_3^{-1}\Delta_4^2.$$



a) b) c)

Figure 2.10:

The meaning of the braid b_A is the following. First, denote by r the number of (real) intersection points of A with the exceptional section, by π the projection $\Sigma_n \to B$, and by π' the restriction of π to $\Sigma_1 \setminus E$. Suppose that there exists a real pseudoholomorphic curve C of bidegree (m, r) realizing the $\mathcal{L}-$ *scheme* A. The set $S_A = (\pi_{|A})^{-1}(\mathbb{R}B)$ is a singular 1-dimensional subvariety of $\pi^{-1}(\mathbb{R}B)$. The set S_A corresponding to the $\mathcal{L}-$ *scheme* A of Figure 2.10a) is depicted on Figure 2.10b), where the non-real points of S_A are drawn in dashed lines. For $\epsilon > 0$, denote by $\gamma_{1,\epsilon}$ and $\gamma_{2,\epsilon}$ the followings paths

$$\begin{array}{rccc} \gamma_{1,\epsilon} : & [-1;1] & \to & \mathbb{C} \\ & t & \mapsto & \frac{1}{\epsilon}t + i\epsilon \end{array} \quad \text{and} \quad \begin{array}{rccc} \gamma_{2,\epsilon} : & [0;1] & \to & \mathbb{C} \\ & t & \mapsto & \frac{1}{\epsilon}e^{i\pi t} + i\epsilon \end{array}.$$

Choose a parametrization γ_ϵ by the segment $[0;1]$ of the union of the images of $\gamma_{1,\epsilon}$ and $\gamma_{2,\epsilon}$ (see Figure 2.10c)). Then, the set $\pi'^{-1}(\gamma_\epsilon([0;1])$ for ϵ small enough is a smooth 1-dimensional subvariety of $[0;1] \times \mathbb{C}$. It can be viewed as a smoothing of S_A and is actually the closure of the braid b_A. That means that the algorithm given in the beginning of this section to obtain b_A from A describes how the singular points of S_A are smoothed.
In [Ore99], Orevkov proved the two following results.

Theorem 2.78 (Orevkov) *The $\mathcal{L}-$ scheme A is realizable by a pseudoholomorphic curve of bidegree (m,r) if and only if the braid b_A is quasipositive.*

Proposition 2.79 (Orevkov) *Let A an $\mathcal{L}-$ scheme, and A' the $\mathcal{L}-$ scheme obtained from A by one of the following elementary operations :*

$$\times_j \supset_{j\pm1} \to \times_{j\pm1} \supset_j \quad , \qquad \subset_{j\pm1} \times_j \to \subset_j \times_{j\pm1} \quad , \qquad \times_j u_k \to u_k \times_j \quad ,$$

$$o_k \subset_k \supset_{k-1} \quad \longleftrightarrow \quad \subset_k \times_{k-1} \supset_k \quad \longleftrightarrow \quad \subset_{k-1} \supset_k o_k \quad ,$$

$$\subset_j \supset_{j\pm1} \to \emptyset \quad , \qquad \subset_j \supset_k \to \supset_k \subset_j \quad , \qquad o_j \to \emptyset \quad ,$$

where $|k-j|>1$ and u stands for $\times$, $\subset$, or $\supset$.
Then if A is realizable by a pseudoholomorphic curve of bidegree (m,r), so is A'.
Moreover, if A is realizable by a dividing pseudoholomorphic curve of bidegree (m,r) and if A' is obtained from A by one of the previous elementary operations except the last one, so is A'.

According to Proposition 2.79, if an $\mathcal{L}$-scheme A is realizable by a pseudoholomorphic curve, so is any $\mathcal{L}$-scheme obtained from A by some $\subset_j \supset_{j\pm1} \to \emptyset$ operations. Orevkov showed ([Ore02b]) that unfortunately this is not the case for algebraic curves.

Remark. To define the braid b_A, we have supposed that A intersects each fiber of Σ_n in m or $m-2$ real points (counted with multiplicities). If this condition is not fulfilled, one cannot associate anymore a unique braid to A. However, one can associate to it a family of braids (see [Ore99], [Ore02a]). We will never consider this case in this thesis, so we just mention it.

2.7.3 The quasipositivity problem

The quasipositivity problem in the braid group is solved only for 3-strings braids (Orevkov, [Ore04]). However, following Rudolph's work (see [Rud83]), Orevkov (see [Ore99]) observed that the quasipositivity of a braid implies that its closure bounds a smoothly and properly embedded surface in the ball B^4, with given Betti numbers. As necessary conditions for this property, Orevkov used elementary arguments on the linking numbers of the components, and the so-called Murasugi-Tristram inequality (see [Tri69], [Mur65], [Ore99]). The Tristram signatures of a link L are constructed from a Seifert form, and the inequality states that the existence of a smooth surface in B^4, with given Betty numbers and boundary L, implies restrictions on the possible values of these signatures. Propositions 2.80 and 2.81 are corollaries of this inequality (see [Ore99]) that we will use in this thesis.

Proposition 2.80 (Orevkov) *If a braid b in B_m is quasipositive and $e(b) < m-1$, then the Alexander polynomial of b is identically zero.*

Proposition 2.81 (Orevkov) *If a braid b in B_m is quasipositive and $e(b) = m-1$, then all the roots of the Alexander polynomial of b situated on the unit circle are of order at least two.*

One may note that, according to Viro's work (see [Vir73]), the Tristram signatures of a link can be interpreted as signatures of intersection forms related to some finite cyclic coverings of B^4, branched along a smooth surface whose boundary is the link. In [Gil81], P. Gilmer gave a proof of the Murasugi-Tristram inequality using this point of view. This illustrates in particular that the work of Orevkov extends the spirit of the work of Arnold (see [Arn71]), which uses arithmetic of the intersection form of the 2-fold covering of $\mathbb{C}P^2$, branched along a curve.

Using the intersection form point of view, Florens [Flo] constructed generalized signatures as an extension of Tristram's signatures to the case of colored links, and showed that they verify also the Murasugi-Tristram inequality. This has allowed him to achieve the classification of the complex schemes realized by real algebraic M-curves of degree 7 in $\mathbb{R}P^2$. More recently, he has showed with D. Cimasoni in [CF] that these generalized signatures can be constructed in terms of generalized Seifert forms for colored links. This gives in particular a very practical way to compute these invariants in terms of a word in the braid group.
Orevkov has also used the generalized Fox-Milnor theorem in [Ore02a] to prove non-quasipositivity of some braids. The following proposition is a corollary of this latter theorem (see [Ore02a]).

Proposition 2.82 (Orevkov) *If a braid b in B_m is quasipositive and $e(b) = m - 1$, then $|det(b)|$ is a square of integer number.*

Orevkov also used unitary representations of the braid group in [Ore01b]. One can mention that in [KT02], the authors propose an efficient probabilistic method to study a particular case of quasipositive braids.

Chapter 3

Real rational graphs on $\mathbb{C}P^1$

3.1 Motivation

This section deals with the following problem : given a real arrangement of roots of three real polynomials (called *a root scheme* below), does there exist two real polynomials P and Q such that the real roots of P, Q and $P+Q$ realize the given arrangement?
This question can be reformulated in terms of existence of a certain graph on $\mathbb{C}P^1$ (called *a real rational graph* below), usually called a *dessin d'enfant*.
We start with the following fact : to any rational map $f : \mathbb{C}P^1 \to \mathbb{C}P^1$, one can associate a natural graph on $\mathbb{C}P^1$, namely $f^{-1}(\mathbb{R}P^1)$. This correspondence is used for example by S. Natanzon, B. Shapiro and A. Vainshtein to classify topologically generic real rational maps (see [NSV02] and [SV03]). An other application of these graphs has been exploited by A. Zvonkin ([Zvo]). He used these graphs to study the minimal degree of $P^3 - Q^2$, where P and Q are complex polynomials of degrees $2k$ and $3k$, respectively. Following Zvonkin, Orevkov proposed in [Ore03] a new way to construct real algebraic trigonal curves in rational geometrically ruled surfaces.
Zvonkin and Orevkov actually used a particular case of rational graphs : *trigonal graphs*. However, arguments used in [Ore03] are valid in a more general context. In the first section, we state definitions in the general setting. In the second section, we focus our attention on real trigonal graphs. As Orevkov as showed, these graphs play an important role in the study of trigonal curves. He has exhibited a correspondence between both set of objects. Here we give an efficient algorithm which deals with real rational graphs.
The results of section 3.2 will be used in section 4 to construct curves in Σ_n with a prescribed position with respect to two basis. We will use in Chapter 5 the algorithm given in section 3.3 to prove the non-realizability of some $\mathcal{L}$-schemes by real algebraic trigonal curves.

3.2 General situation

Definition 3.1 *A root scheme is a k-uplet* $((l_1, m_1), \ldots, (l_k, m_k)) \in (\{p, q, r\} \times \mathbb{N})^k$ *with k a natural number (here, p, q and r are symbols and do not stand for natural numbers).*
A root scheme $((l_1, m_1), \ldots, (l_k, m_k))$ *is realizable by polynomials of degree n if there exist two real polynomials in one variable of degree n, with no common roots,* $P(X)$ *and* $Q(X)$ *such that*

if $x_1 < x_2 < \ldots < x_k$ are the real roots of P, Q and $P+Q$, then $l_i = p$ (resp., q, r) if x_i is a root of P (resp., $Q, P+Q$) and m_i is the multiplicity of x_i.
The polynomials P, Q and $P+Q$ are said to realize the root scheme $((l_1, m_1), \ldots, (l_k, m_k))$.

In a root scheme, we will abbreviate a sequence S repeated u times by S^u.
From now on, let RS be a root scheme and suppose that RS is realized by P, Q and $P+Q$ of degree n. Put $R(X) = P(X) + Q(X)$ and consider the rational function $f(X) = \frac{R(X)}{Q(X)} = \frac{P(X)}{Q(X)} + 1$. Color and orient $\mathbb{R}P^1$ as depicted in Figure 3.1a). Let Γ be $f^{-1}(\mathbb{R}P^1)$ with the coloring and the orientation induced by those chosen on $\mathbb{R}P^1$. Then, Γ is a colored and oriented graph on $\mathbb{C}P^1$, invariant under the action of the complex conjugation. The colored and oriented graph on $\mathbb{R}P^1$ obtained as the intersection of Γ and $\mathbb{R}P^1$ can clearly be extracted from RS.

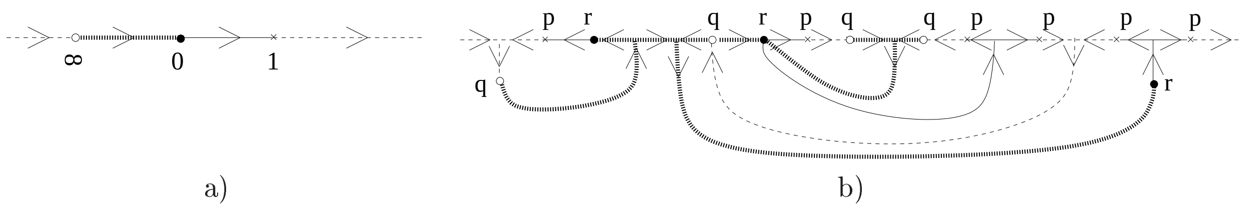


Figure 3.1:

Definition 3.2 *The colored and oriented graph on $\mathbb{R}P^1$ constructed above is called the real graph associated to RS.*

Definition 3.3 *Let Γ be a graph on $\mathbb{C}P^1$ invariant under the action of the complex conjugation and $\pi : \Gamma \to \mathbb{R}P^1$ a continuous map. Then the coloring and orientation of $\mathbb{R}P^1$ shown in Figure 3.1a) defines a coloring and an orientation of Γ via π.*
The graph Γ equipped with this coloring and this orientation is called a real rational graph if

- *any vertex of Γ has an even valence,*
- *any connected component D of $\mathbb{C}P^1 \setminus \Gamma$ is homeomorphic to an open disk,*
- *for any connected component D of $\mathbb{C}P^1 \setminus \Gamma$, one has $\pi_{|\partial D}$ is a covering of $\mathbb{R}P^1$ of degree d_D.*

The sum of the degrees d_D for all connected component D of $\{Im(z) > 0\} \setminus \Gamma$ of is called the degree of Γ.

The importance of real rational graphs is given by the following proposition. The proof is the same as in [Ore03]. However, as we work in a more general setting, we reproduce the proof here.

Theorem 3.4 (Orevkov, [Ore03]) *Let RS be a root scheme and G its real graph. Then RS is realizable by polynomials of degree n if and only if there exists a real rational graph Γ of degree n such that $\Gamma \cap \mathbb{R}P^1 = G$.*

Proof. If the root scheme RS is realizable by polynomials of degree n, then consider $\Gamma = f^{-1}(\mathbb{R}P^1)$ (f is the rational function constructed at the beginning of this section). If the graph Γ is connected, then it is a real rational graph on $\mathbb{C}P^1$ satisfying the conditions of the proposition. If Γ is not connected, we will perform some operations on $\Gamma \cap \{Im(z) > 0\} \subset \mathbb{C}P^1$. The final graph on $\mathbb{C}P^1$ will be obtained by gluing the obtained graph with its image under the complex conjugation.

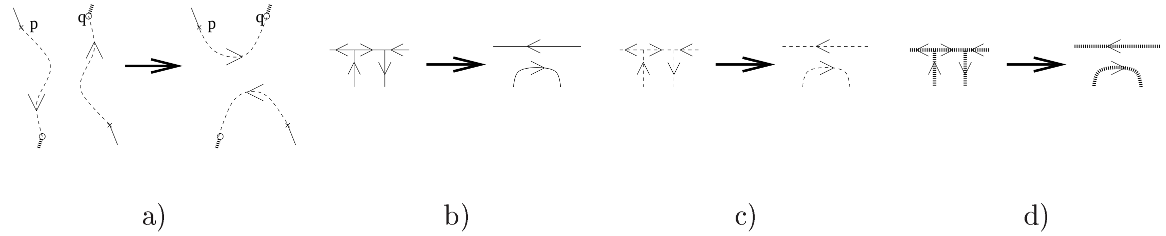


Figure 3.2:

Choose p in $f^{-1}(1) \cap \{Im(z) > 0\}$ and q in $f^{-1}(\infty) \cap \{Im(z) > 0\}$ belonging to different connected components of Γ, and lying on the boundary of one connected components of $\mathbb{C}P^1 \setminus \Gamma$. Then, perform the operation depicted in Figure 3.2a). Perform this operation until the graph obtained is connected. Then, this graph is a real rational graph on $\mathbb{C}P^1$ satisfying the conditions of the proposition.

Now suppose there exists a real rational graph Γ of degree n such that $\Gamma \cap \mathbb{R}P^1 = G$. Let $\pi : \Gamma \to \mathbb{R}P^1$ be a continuous map compatible with the orientation and the coloring of both spaces. Denote by $D_1, \ldots, D_k$ the connected components of $\mathbb{C}P^1 \cap \Gamma \cap \{Im(z) \leq 0\}$ and choose a sign for each connected component of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$. To each D_i, assign

- a sign s_j to each D_i such that if $\overline{D_i} \cap \overline{D_j} \neq \emptyset$, then D_i and D_j have opposite signs,
- an natural integer d_i which is the degree of the covering $\pi_{|\partial D_i} : \partial(D_i) \to \mathbb{R}P^1$.

Now extend on D_i each $\pi_{|Fr(D_i)}$ on a (branched if $d_i > 1$) covering of degree d_i of the half of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$ with the sign s_i. Finally, extend π on a real topological branched covering $S^2 \to \mathbb{C}P^1$. The pull-back structure by f on S^2 makes f to be a real rational function $\frac{R}{Q}$ with R and Q of degree n. The polynomials $R - Q$, Q and R realize the root scheme RS. □

For example, the root scheme $((p,1),(r,1),(q,2),(r,3),(p,1),(q,1)^2,(p,1)^4)$ is realizable by polynomials of degree 6 as it is depicted in Figure 3.1b).

Lemma 3.5 *Let $RS = ((l_1,m_1),\ldots,(l_k,m_k))$ a root scheme such that there exist i and s such that $\forall j \in \{i,\ldots,i+s\}, l_j = l_i$. Define the root scheme $RS' = ((l'_1,m'_1),\ldots,(l'_{k-s},m'_{k-s}))$ by*

- $(l'_t,m'_t) = (l_t,m_t)$ *for* $t < i$,
- $(l'_i,m'_i) = (l_i, m_i + \ldots + m_{i+s})$,
- $(l'_t,m'_t) = (l_{t+s},m_{t+s})$ *for* $t > i$,

Then RS is realizable by polynomials of degree n if and only if RS' is realizable by polynomials of degree n.

Proof. Straightforward. □

3.3 An important special case : real trigonal graphs on $\mathbb{C}P^1$

In [Ore03], Orevkov reformulates the existence of real algebraic trigonal curves realizing a given $\mathcal{L}$-scheme in Σ_n in terms of the existence of a special case of real rational graphs on $\mathbb{C}P^1$. Using these *trigonal graphs*, he obtained a classification of algebraic trigonal curves in Σ_n up to isotopy which respects the fibration, in terms of gluing of cubics ([Ore02b]).
Guided by this article, we give in this section an efficient algorithm to check whether an $\mathcal{L}$-scheme is realizable by a real algebraic trigonal curve in Σ_n.

3.3.1 Root scheme associated to a trigonal curve

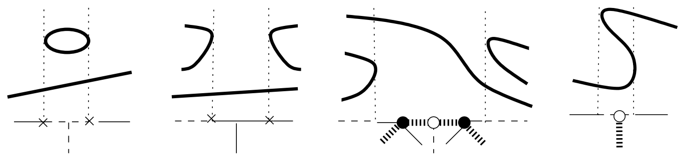
In what follows, n is a positive integer and $C(X,Y) = Y^3 + b_2(X)Y + b_3(X)$ is a real polynomial, where $b_i(X)$ is a real polynomial of degree in in X. By a suitable change of coordinates in Σ_n, any trigonal curve in Σ_n can be put into this form. The aim of this section is to explain how to use real rational graphs to construct the polynomials b_2 and b_3.
Denote by $D = -4b_2^3 - 27b_3^2$ the discriminant of C with respect to the variable Y. The knowledge of the root scheme realized by D, $27b_3^2$ and $-4b_2^3$ allows one to recover the $\mathcal{L}$-scheme realized by C, up to the transformation $Y \mapsto -Y$. Indeed, the position of C with respect to the pencil of lines is given by the sign of the double root of $C(x,Y)$ at each root x of D, which is the sign of $b_3(x)$.
Consider a trigonal $\mathcal{L}$-scheme A in Σ_n such that A intersects some fiber l_∞ in 3 distinct real points. Consider also the coding $s_1 \dots s_r$ of A defined in section 2.7.2, using the symbols $\subset$, $\supset$, o and $\times$. In this coding, replace all the occurrences $\times_k$ (resp., o_k) by $\supset_k \subset_k$ (resp., $\subset_k \supset_k$). This coding is denoted by $r_1 \dots r_q$. Define root scheme $RS_A = (S_1, \dots, S_q)$ as follows :

- $$S_1 = \begin{cases} (p,1) & \text{if } n \text{ is even and } r_1 = \supset_k \text{ and } r_q = \subset_k, \\ & \text{or } n \text{ is odd and } r_1 = \supset_k \text{ and } r_q = \subset_{k\pm1}, \\ (q,2),(p,1) & \text{otherwise ,} \end{cases}$$

- for $i > 1$,
$$S_i = \begin{cases} (p,1) & \text{if } r_i = \subset_k \text{ and } r_{i-1} = \supset_k, \\ & \text{or } r_i = \supset_k \text{ and } r_{i-1} = \subset_k, \\ (q,2),(p,1) & \text{if } r_i = \supset_k \text{ and } r_{i-1} = \subset_{k\pm1}, \\ (r,3),(q,2),(r,3),(p,1) & \text{if } r_i = \subset_k \text{ and } r_{i-1} = \supset_{k\pm1} \text{ .} \end{cases}$$

Definition 3.6 *The root scheme RS_A is called the root scheme associated to A. The real graph associated to RS_A is called the real graph associated to A.*

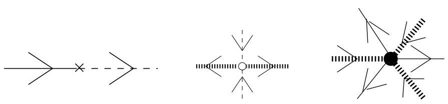
The real graph associated to A is obtained from A as depicted in Figure 3.3.



a) b) c) d)

Figure 3.3:

As we want to construct polynomials with double or triple roots, we need to consider a subclass of real rational graphs.

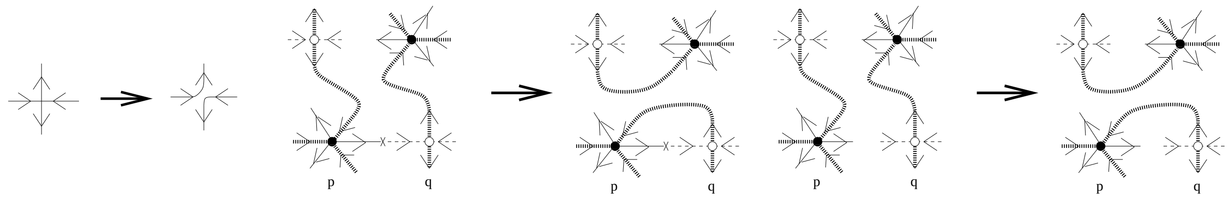


a) b) c)

Figure 3.4:

Definition 3.7 *Let Γ be a real rational graph of degree $6n$, and $\pi : \Gamma \to \mathbb{R}P^1$ a continuous map which induces the orientation and coloring of Γ. The graph Γ is a real trigonal graph of degree n if*

- *Γ has exactly $6n$ vertices of the kind depicted in Figure 3.4a), $3n$ vertices of the kind depicted depicted on 3.4b) and $2n$ vertices of the kind depicted depicted on 3.4c), and no other non-real multiple points,*
- *The set $\pi^{-1}([\infty; 0])$ is connected.*



a) b) c)

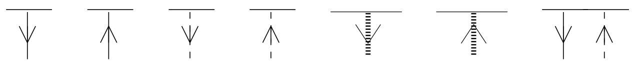
Figure 3.5:

Theorem 3.8 *Let A be a trigonal $\mathcal{L}$-scheme in Σ_n and G its real graph. Then A is realizable by real algebraic trigonal curves in Σ_n if and only if there exists a real trigonal graph Γ of degree n such that $\Gamma \cap \mathbb{R}P^1 = G$.*

Proof. If there exists such a trigonal graph of degree n, according to proposition 3.4, there exists a polynomial b_2 of degree $2n$ and a polynomial b_3 of degree $3n$ such that the polynomials $-4b_2^3 - 27b_3^2$, $27b_3^2$ and $-4b_2^3$ realize the root scheme RS_A. Clearly, the curve define by the equation $Y^3 + b_2(X)Y + b_3(X)$ or $Y^3 + b_2(X)Y - b_3(X)$ realizes the expected $\mathcal{L}$-scheme.
Suppose that there exists a real algebraic curve C realizing A. As in the previous section, let us put $f = \frac{-4b_2^3}{27b_3^2}$ and $\Gamma = f^{-1}(\mathbb{R}P^1)$. We will perform some operations on one of the halves of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$. The final picture will be obtained by gluing the obtained graph with its image under the complex conjugation.
If Γ has non-real double points, smooth them as depicted in Figure 3.5a).
Performing operations depicted in Figures 3.2b), c)and d), minimize the number of real double points of Γ.
If $f^{-1}([\infty; 0])$ is not connected, choose p in $f^{-1}(0)$ and q in $f^{-1}(\infty)$ belonging to different connected components of $f^{-1}([\infty; 0])$. If p and q belong to the same connected components of Γ, choose p and q such that they are connected in Γ by an arc of $f^{-1}(]0; \infty[)$ and perform on Γ the operation depicted in Figure 3.5b). Otherwise, choose p and q lying on the boundary of one connected components of $\mathbb{C}P^1 \setminus \Gamma$ and perform on Γ the operation depicted in Figure 3.5c). Perform these operations until $f^{-1}([\infty; 0])$ is connected. $\square$

Remark. The connexity of $\pi^{-1}([\infty; 0])$ is not necessary to the existence of the algebraic curve. We use this condition only in the next section.

3.3.2 Comb theoretical method



g_1 g_2 g_3 g_4 g_5 g_6 g_1g_4

Figure 3.6:

Denote by CB the semigroup generated by the elements $g_1, \dots g_6$ in $\mathbb{R}^2$ depicted in Figure 3.6. The multiplication of two elements c_1 and c_2 in CB is the attachment of the right endpoint of c_1 to the left endpoint of c_2.

Definition 3.9 *The elements of CB are called combs.*

For example, the comb $g_1 g_4$ is depicted in figure 3.6. The unit element of CB is denoted by 1.

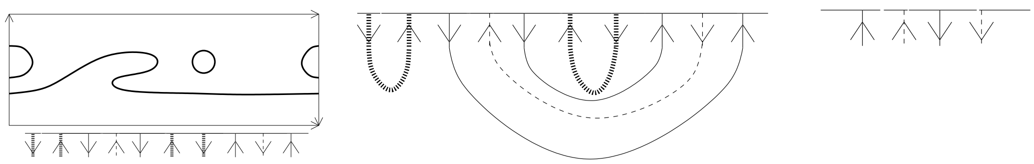
Definition 3.10 *A weighted comb is a quadruplet $(c, \alpha, \beta, \gamma)$ in $CB \times \mathbb{Z}^3$.*

Consider a trigonal $\mathcal{L}$-scheme A in Σ_n satisfying the hypothesis of section 2.7.2, i.e. all the intersections of A and E are real, and A intersects some fiber l_∞ in 3 distinct real points. Consider also the coding $s_1 \dots s_r$ of A defined in section 2.7.2, using the symbols $\subset$, $\supset$, o and $\times$. In this coding, replace all the occurrences $\times_k$ (resp., o_k) by $\supset_k \subset_k$ (resp., $\subset_k \supset_k$). This coding is denoted by $r_1 \dots r_q$. Define the weighted combs $(c_i, \alpha_i, \beta_i, \gamma_i)$ as follows :

- $(c_0, \alpha_0, \beta_0, \gamma_0) = (1, 6n, 3n, 2n)$
- $(c_1, \alpha_1, \beta_1, \gamma_1) = \begin{cases} (g_3, 6n-1, 3n, 2n) & \text{if } n \text{ is even and } r_1 = \supset_k \text{ and } r_q = \subset_k, \\ & \text{or } n \text{ is odd and } r_1 = \supset_k \text{ and } r_q = \subset_{k\pm1}, \\ (g_5, 6n-1, 3n-1, 2n) & \text{otherwise ,} \end{cases}$
- for $i > 1$,

$$(c_i, \alpha_i, \beta_i, \gamma_i) = \begin{cases} (c_{i-1}g_2, \alpha_{i-1} - 1, \beta_{i-1}, \gamma_{i-1}) & \text{if } r_i = \subset_k \text{ and } r_{i-1} = \supset_k, \\ (c_{i-1}g_3, \alpha_{i-1} - 1, \beta_{i-1}, \gamma_{i-1}) & \text{if } r_i = \supset_k \text{ and } r_{i-1} = \subset_k, \\ (c_{i-1}g_5, \alpha_{i-1} - 1, \beta_{i-1} - 1, \gamma_{i-1}) & \text{if } r_i = \supset_k \text{ and } r_{i-1} = \subset_{k\pm1}, \\ (c_{i-1}g_6g_1g_4g_1g_6, \alpha_{i-1} - 1, \beta_{i-1} - 1, \gamma_{i-1} - 2) & \text{if } r_i = \subset_k \text{ and } r_{i-1} = \supset_{k\pm1} . \end{cases}$$

Definition 3.11 *One says that the weighted comb $(c_q, \frac{\alpha_q}{2}, \frac{\beta_q}{2}, \frac{\gamma_q}{2})$ is associated to the $\mathcal{L}$-scheme A.*



a) b) c)

Figure 3.7:

Definition 3.12 *Let c be a comb. A closure of c is a subset of $\mathbb{R}^2$ obtained by joining each generator g_1 (resp., g_3, g_5) in c to a generator g_2 (resp., g_4, g_6) in c by a path in $\mathbb{R}^2$ in such a way that these paths do not intersect.*
If there exists a closure of c, one says that c is closed.

For example, the weighted comb associated to the $\mathcal{L}$-scheme in Σ_1 depicted in Figure 3.7a) is $(g_5g_6g_1g_4g_1g_6g_5g_2g_3g_2, 0, 0, 0)$. A closure of this comb is shown in Figure 3.7b). The comb depicted in Figure 3.7c) is not closed.

Definition 3.13 *A chain of weighted combs is a sequence $(w_i)_{1 \leq i \leq k}$ of weighted combs, with $w_i = (c_i, \alpha_i, \beta_i, \gamma_i)$, such that :*

$w_k = (c, 0, 0, 0)$, *where* c *is a closed comb,*
$\forall i \in \{1 \dots k-1\}$, *the weighted comb* w_{i+1} *is obtained from* w_i *by one of the following operations :*

(1) *if* $\gamma_i > 0$: $g_2 \to (g_6 g_1)^2 g_6$, $\alpha_{i+1} = \alpha_i$, $\beta_{i+1} = \beta_i$, $\gamma_{i+1} = \gamma_i - 1$
or $g_5 \to (g_3 g_6)^2 g_3$, $\alpha_{i+1} = \alpha_i - 3$, $\beta_{i+1} = \beta_i$, $\gamma_{i+1} = \gamma_i - 1$
(2) *if* $\alpha_i > 0$: $g_1 \to g_3$, $\alpha_{i+1} = \alpha_i - 1$, $\beta_{i+1} = \beta_i$, $\gamma_{i+1} = 0$
(3) *else :* $g_5 \to g_4 g_5 g_4$, $\alpha_{i+1} = 0$, $\beta_{i+1} = \beta_i - 1$, $\gamma_{i+1} = 0$.

One says that the chain $(w_i)_{1 \leq i \leq k}$ *starts at* w_1.

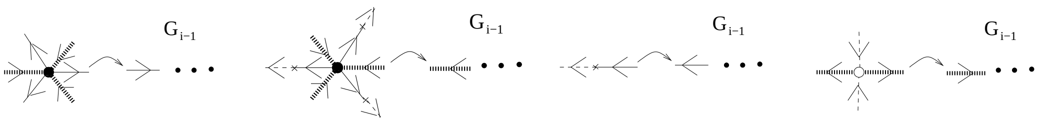
Definition 3.14 *Let* w *be a weighted comb. The multiplicity of* w*, denoted by* $\mu(w)$*, is defined as the number of chains of weighted combs which start at* w.

Theorem 3.15 *Let* A *be a trigonal* $\mathcal{L}$*-scheme in* Σ_n*, and* w *its associated weighted comb. Then* A *is realizable by real algebraic trigonal curves in* Σ_n *if and only if* $\mu(w) > 0$ *or* $w = (1, 6n, 3n, 2n)$.

Proof. Let $w = (c, \alpha, \beta, \gamma)$, and G be the real graph associated to the $\mathcal{L}$-scheme A. If $c = 1$, it is well known that A is realizable by a real algebraic trigonal curve in Σ_n. Otherwise, a chain of weighted combs starting at w is a reformulation of the statement :
"choose a half D of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$; then there exists a finite sequence $(G_i)_{0 \leq i \leq k}$ of subsets of $\mathbb{C}P^1$ such that:

- $G_0 = G$,
- for i in $\{1, \dots, \alpha\}$, the subset G_i is obtained from G_{i-1} by one of the gluings in D depicted in Figures 3.8a) and b); denote by c the number of times that G_i is obtained from G_{i-1} by the gluing depicted in Figure 3.8b) for i in $\{1, \dots, \alpha\}$,
- for i in $\{\alpha+1, \dots, \alpha+\beta-3c\}$, the subset G_i is obtained from G_{i-1} by the gluing in D depicted in Figure 3.8c),
- for i in $\{\alpha + \beta - 3c + 1, \dots, k-2\}$, the subset G_i is obtained from G_{i-1} by the gluing in D depicted in Figure 3.8d),
- G_{k-1} has no boundary and contains G_{k-2},
- $k = \alpha + \beta + \gamma - 3c + 2$,
- G_k is the gluing of G_{k-1} and its image under the complex conjugation,
- G_k is a trigonal graph such that $G_k \cap \mathbb{R}P^1 = G$".

So, according to Theorem 3.8, there exists a chain of weighted combs starting at w if and only if A is realizable by a real algebraic trigonal curve in Σ_n. □



a) b) c) d)

Figure 3.8:

Theorem 3.15 provides an algorithm to check whether an $\mathcal{L}$-scheme is realizable by a real algebraic trigonal curve in Σ_n. In order to reduce computations, one can use the following observations.

Lemma 3.16 *Let c be a closed comb, and Cl one of its closures. Suppose that $c = c_1 g_{i_1} c_2 g_{i_2} c_3$, and that g_{i_1} and g_{i_2} are joined in Cl. Then the combs c_1c_3 and c_2 contain the same number of generators g_1 (resp., g_3, g_5) and g_2 (resp., g_4, g_6).*

Proof. Straightforward. □

Lemma 3.17 *Let $(c, 0, \beta, 0)$ be an element of a chain of weighted combs. Then it is possible to join each g_1 in c to a g_2 in c by pairwise non-intersecting paths in $\mathbb{R}^2$ such that if $c = c_1 g_{i_1} c_2 g_{i_2} c_3$ with g_{i_1} and g_{i_2} joined, then the combs c_1c_3 and c_2 contain the same number of generators g_1 and g_2.*

Proof. Straightforward. □

Lemma 3.18 *Let $(c, \alpha, \beta, 0)$ be an element of a chain of weighted combs, where $c = \prod_{j=1}^{k} g_{i_j}$. Define the equivalence relation $\sim$ on $\{j \mid i_j = 1 \text{ or } 2\}$ as follows :*

$$r \sim s \text{ if the number of } g_1,\ g_2,\ g_3, \text{ and } g_4 \text{ in } \{g_{i_j} \mid j = r \dots s\} \text{ is odd.}$$

Denote by E_1^c and E_2^c the two equivalence classes of $\sim$. Then

$$|Card(E_1^c) - Card(E_2^c)| \leq \alpha.$$

Proof. Choose a chain of weighted combs $(w_i)_{1 \leq i \leq k}$ which contains $(c, \alpha, \beta, 0)$. Let $(\widetilde{c}, 0, \beta, 0)$ be an element of this chain. Then there exists $l \in \{1 \dots \alpha\}$ such that

$$Card(E_1^{\widetilde{c}}) = Card(E_1^c) - l \text{ and } Card(E_2^{\widetilde{c}}) = Card(E_2^c) - \alpha + l.$$

It is obvious that in a closure of $\widetilde{c}$, an element of $E_1^{\widetilde{c}}$ has to be joined to an element of $E_2^{\widetilde{c}}$, hence the cardinal of these two sets are equal. □

The algorithm given by Theorem 3.15 improved by Lemmas 3.16, 3.17, and 3.18, is quite efficient. It will allow us in section 5.5 to prohibit algebraically two $\mathcal{L}$-schemes pseudoholomorphically realizable.

Chapter 4

Real plane algebraic curves with asymptotically maximal number of even ovals

4.1 Motivation

An oval of a real algebraic curve of even degree is called *even* (resp., *odd*) if it is contained in an even (resp., odd) number of ovals. The number of even (resp., odd) ovals of a real plane algebraic curve of degree $2k$ is denoted by p (resp., n). This separation of ovals in two groups is important for many reasons. One of them is the fact that curves with many even ovals can be used to construct real algebraic surfaces with big Betti numbers (see section 4.5).

What are the maximal possible values for p and n with respect to k? The first step in the study of this problem is due to Ragsdale. In 1906, she conjectured in [Rag06] that $p \leq \frac{3k(k-1)}{2} + 1$ and $n \leq \frac{3k(k-1)}{2}$. About 30 years later, Petrovsky proved in [Pet33] that $p - n \leq \frac{3k(k-1)}{2} + 1$ and $n - p \leq \frac{3k(k-1)}{2}$ (these inequalities were also conjectured by Ragsdale), and formulated a conjecture similar to Ragsdale's one (it seems clear that Petrovsky was not familiar with Ragsdale's work). Combining the Petrovsky inequalities with the Harnack theorem one can obtain the following upper bounds for p and n :

$$p \leq \frac{7k^2}{4} - \frac{9k}{4} + \frac{3}{2} \text{ and } n \leq \frac{7k^2}{4} - \frac{9k}{4} + 1.$$

The first counterexamples to Ragsdale's conjecture for n (but not to Petrovsky's one) were constructed by Viro in the late 70's (see [Vir89]). In 1993, Itenberg gave in [Ite93] counterexamples to Ragsdale's and Petrovsky's conjectures. He has constructed for every positive integer k curves of degree $2k$ with $\frac{13k^2}{8} + O(k)$ even and curves of degree $2k$ with $\frac{13k^2}{8} + O(k)$ odd ovals. These lower bounds for maximal values of p and n with respect to the degree were successively improved by Haas (see [Haa95]) and Itenberg (see [Ite01]). The best lower bound known before this thesis was $\frac{81k^2}{48} + O(k)$ for both p and n. We point out the fact that no counterexample of Ragsdale's conjectures is known among M-curves.

All these constructions are based on a particular case of the Viro method, the combinatorial patch-

working. One can note that dealing with non-convex triangulations (and so with pseudo-holomorphic curves, see [IS02]), Santos ([San]) has constructed curves with $\frac{17k^2}{10} + O(k^{\frac{3}{2}})$ even ovals.
It seemed to us that the T-construction is more or less "rigid" and that the general Viro method gives one more flexibility and possibilities to construct real algebraic curves. Then, we resumed the work of Itenberg and Santos in this way, trying to increase the density of even ovals. It turned out that gluing curves whose Newton polygon is not anymore a triangle but an hexagon, it was possible to prove that the upper bounds given by the Harnack theorem and the Petrovsky inequalities are asymptotically sharp.

4.2 Main result

Theorem 4.1 *For any integer k, there exists a real algebraic curve of degree $2k$ in $\mathbb{R}P^2$ with $p = \frac{7k^2}{4} + O(k^{\frac{3}{2}})$.*
For any integer k, there exists a real algebraic curve of degree $2k$ in $\mathbb{R}P^2$ with $n = \frac{7k^2}{4} + O(k^{\frac{3}{2}})$.

Proof. The assertion relative to p is a direct consequence of corollary 4.6. The assertion relative to n can be proved, as in [Ite93] and [Ite01], by a small modification of the construction given in section 4.3 (see Figure 4.6). □

This chapter is organized as follows: in section 4.3, we prove the first part of Theorem 4.1. The constructions in this section are based on the Viro method. We assume in this section the existence of some special curves in rational geometrically ruled surfaces. The construction of the latter curves are based on the real rational graphs theoretical method and is performed in section 4.4. In section 4.5, we give some applications of Theorem 4.1 to real algebraic surfaces.

4.3 Construction of real algebraic curves with many even ovals

In this section, we use the following proposition which will be proved in section 4.4.

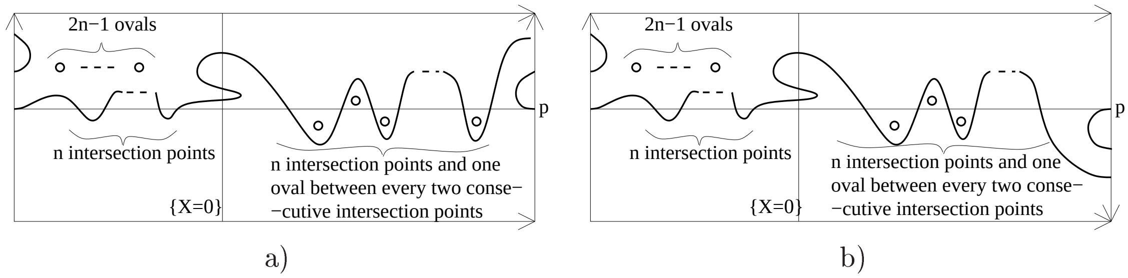


Figure 4.1:

Proposition 4.2 *For any $n \geq 1$, there exists a maximal real algebraic trigonal curve C_n in Σ_n realizing the $\mathcal{L} -$ scheme and whose position with respect to the axis $\{Y = 0\}$ and $\{X = 0\}$ is depicted in Figure 4.1a) if n is even and 4.1b) if n is odd, where p is a tangency point of order n of C_n and the axis $\{Y = 0\}$.*

For example, the curve for C_4 is depicted in figure 4.2.

Let us fix an even integer n.

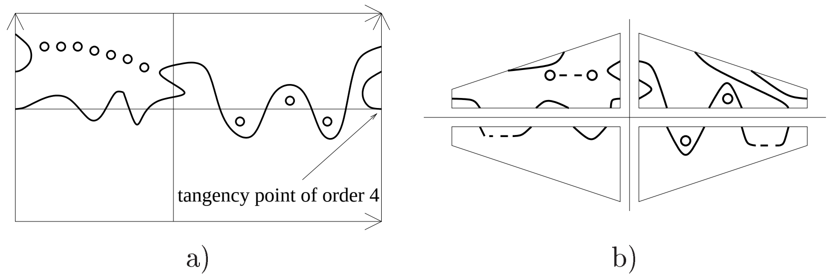


Figure 4.2:

The Newton polygon of the curve C_n is the quadrangle with vertices $(0,0)$, $(2n,0)$, $(2n,1)$ and $(0,3)$ and the chart of C_n is depicted in Figure 4.2b) (we have disjointed the 4 symmetric copies of the Newton polygon of C_n for convenience). Moreover, performing the transformation $\widetilde{Y}=\lambda Y$ if necessary, we can assume that the truncation of C_n on $[(0,3);(2n,1)]$ is $\alpha Y^3+\beta Y^2X^n+\alpha YX^{2n}$ with α and β two real numbers. Let us denote by H_n the hexagon obtained by gluing the charts of the polynomials (see Figure 4.3a))

$$X^{2n}Y^3C_n(X,Y),\ X^{2n}Y^3C_n(\tfrac{1}{X},Y),\ X^{2n}Y^3C_n(\tfrac{1}{X},\tfrac{1}{Y}),\text{ and }X^{2n}Y^3C_n(X,\tfrac{1}{Y})\ .$$

Let us fix an integer k and denote by T_{2k} the triangle with vertices $(0,0)$, $(2k,0)$ and $(0,2k)$. We start a subdivision of T_{2k} in the following way : for each integers l and h, we put the hexagon H_n centered in the point $(1+2n+4l,3+8h)$ or $(1+4n+4l,7+8h)$ if this hexagon is contained in T_{2k}. In this way, we obtain the beginning of a patchwork of a real plane curve of degree $2k$ as depicted in Figure 4.4 (here were chose n=4 for convenience).

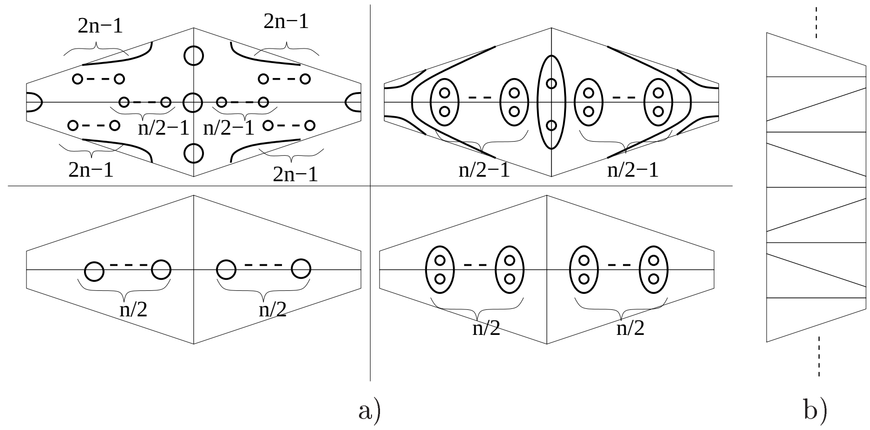


Figure 4.3:

Lemma 4.3 *The union of all the hexagons is a part of a convex subdivision of T_{2k}.*

Proof. The union of the hexagons can be decomposed into vertical strips as depicted in Figure 4.3b). Given any convex function on the left edge of the strip, one can extend it to a convex function on the whole strip which induces this subdivision. □

Let us denote by O the set composed of all the ovals obtained in our beginning of patchwork. Note that the decomposition of O in 2 disjoint subsets (ovals in one of the subset will be even in the final patchwork, and the other ovals will be odd) is already given, whatever the final patchwork is. The

proof is obvious dealing with T-constructions (the signs of the vertices of the triangulation contained in the interior of an empty oval are the same) and extends immediately to the general case.
Suppose we are given an extension of our beginning of patchwork to the whole T_{2k}, satisfying the hypothesis of the Viro Theorem. Then, by Viro Theorem, we obtain a real algebraic curve of degree $2k$ in $\mathbb{R}P^2$, which we will denote by C_k^n. Let us choose such an extension that (see Figure 4.5):

- each oval of C_k^n lying in the half plane $\{x < 0\}$ and coming from an hexagon is even and not contained in another oval of the curve,
- each oval of C_k^n lying in the quadrant $\{x > 0\}$ and coming from an empty oval of an hexagon is even and contained in two other ovals of the curve,

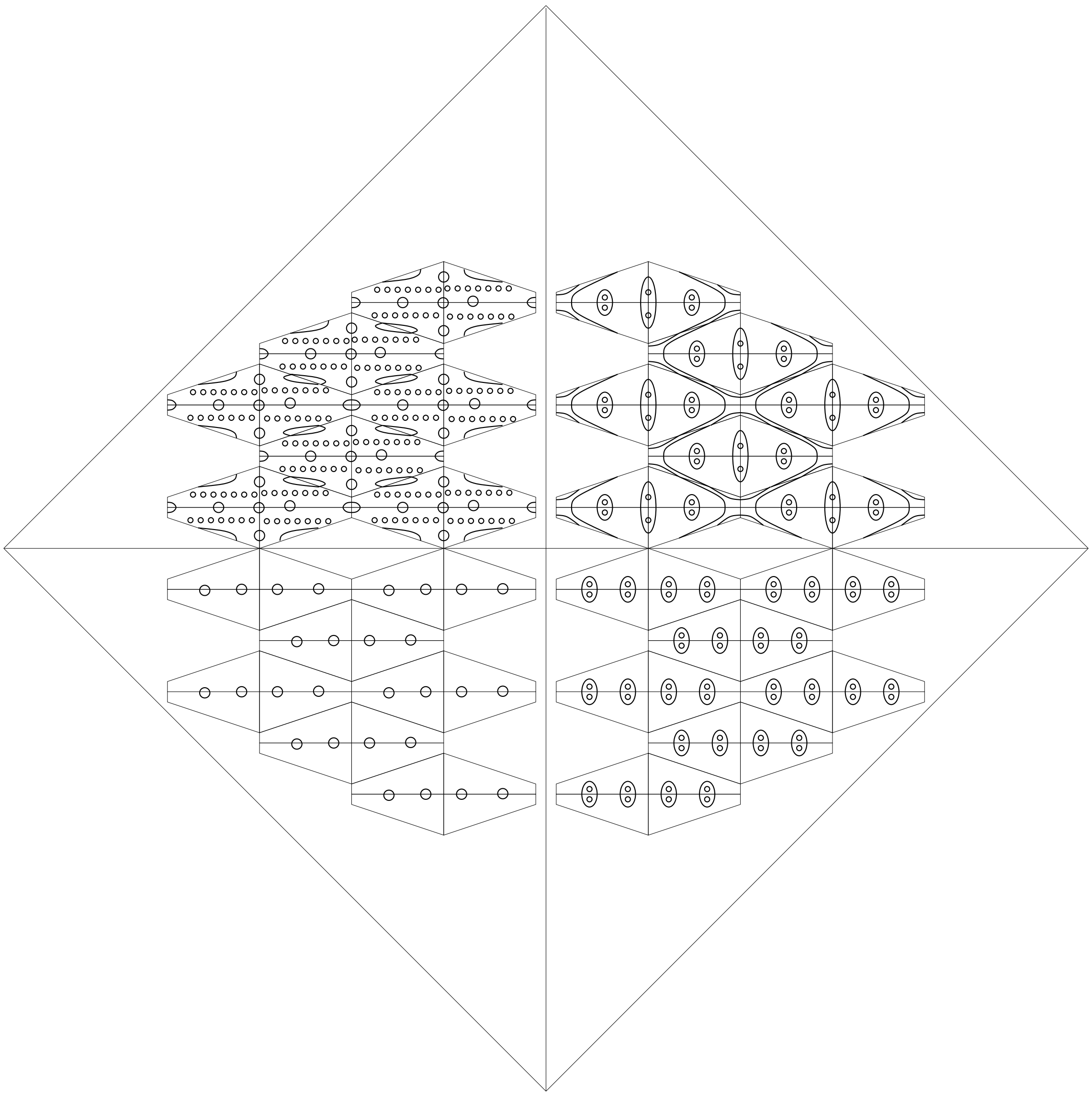


Figure 4.4:

It is clear that such an extension exists. One can construct curves we need to complete our patchwork by the classical small perturbation method (see, for example, [Vir89]). The convexity condition can be ensured, for example, by keeping on decomposing T_{2k} in strips.

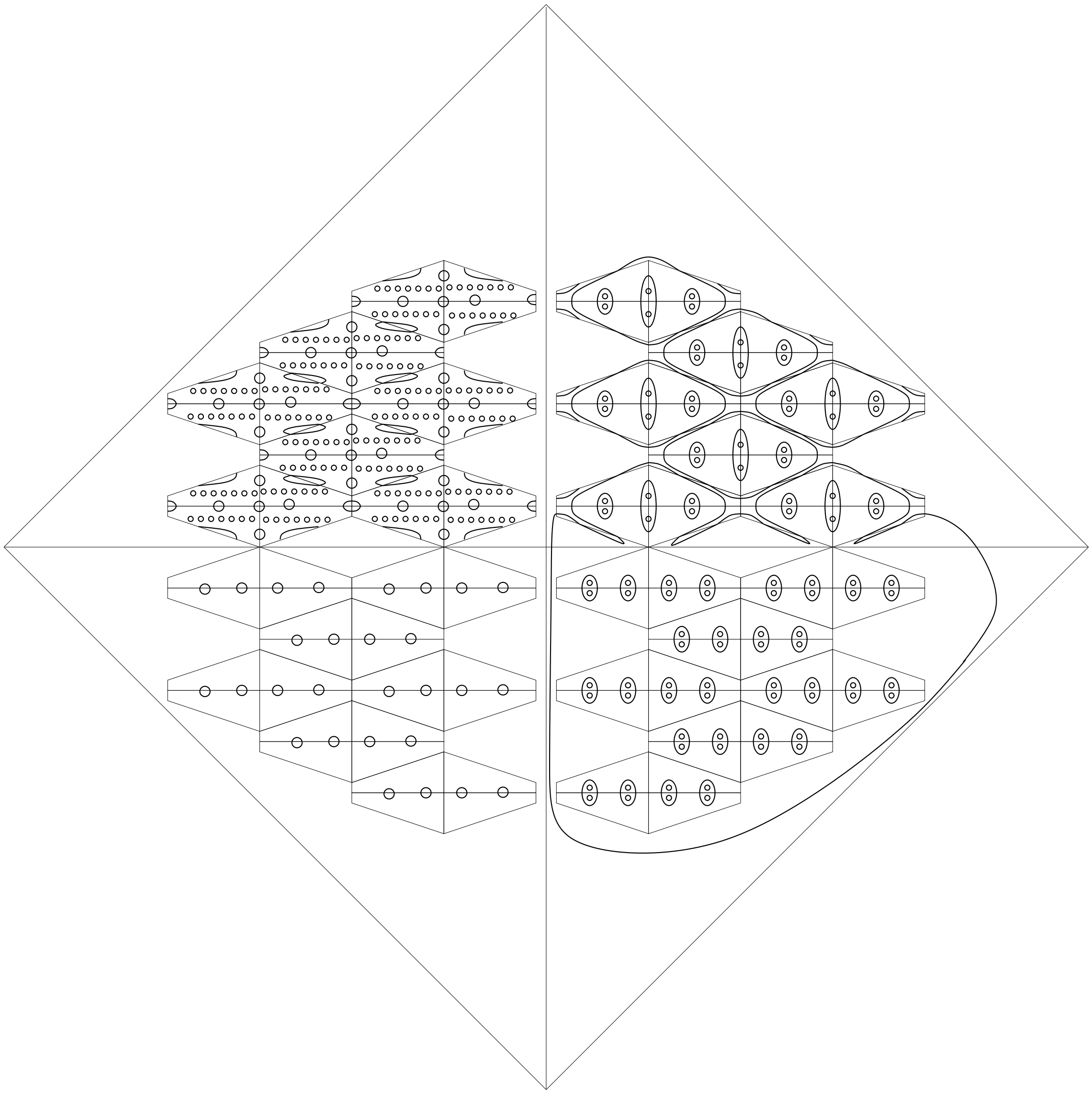


Figure 4.5:

Lemma 4.4 *Each hexagon contributes of at least* $14n - 5$ *even ovals to the curve* C_k^n.

Proof. Straightforward. □

Lemma 4.5 *The curve* C_k^n *has at least* $\frac{7k^2}{4} - \frac{5k^2}{8n} - \frac{63nk}{2} + \frac{45k}{4} + \frac{567n^2}{4} - \frac{405n}{8}$ *even ovals.*

Proof. According to Lemma 4.4, each hexagon H_n in the patchwork of the curve C_k^n gives at least $14n - 5$ even ovals. Then, if the patchwork contains N hexagons H_n, the curve C_k^n will have at least $N(14n - 5)$ even ovals. The triangle T'_{2k} with vertices $(6n, 6n)$, $(2k - 12n, 6n)$ and $(6n, 2k - 12n)$ is contained in the union of the hexagons, so

$$N \geq \frac{Area(T'_{2k})}{Area(H_n)} = \frac{(k - 9n)^2}{8n}.$$

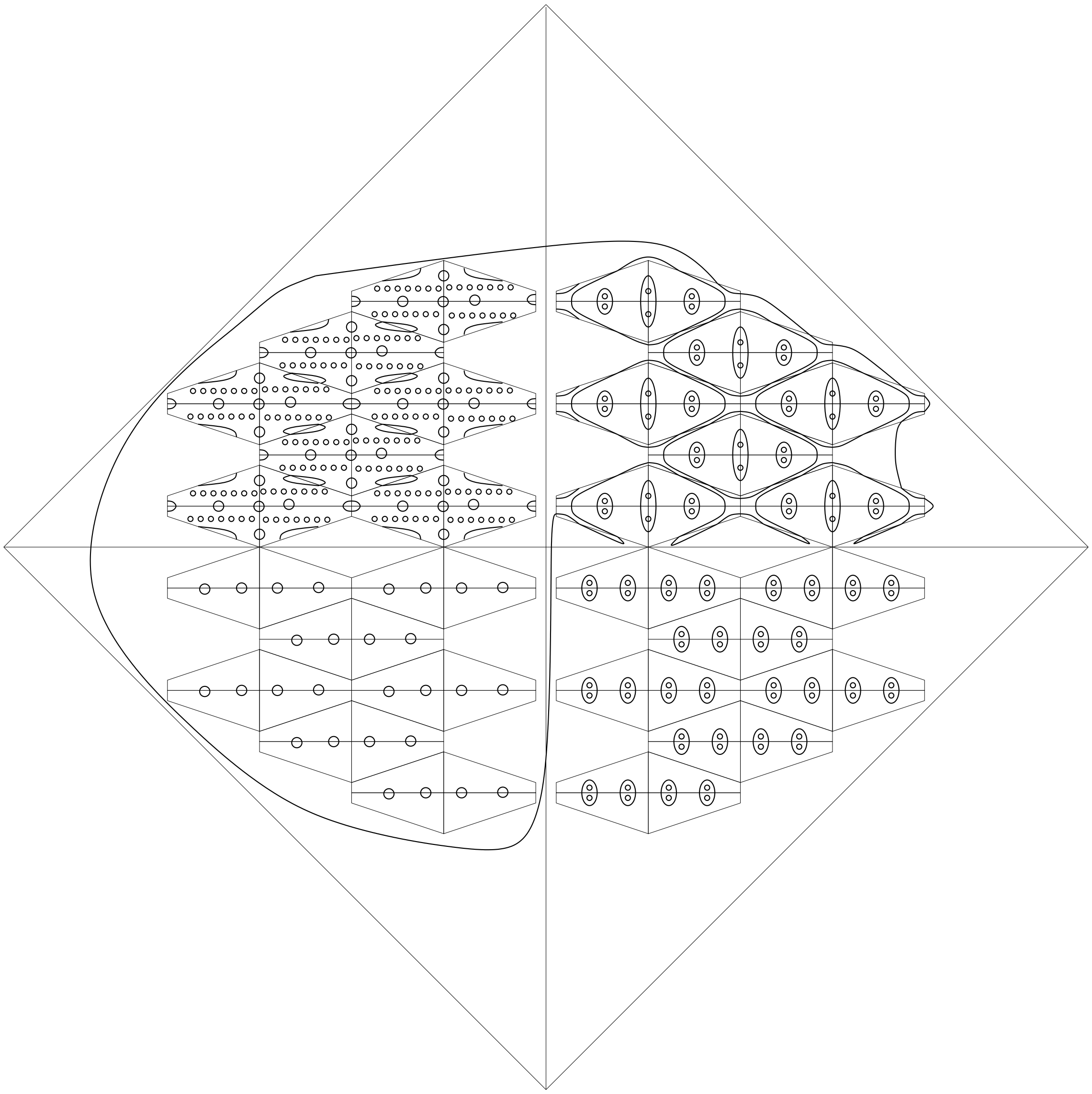


Figure 4.6:

Hence the number of even ovals of C_k^n is at least $\frac{(k-9n)^2}{8n}(14n-5)$. Developing this quantity, we obtain the lower bound stated in the lemma. □

The same construction can be done with an odd integer n. The curve obtained is also denoted by C_k^n and the lower bound of lemma 4.5 for its number of even oval is still valid.
Now we are able to prove the main theorem of this chapter. We denote the integer part of a real r by $[r]$.

Corollary 4.6 *The curve $C_k^{[\sqrt{k}]}$ has $\frac{7k^2}{4} + O(k^{\frac{3}{2}})$ even ovals.* □

4.4 Construction of reducible curves with a deep tangency point

Let us define the root schemes RS_n by

- $\Big((p,n),\left[(q,1),(r,1)^2,(q,1)\right]^k,\left[(p,1),(r,1)^2,(p,1)\right]^k,(q,1)^n\Big)$
  if $n=2k$,
- $\Big((p,n),\left[(r,1),(q,1)^2,(r,1)\right]^k,(r,1),(q,1),(p,1),(r,1),\left[(r,1),(p,1)^2,(r,1)\right]^k,(q,1)^n\Big)$
  if $n=2k+1$.

Proposition 4.7 *For any n in $\mathbb{N}$, the root scheme RS_n is realizable by polynomials of degree $2n$.*

Proof. According to lemma 3.5, one can replace (p,n) by $(p,1)^n$ in RS_n, and according to proposition 3.4, one has just to construct a rational graph on $\mathbb{C}P^1$ with a real part corresponding to the real graph of RS_n. We will prove it by induction on n. All the pictures will represent the half $\{Im(z)\le 0\}$ of $\mathbb{C}P^1$.
The rational graph corresponding to RS_1 is depicted in Figure 4.7a).

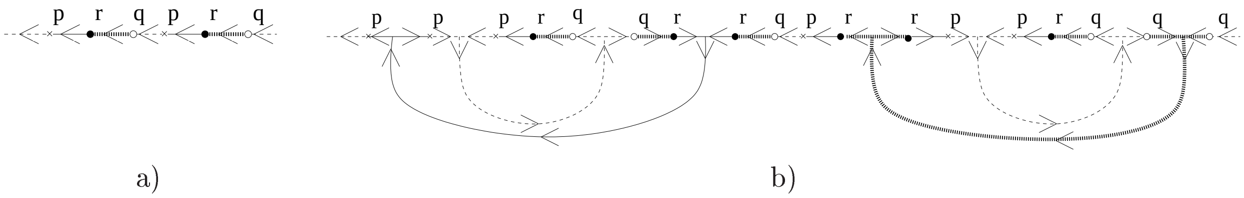


Figure 4.7:

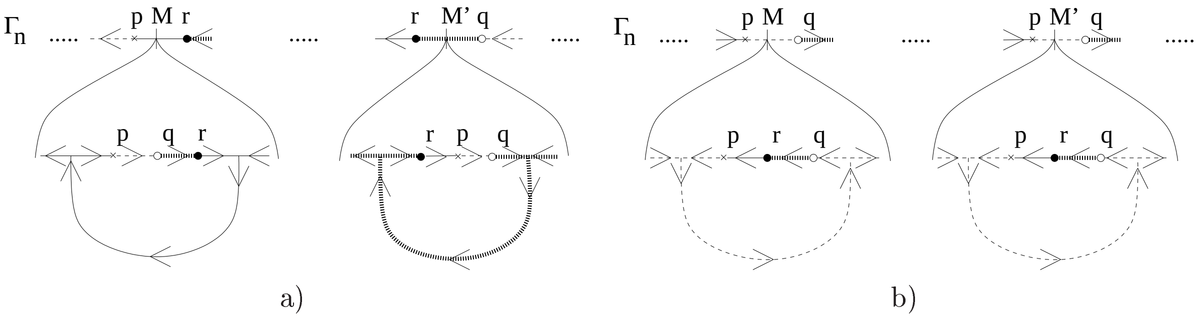


Figure 4.8:

Suppose that a rational graph Γ_n corresponding to RS_n is constructed. Let us examine $\Gamma_n\cap\mathbb{R}P^1$ from the left to the right.
Consider, first, the case $n=2k+1$.
Let M be a point on $\Gamma_n\cap\mathbb{R}P^1$ between the n^{th} point corresponding to p in RS_n and the first point corresponding to r in RS_n. Then cut Γ_n at M and glue the piece of graph depicted in Figure 4.8a).
Let M' be a point on $\Gamma_n\cap\mathbb{R}P^1$ between the last point corresponding to r in RS_n and the $n^{th}+1$ point corresponding to q in RS_n. Then cut Γ_n at M' and glue the piece of graph depicted in Figure 4.8a).
Consider now the case $n=2k$.
Let M be a point on $\Gamma_n\cap\mathbb{R}P^1$ between the n^{th} point corresponding to p in RS_n and the first point corresponding to q in RS_n. Then cut Γ_n at M and glue the piece of graph depicted in Figure 4.8b).
Let M' be a point on $\Gamma_n\cap\mathbb{R}P^1$ between the last point corresponding to p in RS_n and the $n^{th}+1$ point corresponding to q in RS_n. Then cut Γ_n at M' and glue the piece of graph depicted in Figure 4.8b).

For example, Γ_3 is depicted in Figure 4.7b). According to proposition 3.4, the rational graphs Γ_n ensure the realizability of the root schemes RS_n by polynomials of degree $2n$. $\square$

Corollary 4.8 *For any n in $\mathbb{N}$, there exist three real polynomials $a_1(X)$, $a_2(X)$ and $b(X)$ of degree n such that*

- *all the roots of a_1, a_2, b, and $a_1b + a_2$ are real,*
- *any root of a_2 and $a_1b + a_2$ is smaller than any root of b.*

Proof. Let $P(X)$, $Q(X)$, and $R(X) = P(X) + Q(X)$ be three polynomials of degree $2n$ realizing the root scheme RS_n. Then

- $Q(X) = \prod_{i=1}^{2n}(X - y_i)$ with $y_1 < y_2 < \ldots < y_{2n}$,
- $P(X) = (X - \alpha)^n \prod_{i=1}^{n}(X - x_i)$ with $\alpha < x_1 < x_2 < \ldots < x_n < y_{n+1}$,
- $R(X) = \prod_{i=1}^{2n}(X - z_i)$ with $\alpha < z_1 < z_2 < \ldots < z_{2n} < y_{n+1}$.

Let us put

- $a_2(X) = (X)^{2n} P\left(-\frac{1}{X} + \alpha\right)$,
- $A_1(X) = \prod_{i=1}^{n}(X - y_i)$,
- $a_1 = (X)^n A_1\left(-\frac{1}{X} + \alpha\right)$,
- $B(X) = \prod_{i=n+1}^{2n}(X - y_i)$,
- $b = (X)^n B\left(-\frac{1}{X} + \alpha\right)$.

As $a_1b + a_2 = (X)^{2n} R\left(-\frac{1}{X} + \alpha\right)$, the corollary follows from the definition of P, Q, and R. $\square$

Now we are able to prove proposition 4.2.
Proof of proposition 4.2. We construct here explicitly only curves in Σ_{2k}. The construction of curves in Σ_{2k+1} is done in the same way. Let us fix an even $n \geq 1$ and consider the polynomials $a_1(X)$, $a_2(X)$, and $b(X)$ of degree n constructed in corollary 4.8. Multiplying these three polynomials by -1 and performing a linear change of coordinates if necessary, we can assume that the leading coefficient of b is positive, all the roots of b are positive, and all the roots of a_2 and $a_1b + a_2$ are negative. Then, the curve $Y(Y - b(X))$ in Σ_n is depicted in Figure 4.9a).

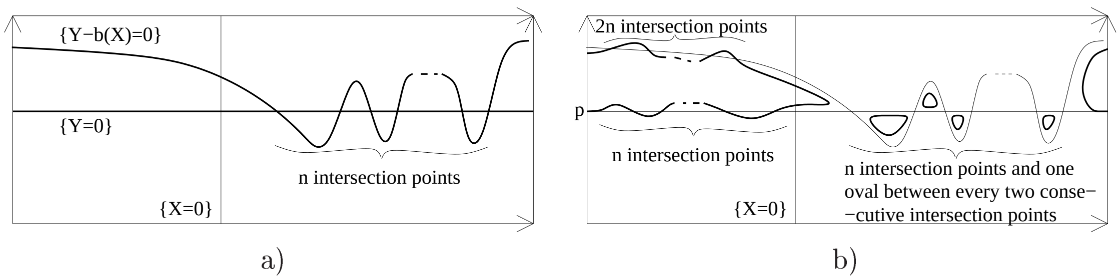


Figure 4.9:

Put $D_n(X,Y) = Y(Y - b(X)) + t(a_1(X)Y + a_2(X))$. For t small enough and of suitable sign, the relative positions of the curves $D_n(X,Y)$, $\{Y = 0\}$ and $\{Y - b(X) = 0\}$ are as depicted in Figure 4.9b), where p is a tangency point of order n of D_n and the axis $\{Y = 0\}$. Indeed, the definition of $a_1(X)$, $a_2(X)$, and $b(X)$ exactly means that the intersection points of $a_1(X)Y + a_2(X)$ and $Y(Y - b(X))$ have negative abscissa. Perturbing all the double points of $D_n(X,Y)(Y - b(X))$ in order to have the maximal number of ovals and keeping the tangency point of order n with the axis $\{Y = 0\}$, we obtain a curve C_n with n even, satisfying the conditions of proposition 4.2. $\square$

4.5 Applications to real algebraic surfaces

Here we recall and follow the notations proposed in [Bih]. We consider only $\mathbb{Z}/2\mathbb{Z}$-homology.
Let d, i, k and n be integers, $\beta_i(\mathbb{R}X_d^n)$ the i^{th} Betti number of the real part of a nonsingular real hypersurface of degree d in $\mathbb{C}P^n$, and $\beta_i(\mathbb{R}Y_{2k}^n)$ the i^{th} Betti number of the real part (for some real structure) of a double covering of $\mathbb{C}P^n$ branched along a nonsingular real hypersurface of degree $2k$. An interesting question concerns the asymptotic behavior of $max\ \beta_i(\mathbb{R}X_d^n)$ and $max\ \beta_i(\mathbb{R}Y_{2k}^n)$ when d and k go to infinity. In [Bih], Bihan has showed that there exist two sequences of real numbers $(\zeta_{i,n})_{i,n\in\mathbb{N}^2}$ and $(\delta_{i,n})_{i,n\in\mathbb{N}^2}$ such that

$$max\ \beta_i(\mathbb{R}X_d^n) \underset{d\to\infty}{\sim} \zeta_{i,n} d^n \qquad \text{and} \qquad max\ \beta_i(\mathbb{R}Y_{2k}^n) \underset{k\to\infty}{\sim} \delta_{i,n} k^n.$$

The exact value of the numbers $\zeta_{i,n}$ and $\delta_{i,n}$ are known only for small n. The following equalities are well known (see [Bih], [Ite01]).

$$\delta_{0,0} = 2,\ \zeta_{0,1} = \delta_{0,1} = \delta_{1,1} = 1,\ \zeta_{0,2} = \zeta_{1,2} = \frac{1}{2},$$

$$\delta_{0,2} \le \frac{7}{4},\ \delta_{1,2} \le \frac{7}{2},\ \zeta_{0,3} \le \frac{5}{12} \text{ and } \zeta_{1,3} \le \frac{5}{6}.$$

The upper bounds are classical and are obtained using the Harnack and Comessati-Petrovsky-Oleinik inequalities. Lower bounds for $\delta_{0,2}$ and $\delta_{1,2}$ are directly related to the asymptotically maximal number of even ovals of a curve of even degree in $\mathbb{R}P^2$, and before the results of the present paper, the best known lower bounds for these two numbers were, respectively, $\frac{27}{16}$ and $\frac{27}{8}$ (see [Ite01]). In [Bih], Bihan has constructed nonsingular real algebraic surfaces in $\mathbb{R}P^3$ with Betti numbers related to $\delta_{0,2}$ and $\delta_{1,2}$.

Theorem 4.9 (Bihan) *One has* $\frac{\delta_{0,2}}{6} + \frac{1}{12} \le \zeta_{0,3}$ *and* $\frac{\delta_{1,2}}{6} + \frac{1}{6} \le \zeta_{1,3}$.

Theorem 4.1 gives as immediate corollaries the exact values of $\delta_{0,2}$ and $\delta_{1,2}$ and improves the known lower bounds for $\zeta_{0,3}$ and $\zeta_{1,3}$.

Proposition 4.10 *One has* $\delta_{0,2} = \frac{7}{4}$ *and* $\delta_{1,2} = \frac{7}{2}$.

Corollary 4.11 *One has* $\frac{9}{24} \le \zeta_{0,3} \le \frac{5}{12}$ *and* $\frac{9}{12} \le \zeta_{1,3} \le \frac{5}{6}$.

Chapter 5

Symmetric curves of degree 7 in $\mathbb{R}P^2$

5.1 Motivation

The real symplectic isotopy problem in $\mathbb{R}P^2$ turns out to be a very difficult problem for nonsingular curves. Until now, for each known algebraic classification of nonsingular curves, the pseudoholomorphic classification is the same (and even the proofs for both classifications are alike!). So, one can tackle a simpler question, looking for example at complex curves which admit more symmetries than the action of $\mathbb{Z}/2\mathbb{Z}$ given by the complex conjugation. The first natural action is an additional holomorphic action of $\mathbb{Z}/2\mathbb{Z}$, which can be given by a symmetry of the projective plane. Such a real plane curve, invariant under a symmetry (see the definition below), is called a *symmetric curve*. The systematic study of symmetric curves was initiated by Fiedler ([Fie]) and continued by Trille (see [Tri01]). The *rigid isotopy classes* of nonsingular sextic in $\mathbb{R}P^2$ which contain a symmetric curve can be obtain from [Ite95]. Recently (see [II01]), using auxiliary conics, I. Itenberg and V. Itenberg have found an elementary proof of this classification. Once again, algebraic and pseudoholomorphic classifications coincide. On the other hand, in [OS], Orevkov and Shustin have showed that there exists a real scheme which is realizable by nonsingular symmetric pseudoholomorphic M-curves of degree 8, but which is not realizable by real symmetric algebraic curves of degree 8.

Hence, it is natural to wonder about the degree 7 and this is the subject of this chapter. It turns out that the classification of real schemes which are realizable by nonsingular symmetric curves of degree 7 in $\mathbb{R}P^2$ are again the same in both algebraic and pseudoholomorphic cases, as well as the classification of complex schemes which are realizable by nonsingular symmetric M-curves of degree 7 in $\mathbb{R}P^2$ (Theorems 5.2 and 5.3). However, if we look at real schemes which are realizable by *nonsingular dividing symmetric curves* of degree 7 in $\mathbb{R}P^2$, the answers are different. In Theorems 5.4 and 5.5, we exhibit two real schemes which are realizable by symmetric dividing pseudoholomorphic curves of degree 7 in $\mathbb{R}P^2$ but not by algebraic ones.

We begin by stating our classification results. Then the following sections are devoted to the proofs of these statements. First we prove in section 5.4 results related to the pseudoholomorphic category, using mostly the braid theoretical approach and the Rokhlin-Mischachev orientation formula. Then in section 5.5, we deal with algebraic statements, using various methods.

5.2 Definitions and statement of results

Denote by s the holomorphic involution of $\mathbb{C}P^2$ given by $[x : y : z] \mapsto [x : -y : z]$.

Definition 5.1 *A real pseudoholomorphic curve C in $\mathbb{R}P^2$ is said to be symmetric if $s(C) = C$.*

Theorem 5.2 *The following real schemes are not realizable by symmetric real pseudoholomorphic curves of degree 7 in $\mathbb{R}P^2$:*

- $\langle J \amalg (14-\alpha) \amalg 1\langle\alpha\rangle\rangle$ *with* $\alpha = 6, 7, 8, 9$,
- $\langle J \amalg (13-\alpha) \amalg 1\langle\alpha\rangle\rangle$ *with* $\alpha = 6, 7, 9$.

Moreover, any real scheme realizable by real algebraic curves of degree 7 in $\mathbb{R}P^2$ and not mentioned in the list above is realizable by symmetric real algebraic curves of degree 7 in $\mathbb{R}P^2$.

Proof. The pseudoholomorphic prohibitions are proved in Propositions 5.11 and 5.16. All the other curves are constructed algebraically in Propositions 5.36, 5.41, 5.42, and Corollary 5.40. □

Theorem 5.3 *A complex scheme is realizable by symmetric real algebraic (or pseudoholomorphic) M-curves of degree 7 in $\mathbb{R}P^2$ if and only if it is contained in the following list :*

- $\langle J \amalg 9_+ \amalg 6_-\rangle_I$
- $\langle J \amalg (7-k)_+ \amalg (6-k)_- \amalg 1_-\langle (k+1)_+ \amalg k_-\rangle\rangle_I$ *with* $k = 0, 2, 6$,
- $\langle J \amalg (7-k)_+ \amalg (6-k)_- \amalg 1_+\langle k_+ \amalg (k-1)_-\rangle\rangle_I$ *with* $k = 1, 4$,
- $\langle J \amalg (5-k)_+ \amalg (7-k)_- \amalg 1_-\langle (k+2)_+ \amalg k_-\rangle\rangle_I$ *with* $k = 0, 1, 4, 5$.

Proof. This is a direct consequence of Theorem 5.2, Corollary 5.13 and 5.15, and of the Rokhlin-Mischachev orientation formula. □

Theorem 5.4 (Pseudoholomorphic classification) *The following real schemes are not realizable by dividing symmetric real pseudoholomorphic curves of degree 7 in $\mathbb{R}P^2$:*

$$\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle \text{ with } \alpha = 2, 6 \text{ and } \alpha + \beta = 12,\ \alpha = \beta = 4,$$

Moreover, any real scheme mentioned in Proposition 2.54 and not forbidden by Theorem 5.2 and the above list is realizable by dividing symmetric real pseudoholomorphic curves of degree 7 in $\mathbb{R}P^2$; any real scheme mentioned in Proposition 2.55 which is not forbidden by Theorem 5.2 is realizable by non-dividing symmetric real pseudoholomorphic curves of degree 7 in $\mathbb{R}P^2$.

Proof. All the pseudoholomorphic prohibitions are proved in Propositions 5.22 and 5.24. All the constructions are done in Propositions 5.37, 5.46, 5.47, 5.48, and Corollaries 5.39 and 5.28. □

Theorem 5.5 (Algebraic classification) *The real schemes*

$$\langle J \amalg 8 \amalg 1\langle 4\rangle\rangle \text{ and } \langle J \amalg 4 \amalg 1\langle 8\rangle\rangle$$

are not realizable by a dividing symmetric real algebraic curve of degree 7 in $\mathbb{R}P^2$. Any other real scheme which is realizable by dividing symmetric real pseudoholomorphic curves of degree 7 in $\mathbb{R}P^2$ is realizable by dividing symmetric real algebraic curves of degree 7 in $\mathbb{R}P^2$.
Any real scheme which is realizable by non-dividing symmetric real pseudoholomorphic curves of degree 7 in $\mathbb{R}P^2$ is realizable by non-dividing symmetric real algebraic curves of degree 7 in $\mathbb{R}P^2$.

Proof. The two algebraic prohibitions are proved in Propositions 5.23, 5.31, and 5.35. All the constructions are done in Propositions 5.37, 5.46, 5.47, 5.48, and Corollary 5.39. □

5.3 General facts about symmetric curves in the real plane

We denote by B_0 the line $\{Y = 0\}$. The involutions s and $conj$ commute, so $s \circ conj$ is an anti-holomorphic involution of $\mathbb{C}P^2$. The real part of this real structure is a real projective plane

$$\widetilde{\mathbb{R}P^2} = \{[x_0 : ix_1 : x_2] \in \mathbb{C}P^2 \mid (x_0, x_1, x_2) \in \mathbb{R}^3 \setminus \{0\}\}.$$

It is clear that $\widetilde{\mathbb{R}P^2} \cap \mathbb{R}P^2 = \mathbb{R}B_0 \cup \{[0 : 1 : 0]\}$. A symmetric pseudoholomorphic curve C is real for the structures defined by $conj$ and $s \circ conj$. Denote by $\widetilde{\mathbb{R}C}$ the real part of C with respect to $s \circ conj$, and call this real part the *mirror curve* of $\mathbb{R}C$.
For a maximal symmetric pseudoholomorphic curve, the real scheme realized by the mirror curve is uniquely determined.

Theorem 5.6 (Fiedler, [Fie], [Tri01]) *The mirror curve of a maximal symmetric pseudoholomorphic curve of degree $2k + 1$ is a nest of depth k with a pseudo-line.*

Denote by $\mathcal{L}_p$ the pencil of lines through the point $[0 : 1 : 0]$ in $\mathbb{C}P^2$. If C is a real symmetric pseudoholomorphic curve of degree 7 in $\mathbb{R}P^2$, the curve $X = C/s$ is a pseudoholomorphic curve of bidegree $(3, 1)$ in Σ_2 and is called the *quotient curve* of C. The $\mathcal{L}$-scheme realized by $\mathbb{R}X$ is obtained by gluing the $\mathcal{L}_p$-schemes realized by $\mathbb{R}C$ and $\widetilde{\mathbb{R}C}$ along B_0.
Conversely, a symmetric pseudoholomorphic curve is naturally associated to an arrangement of a curve X of bidegree $(3, 1)$ and a base in Σ_2.

Proposition 5.7 (Fiedler, [Fie], [Tri01]) *If C is a dividing symmetric pseudoholomorphic curve of degree d in $\mathbb{R}P^2$, then*

$$Card(\mathbb{R}C \cap B_0) = d \text{ or } Card(\mathbb{R}C \cap B_0) = 0$$

Thus, in the case of a dividing symmetric curve of odd degree, all the common points of the curve and B_0 are real.

Proposition 5.8 *Let C be a nonsingular real pseudoholomorphic symmetric curve in $\mathbb{R}P^2$. Then C is smoothly isotopic to a nonsingular real pseudoholomorphic symmetric curve C' in $\mathbb{R}P^2$ such that all tangency points between the lines of $\mathcal{L}_p$ and the invariant components of $\widetilde{\mathbb{R}C'}$ lie on B_0.*

Proof. Suppose that some tangency points of the invariant components of $\widetilde{\mathbb{R}C'}$ with a line of $\mathcal{L}_p$ do not lie on B_0. Then push all the corresponding tangent points of the quotient curve above B_0 applying the first elementary operation of Proposition 2.79. The resulting symmetric curve satisfies the conditions of the proposition. □

For example, the symmetric pseudoholomorphic curves of degree 4 depicted in Figures 5.1a) and d) are isotopic in $\mathbb{C}P^2$. The dashed curve represents their mirror curves. The corresponding quotient curves are depicted in Figures 5.1b) and c).
The curves $\mathbb{R}C$ and $\mathbb{R}C'$ have the same complex orientations. Using the invariant components of $\widetilde{\mathbb{R}C'}$, one can apply the Fiedler orientations alternating rule to $\mathbb{R}C'$.

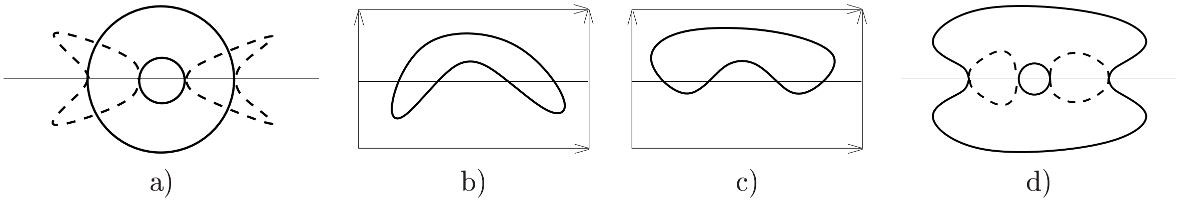


Figure 5.1:

Proposition 5.9 *If C is a dividing symmetric pseudoholomorphic curve of odd degree in $\mathbb{R}P^2$, then an oval of $\mathbb{R}C$ and an oval of $\widetilde{\mathbb{R}C}$ cannot intersect in more than 1 point.*

Proof. Suppose that there exists a symmetric dividing curve C such that some oval of $\mathbb{R}C$ intersects some oval of $\widetilde{\mathbb{R}C}$ in 2 points. Then, because both $\mathbb{C}$ and $\widetilde{\mathbb{R}C'}$ have a pseudo-line and according to Proposition 5.8, the curve C is isotopic to a dividing pseudoholomorphic symmetric curve C' with an invariant oval O of $\mathbb{R}C'$, and an oval of $\widetilde{\mathbb{R}C'}$ is as shown in Figure 5.2, which is impossible according to the Fiedler alternating rule. □

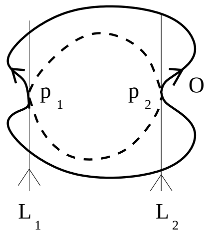


Figure 5.2:

In general, there is no link between the type of the symmetric curve and the type of its quotient curve. However, if both the symmetric curve and its mirror curve are of type I, there is no ambiguity.

Proposition 5.10 (Trille, [Tri01]) *If a symmetric pseudoholomorphic curve in $\mathbb{R}P^2$ and its mirror curve are of type I, so is the quotient curve.*

5.4 Pseudoholomorphic statements

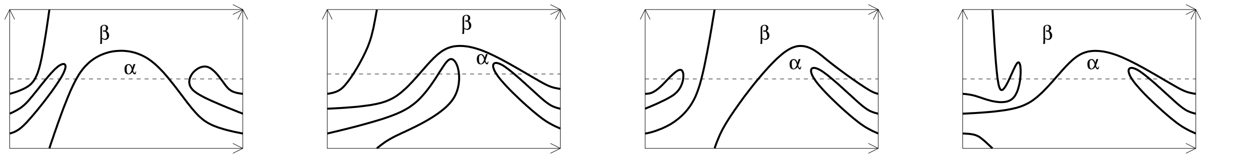
5.4.1 Prohibitions for curves of bidegree $(3, 1)$ in Σ_2

Proposition 5.11 *The real schemes $\langle J \amalg \beta \amalg 1\langle\alpha\rangle\rangle$ with*

- $\alpha = 6, 7, 8, 9$ *and* $\beta = 14 - \alpha$,
- $\alpha = 7, 9$ *and* $\beta = 13 - \alpha$,

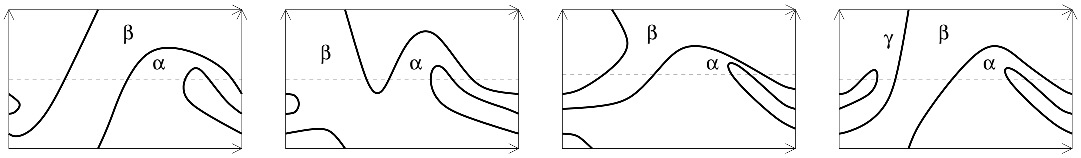
cannot be realized by symmetric pseudoholomorphic curves of degree 7 in $\mathbb{R}P^2$.

Proof. According to Theorem 5.6, Proposition 5.7, the Bézout theorem, Lemmas 2.49 and 2.51, the only possibilities for the $\mathcal{L}$-schemes of the quotient curves are depicted in Figure 5.3 for the four M-curves, and in Figures 5.4a), b) and c), for the two $(M-1)$-curves.
The curves in Figures 5.4a) (resp., c)) and 5.3c) (resp., d)) have the same position with respect to $\mathcal{L}$, so they give the same braid. Moreover, the braid corresponding to the curves in Figures 5.3d) and b) are the same. Thus, according to Lemma 2.49, the corresponding braids are :



$(\alpha,\beta) = (3,3)$ or $(4,2)$ a) $(\alpha,\beta) = (3,3)$ or $(2,4)$ b) $(\alpha,\beta) = (3,3)$ or $(4,2)$ c) $(\alpha,\beta) = (3,3)$ or $(4,2)$ d)

Figure 5.3:



a) b) c) d)

$(\alpha,\beta) = (3,3)$ or $(4,2)$, $\gamma+\delta = 6$

Figure 5.4:

$$
\begin{aligned}
b^1_{\alpha,\beta} &= \sigma_2\sigma_1\sigma_2^{-(1+\beta)}\sigma_1^{-1}\sigma_2\sigma_1^{-\alpha}\sigma_2^{-1}\sigma_1\Delta_3^2 \text{ with } (\alpha,\beta) = (3,3),(4,2),\\
b^2_{\alpha,\beta} &= \sigma_1\sigma_2^{-\beta}\sigma_1^{-1}\sigma_2\sigma_1^{-\alpha}\Delta_3^2 \text{ with } (\alpha,\beta) = (3,3),(2,4),(4,2),\\
b^3_{\alpha,\beta} &= \sigma_2^{-1}\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_2^{-\beta}\sigma_1^{-1}\sigma_2\sigma_1^{-\alpha}\Delta_3^2 \text{ with } (\alpha,\beta) = (3,3),(4,2),\\
b^4 &= \sigma_2^{-7}\sigma_1\sigma_2\Delta_3^2.
\end{aligned}
$$

The Alexander polynomials of these braids are

$$
\begin{aligned}
&p^1_{3,3} = p^2_{3,3} = p^3_{4,2} = (-1+t)^3, \qquad p^1_{4,2} = p^2_{2,4} = p^2_{4,2} = p^3_{3,3} = (-1+t)(t^2-t+1),\\
&p^4 = (-1+t)(t^4-t^3+t^2-t+1).
\end{aligned}
$$

In each case, $e(b) = 1$ and the Alexander polynomial is not identically zero. So, according to Proposition 2.80, none of these braids is quasipositive. □

Proposition 5.12 *The topology of the quotient curve of a maximal nonsingular symmetric pseudoholomorphic curve in* $\mathbb{R}P^2$ *of degree* 7 *with a nest containing at least* 2 *ovals, with respect to the base* $\{y=0\}$ *of* Σ_2 *is uniquely determined by the real scheme of the symmetric curve.*

Proof. Let C be a symmetric M-curve of degree 7 with a nest containing at least 2 ovals. Then the possible $\mathcal{L}$-schemes for the quotient curves of C are depicted in Figures 5.3a) and b) with $\alpha = 1,2,3,4,5,6$ if the number of inner ovals is even and are depicted in Figures 5.3c) and d) with $\alpha = 0,1,2,3,4,5,6$ if the number of inner ovals is odd. So, according to Lemma 2.49, the corresponding braids are :

$$
\begin{aligned}
b^1_{\alpha,\beta} &= \sigma_2\sigma_1\sigma_2^{-(1+\beta)}\sigma_1^{-1}\sigma_2\sigma_1^{-\alpha}\sigma_2^{-1}\sigma_1\Delta_3^2 \text{ with } \alpha+\beta = 6 \text{ and } \alpha = 1,2,3,4,5,6,\\
b^2_{\alpha,\beta} &= \sigma_1\sigma_2^{-\beta}\sigma_1^{-1}\sigma_2\sigma_1^{-\alpha}\Delta_3^2 \text{ with } \alpha+\beta = 6 \text{ and } \alpha = 0,1,2,3,4,5,6,\\
b^3_{\alpha,\beta} &= \sigma_2^{-1}\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_2^{-\beta}\sigma_1^{-1}\sigma_2\sigma_1^{-\alpha}\Delta_3^2 \text{ with } \alpha+\beta = 6 \text{ and } \alpha = 0,1,2,3,4,5,6.
\end{aligned}
$$

The braids $b^1_{3,3}$, $b^1_{4,2}$, $b^2_{2,4}$, $b^2_{3,3}$, $b^2_{4,2}$, $b^3_{3,3}$, and $b^3_{4,2}$ were already shown to be not quasipositive in the proof of Proposition 5.11. The Alexander polynomials of the remaining braids are :

$p^1_{2,4} = (1-t)(1-t+t^2)$, $\quad p^1_{6,0} = p^2_{0,6} = p^2_{6,0} = p^3_{1,5} = (-1+t)(t^4-t^3+t^2-t+1)$,
$p^3_{0,6} = (t^2+t+1)(t^2-t+1)(-1+t)^3$, $\quad p^3_{5,1} = (-1+t)(t^2-t+1)$,
$p = 0$ for the other braids.

In each case, $e(b) = 1$, so if the Alexander polynomial of the braid is not identically null, according to Proposition 2.80, the braid is not quasipositive. Thus, $b^1_{\alpha,\beta}$ is not quasipositive for $\alpha = 2,3,4,6$, $b^2_{\alpha,\beta}$ is not quasipositive for $\alpha = 0,2,3,4,6$, and $b^3_{\alpha,\beta}$ is not quasipositive for $\alpha = 0,1,3,4,5$. The other braids are quasipositive, as the constructions performed in section 5.5 showed (however, it is not difficult to check this directly on the braids). Hence, the topology of the quotient curve with respect to the base $\{y=0\}$ of Σ_2 is uniquely determined by the topology of the symmetric curve. □

Corollary 5.13 *A complex scheme is realizable by symmetric real algebraic (or pseudoholomorphic) M-curves of degree 7 in $\mathbb{R}P^2$ with a nest containing at least 2 ovals if and only if it is contained in the following list :*

- $\langle J \amalg (7-k)_+ \amalg (6-k)_- \amalg 1_-\langle (k+1)_+ \amalg k_-\rangle\rangle_I$ *with* $k = 2,6$,
- $\langle J \amalg (7-k)_+ \amalg (6-k)_- \amalg 1_+\langle k_+ \amalg (k+1)_-\rangle\rangle_I$ *with* $k = 1,4$,
- $\langle J \amalg (5-k)_+ \amalg (7-k)_- \amalg 1_-\langle (k+2)_+ \amalg k_-\rangle\rangle_I$ *with* $k = 0,1,4,5$.

Proof. If C is a maximal nonsingular symmetric curve in $\mathbb{R}P^2$ of degree 7 with a nest containing at least 2 ovals, all the possibilities for the $\mathcal{L}$-scheme of the arrangement of its quotient curve and the section $\{y=0\}$ of Σ_2 are described in Proposition 5.12. Such an arrangement determines uniquely the complex orientations of the maximal symmetric curve of degree 7. Now, it is not difficult to check that the complex orientations are exactly those stated in the corollary. □

Proposition 5.14 *If C is a maximal nonsingular symmetric pseudoholomorphic curve in $\mathbb{R}P^2$ of degree 7 with a nest containing only one oval, then the only possibilities for the arrangement of its quotient curve and the base $\{y=0\}$ of Σ_2 are depicted in Figure 5.4d) with $(\gamma,\delta) = (4,2)$ or $(0,6)$.*

Proof. If C has exactly one inner oval, the $\mathcal{L}$-schemes of the arrangement of its quotient curve and the base $\{y=0\}$ of Σ_2 can only be as depicted in 5.4d). The braids corresponding to these curves are

$$b^5_{\gamma,\delta} = \sigma_2^{-(1+\delta)}\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_2^{-\gamma}\sigma_1^{-1}\sigma_2\Delta_3^2 \text{ with } \gamma+\delta = 6 \text{ and } \gamma = 0,1,2,3,4,5,6.$$

The corresponding Alexander polynomials are :

$p^5_{1,5} = p^5_{3,3} = (-1+t)(t^2-t+1)$, $\quad p^5_{2,4} = (-1+t)^3$,
$p^5_{5,1} = (-1+t)(t^4-t^3+t^2-t+1)$, $\quad p^5_{6,0} = (t^2-t+1)(t^2+t+1)(-1+t)^3$,
$p^5_{0,6} = p^5_{4,2} = 0$.

In each case, $e(b) = 1$, so according to Proposition 2.80, only the braids $b^5_{0,6}$ and $b^5_{4,2}$ can be quasipositive. □

Corollary 5.15 *If C is a maximal nonsingular symmetric pseudoholomorphic curve in $\mathbb{R}P^2$ of degree 7 with a nest containing only one oval, then its complex scheme is*

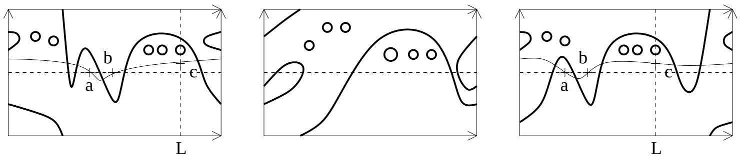
$$\langle J \amalg 7_+ \amalg 6_- \amalg 1_-\langle 1_+ \amalg\rangle\rangle_I$$

Proof. If C is a maximal nonsingular symmetric curve in $\mathbb{R}P^2$ of degree 7 with a nest containing only 1 oval, the only two possibilities for the $\mathcal{L}$-scheme of the arrangement of its quotient curve and the section $\{y = 0\}$ of Σ_2 are described in Proposition 5.14. The symmetric curves corresponding to these arrangements have the complex orientations stated in the corollary. □

5.4.2 Prohibitions for reducible curves of bidegree $(4, 1)$ in Σ_2

Proposition 5.16 *The real scheme $\langle J \amalg 7 \amalg 1\langle 6 \rangle\rangle$ is not realizable by a nonsingular symmetric pseudoholomorphic curves of degree 7 in $\mathbb{R}P^2$.*

Proof. Here, as it is not sufficient to look only at the $\mathcal{L}$-schemes of the quotient curves in Σ_2, we consider the mutual arrangement of the quotient curves and a base of Σ_2. The possible $\mathcal{L}$-schemes of the quotient curves for the symmetric curve announced above with respect to the base $\{y = 0\}$ are depicted in Figure 5.5.



a) b) c)

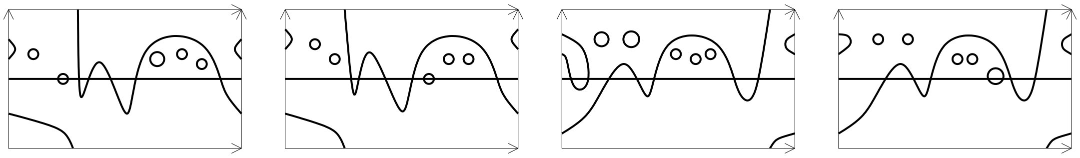
Figure 5.5:

The $\mathcal{L}$-scheme in Figures 5.5b) cannot be realized pseudoholomorphically : the braid corresponding to this $\mathcal{L}$-scheme and its Alexander polynomial are respectively :

$$\sigma_1^{-4}\sigma_2^2\sigma_1^{-3}\sigma_2^{-1}\sigma_1\Delta_3^2 \quad \text{and } (-1+t)^3.$$

The Alexander polynomial is not null although $e(b) = 1$, so according to Proposition 2.80, the braid is not quasipositive.

Consider the base H passing through the points a, b and c in Figures 5.5a) and c), where c is a point of the fiber L. If the point c ranges on L from 0 to ∞, then, because of the choice of L, for some c, the base H passes through an oval. Now, we want H to pass through the first oval we meet as c ranges from 0 to ∞. The only possible mutual arrangements for H and the quotient curves which do not contradict the Bézout theorem are shown in Figure 5.6. The corresponding braids are :



a) b) c) d)

Figure 5.6:

$$\begin{aligned}
b^6 &= \sigma_3^{-2}\sigma_2^{-2}\sigma_3^{-1}\sigma_1\sigma_2^2\sigma_1^{-4}\sigma_2^{-1}\sigma_3\sigma_2^{-3}\sigma_3^{-1}\sigma_2\sigma_1^{-1}\Delta_4^2,\\
b^7 &= \sigma_3^{-3}\sigma_1\sigma_2^2\sigma_1^{-4}\sigma_2^{-1}\sigma_3\sigma_1^{-2}\sigma_2^{-3}\sigma_3^{-1}\sigma_2\sigma_1^{-1}\Delta_4^2,\\
b^8 &= \sigma_2^{-2}\sigma_3^{-3}\sigma_1^{-3}\sigma_2^{-1}\sigma_3\sigma_2^{-3}\sigma_3^{-1}\sigma_2\sigma_1^{-2}\sigma_2^{-1}\sigma_3^2\sigma_2\sigma_1\Delta_4^2,\\
b^9 &= \sigma_3^{-3}\sigma_1^{-3}\sigma_2^{-1}\sigma_3\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1^{-2}\sigma_2^{-1}\sigma_3^2\sigma_2\sigma_1\Delta_4^2.
\end{aligned}$$

The computation of the corresponding Alexander polynomials gives :

$$p^6 = (t^2 - t + 1)(t^6 - 3t^5 + 6t^4 - 5t^3 + 6t^2 - 3t + 1)(-1 + t)^3, \quad p^8 = (-1 + t)^7,$$
$$p^7 = (2t^4 - 2t^3 + 3t^2 - 2t + 2)(t^2 - t + 1)^2(-1 + t)^3, \quad p^9 = (t^2 - t + 1)(-1 + t)^3.$$

In each case, $e(b) = 2$, so according to Proposition 2.80, none of these braids is quasipositive. □

5.4.3 Utilization of the Rokhlin-Mischachev orientation formula

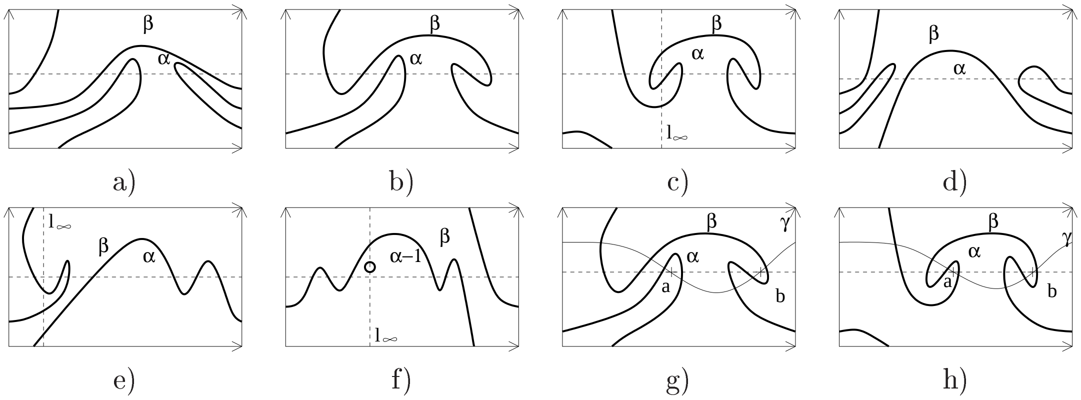


Figure 5.7:

Lemma 5.17 *There does not exist a symmetric dividing pseudoholomorphic curve of degree 7 in $\mathbb{R}P^2$ with a quotient curve realizing the $\mathcal{L}$-scheme depicted in Figures 5.7a) and d) with $\alpha + \beta$ odd.*

Proof. Such a quotient curve is of type II, because it is an $(M-1)$-curve. The mirror curve of the initial symmetric curve is a nest of depth 3 with a pseudo-line, and so is of type I. Thus, according to Proposition 5.10, the initial symmetric curve cannot be of type I. □

Lemma 5.18 *There does not exist a symmetric dividing pseudoholomorphic curve of degree 7 in $\mathbb{R}P^2$ with a quotient curve realizing the $\mathcal{L}$-scheme depicted in Figure 5.7b) with $\alpha + \beta$ odd.*

Proof. According to the Fiedler orientation alternating rule on symmetric curves corresponding to these quotient curves, the symmetric curves cannot be of type I if $\alpha + \beta$ is odd, as the two invariant empty ovals have opposite orientations. □

Lemma 5.19 *There does not exist a symmetric dividing pseudoholomorphic curve of degree 7 in $\mathbb{R}P^2$ with a quotient curve realizing the $\mathcal{L}$-scheme depicted in Figure 5.7c) with $(\alpha, \beta) = (4, 1)$, $(3, 2)$, $(2, 3)$, and $(1, 2)$.*

Proof. According to the Fiedler orientation alternating rule on symmetric curves corresponding to these quotient curves, the three invariant ovals are positive, and hence we have $\Lambda_+ - \Lambda_- = 1$, $\Pi_+ - \Pi_- = 0$ if α is odd, and $\Pi_+ - \Pi_- = -2$ if α is even. Thus, the Rokhlin-Mischachev orientation formula is fulfilled only for $(\alpha, \beta) = (3, 2)$. Choose l_∞ as depicted in Figure 5.7c), and the corresponding braid and its Alexander polynomial are respectively :

$$\sigma_1^{-4}\sigma_2^{-1}\sigma_1\sigma_2^{-2}\sigma_1\Delta_3^2 \quad \text{and } (-1 + t)^3.$$

Since $e(b) = 1$, according to Proposition 2.80, the braid is not quasipositive. □

Lemma 5.20 *There does not exist a symmetric dividing pseudoholomorphic curve of degree* 7 *in* $\mathbb{R}P^2$ *with a quotient curve realizing the* $\mathcal{L}$*-scheme depicted in Figure 5.7e) with* $(\alpha, \beta) = (5, 0)$, $(4, 1)$, $(3, 2)$, $(2, 3)$, *and* $(2, 1)$.

Proof. According to the Fiedler orientation alternating rule on symmetric curves corresponding to these quotient curves, the two invariant empty ovals have opposite orientations, the non-empty oval is negative, and we have $\Lambda_+ - \Lambda_- = 1$, $\Pi_+ - \Pi_- = 0$ if α is even, and $\Pi_+ - \Pi_- = -2$ if α is odd. Hence, the Rokhlin-Mischachev orientation formula is fulfilled only for $(\alpha, \beta) = (4, 1)$ or $(2, 3)$. Choose l_∞ as depicted in Figure 5.7e), and the corresponding braids are :

$$b^{10}_{\alpha,\beta} = \sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-(1+\beta)}\sigma_2^{-1}\sigma_3\sigma_2^{-\alpha}\sigma_3^{-1}\sigma_2\sigma_1^{-3}\sigma_3\sigma_2\sigma_1\sigma_3^{-1}\Delta_4^2.$$

The computation of the corresponding Alexander polynomials gives

$$p^{10}_{4,1} = (t^2+1)(t^2-t+1)(-1+t)^3, \quad p^{10}_{2,3} = (t^4-2t^3+4t^2-2t+1)(-1+t)^3.$$

In each case we have $e(b) = 2$, so according to Proposition 2.80, both braids are not quasipositive. □

Lemma 5.21 *There does not exist a symmetric dividing pseudoholomorphic curve of degree* 7 *in* $\mathbb{R}P^2$ *with a quotient curve realizing the* $\mathcal{L}$*-scheme depicted in Figure 5.7f) with* $(\alpha, \beta) = (5, 0)$, $(3, 2)$, *and* $(2, 3)$.

Proof. Choose l_∞ as depicted in Figure 5.7f), and the corresponding braids are :

$$b^{11}_{\alpha,\beta} = \sigma_2^{-\alpha}\sigma_3^{-1}\sigma_2\sigma_3^{-\beta}\sigma_1^{-3}\sigma_1\sigma_2^2\sigma_1^{-4}\sigma_2^{-1}\sigma_3\Delta_4^2.$$

The computation of the determinant gives 976 for $b^{11}_{5,0}$ and 592 for $b^{11}_{3,2}$ which are not squares in $\mathbb{Z}$ although $e(b) = 3$. So according to Proposition 2.81, these two braids are not quasipositive.
The computation of the Alexander polynomial of $b^{11}_{2,3}$ gives

$$(t^2+1)(t^6-5t^5+12t^4-14t^3+12t^2-5t+1)(-1+t)^2.$$

The number i is a simple root of this polynomial and $e(b) = 3$. Thus, according to Proposition 2.82, this braid is not quasipositive. □

Proposition 5.22 *The real schemes* $\langle J \amalg 2 \amalg 1\langle 10\rangle\rangle$ *and* $\langle J \amalg 6 \amalg 1\langle 6\rangle\rangle$ *are not realizable by non-singular symmetric dividing pseudoholomorphic curves of degree* 7 *in* $\mathbb{R}P^2$.

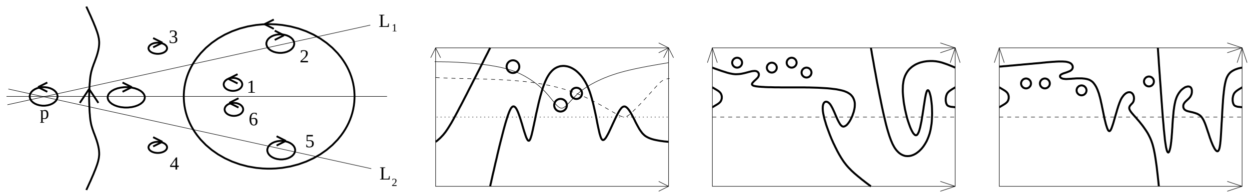
Proof. According to the Bézout theorem, Proposition 5.9, Lemma 2.49 and Lemma 2.51, the only possibilities for the $\mathcal{L}$-scheme of the quotient curve of such a dividing symmetric curve of degree 7 in $\mathbb{R}P^2$ are depicted in Figures 5.7a), g) and h) with $(\alpha, \beta + \gamma) = (4, 1)$ and $(2, 3)$ and in Figures 5.7d), e) and f) with $(\alpha, \beta) = (5, 0)$, and $(3, 2)$. If a trigonal curve realizes one of the two $\mathcal{L}$-schemes depicted in Figures 5.7g) and h) then $\gamma = 0$. Otherwise, the base passing through the points a and b and through an oval γ intersects the quotient curve in more than 7 points, and we get a contradiction with the Bézout theorem.
The remaining quotient curves have been prohibited in Lemmas 5.17, 5.18, 5.19, 5.20, and 5.21 □

Proposition 5.23 *If a nonsingular symmetric dividing pseudoholomorphic curve of degree 7 in $\mathbb{R}P^2$ realizes the real scheme $\langle J \amalg 8 \amalg 1\langle 4\rangle\rangle$, then the $\mathcal{L}$-scheme of its quotient curve is depicted in Figure 5.7c) with $(\alpha,\beta)=(1,4)$.*
If a nonsingular symmetric dividing pseudoholomorphic curve of degree 7 in $\mathbb{R}P^2$ realizes the real scheme $\langle J \amalg 2\beta + 2 \amalg 1\langle 2\alpha\rangle\rangle$ with $(\alpha,\beta)=(4,1)$ or $(2,1)$, then the $\mathcal{L}$-scheme of its quotient curve is as depicted in Figure 5.7f).

Proof. The proof is the same as for the previous proposition. □

Proposition 5.24 *The real scheme $\langle J\amalg 4 \amalg 1\langle 4\rangle\rangle$ is not realizable by nonsingular symmetric dividing pseudoholomorphic curves of degree 7 in $\mathbb{R}P^2$.*

Proof. Suppose the contrary. Then, according to Proposition 5.23, the quotient curve X of such a symmetric curve would be depicted in Figure 5.7f) with $(\alpha,\beta)=(2,1)$. Using the Fiedler orientation alternating rule, and denoting by ϵ the sign of the two non-invariant outer ovals of the symmetric curve, we would have $\Lambda_+ - \Lambda_- = -1+2\epsilon$ and $\Pi_+ - \Pi_- = 0$. Thus, the Rokhlin-Mischachev orientation formula would be fulfilled only if $\epsilon=-1$. Hence, one of the two complex orientations of the curve would be depicted in Figure 5.8a). Using once again the Fiedler orientation alternating rule, we see that the pencil of lines through the point p would induce an order on the 6 non-invariant ovals of the symmetric curve as depicted in Figure 5.8a). So the ovals 4 and 1 would not be on the same connected component of $\mathbb{R}P^2 \setminus (L_1 \cup L_2)$. The mutual arrangement of the curve X and the quotient curve of $(L_1 \cup L_2)$ (in bold line) would be depicted in Figure 5.8b). Now, consider the base of Σ_2 which passes through the three ovals of X. It would have 9 intersection points with X which contradicts the Bézout theorem. □



a) b) c) d)

Figure 5.8:

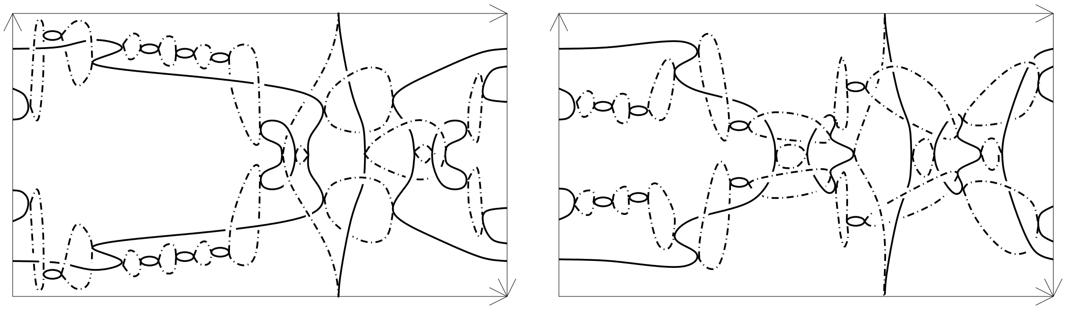
5.4.4 Constructions

Proposition 5.25 *There exist nonsingular real pseudoholomorphic curves of bidegree $(3,1)$ in Σ_2 such that the $\mathcal{L}$-scheme realized by the union of this curve and a base is as shown in Figures 5.8c) and d). In particular, all the real tangency points of the curve with the pencil $\mathcal{L}$ are above the base $\{y=0\}$.*

Proof. The braids associated to these $\mathcal{L}$-schemes are

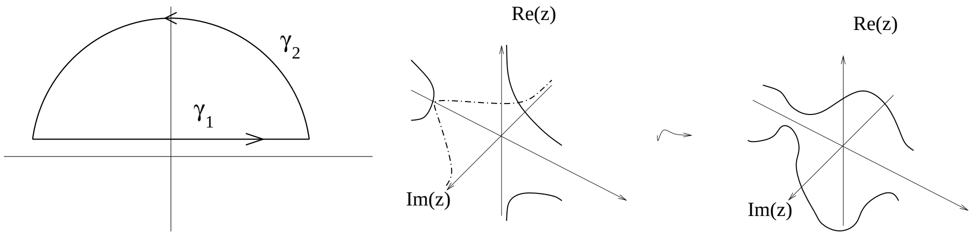
$$\begin{aligned} b^{12} &= \sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-3}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2^{-2}\sigma_3^{-1}\sigma_1\sigma_2^2\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}\sigma_2^{-1}\Delta_4^2,\\ b^{13} &= \sigma_2^{-3}\sigma_3^{-1}\sigma_2^{-1}\sigma_3\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1^{-2}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_1\sigma_2^2\sigma_1^{-2}\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_3\Delta_4^2. \end{aligned}$$

The Garside normal forms of both these braids are the trivial one, so Lemma 2.75 tells us that they are are trivial, so quasipositive. □



a) b)

Figure 5.9:



a) b)

Figure 5.10:

Denote by C (resp., C') the symmetric nonsingular pseudoholomorphic curve of degree 7 in $\mathbb{R}P^2$ corresponding to the quotient curve depicted in Figure 5.8c) (resp., d)). Let us still denote by C (resp., C') their strict transform by the blow up of $\mathbb{C}P^2$ at $[0 : 1 : 0]$. Also denote by p the intersection point of C with the exceptional section E and F_p the fiber of Σ_1 through p. Then, the curve C has a tangency point of order 2 with F_p at p.

In order to prove Proposition 5.26, we recall some of notations of section 2.7 and introduce some others. For $\epsilon > 0$, denote by $\gamma_{1,\epsilon}$ and $\gamma_{2,\epsilon}$ the followings paths

$$\begin{array}{cccc} \gamma_{1,\epsilon} : & [-1;1] & \rightarrow & \mathbb{C} \\ & t & \mapsto & \frac{1}{\epsilon}t + i\epsilon \end{array} \quad \text{and} \quad \begin{array}{cccc} \gamma_{2,\epsilon} : & [0;1] & \rightarrow & \mathbb{C} \\ & t & \mapsto & \frac{1}{\epsilon}e^{i\Pi t} + i\epsilon \end{array}.$$

Choose γ_ϵ a parametrization of the union of the image of $\gamma_{1,\epsilon}$ and $\gamma_{2,\epsilon}$ (see Figure 5.10a)). Denote also by :

- π the projection $\Sigma_1 \to \mathbb{C}P^1$ on the base $\{Y = 0\}$,
- π' the restriction of π to $\Sigma_1 \setminus E$,
- $S_A = (\pi_{|A})^{-1}(\mathbb{R}P^1)$,
- D_ϵ the compact region of $\mathbb{C}$ bounded by γ_ϵ,
- $b_{\mathbb{R},\epsilon} = \pi'^{-1}(\gamma_{1,\epsilon}([-1;1]) \cap A$,
- $b_{\infty,\epsilon} = \pi'^{-1}(\gamma_{2,\epsilon}([0;1]) \cap A$,
- $b_\epsilon = b_{\mathbb{R},\epsilon} b_{\infty,\epsilon}$,
- $N_\epsilon = \pi'^{-1}(D_\epsilon) \cap A$.

Thus , $b_{\mathbb{R},\epsilon}$ and $b_{\infty,\epsilon}$ are braids, and $b_{\infty,\epsilon} = \Delta_6$.
As C is a real curve, the real part of S_C is $\mathbb{R}C$ and the non-real part of S_C has several connected components which are globally invariant by the complex conjugation. One can deduce S_C (resp., $S_{C'}$) from the quotient curve of C. The curve S_C (resp., $S_{C'}$) is depicted in Figure 5.9a) (resp., b)), where the bold lines represent $\mathbb{R}C$ ant the dash lines represent $S_C \setminus \mathbb{R}C$. The braid $b_{\mathbb{R},\epsilon}$ for ϵ small enough can be viewed as a smoothing of S_C.

Proposition 5.26 *The real pseudoholomorphic curve C constructed above is a dividing curve.*

Proof. Except for the point p, all singular points of S_C are smoothed as explained in Section 2.7. There exist local coordinates (z, w) in a neighborhood of p in Σ_1 such that in this neighborhood, a line of $\mathcal{L}$ has equation $z = const$ and C has equation $z = \frac{1}{w^2}$. Then, for ϵ small enough, the smoothing of S_C at p is given by the parametrization

$$\begin{array}{ccc} [0;1] & \to & [0;1] \times \mathbb{C} \\ t & \mapsto & (\epsilon e^{i\Pi(1-t)}, \pm\frac{1}{\epsilon} e^{-\frac{i\Pi(1-t)}{2}}). \end{array}$$

So the smoothing of the connected component of C which contains p is depicted in Figure 5.10b). The closure of the braid b_ϵ corresponding to C has 6 components denoted by $L_1, \ldots, L_6$ as depicted in Figure 5.11. By the Riemann-Hurwitz formula, we have

$$\mu(N_\epsilon) = g(N_\epsilon) + \frac{\mu(\hat{b}_\epsilon) + 6 - e(b_\epsilon)}{2},$$

where $\mu(N_\epsilon)$ is the number of connected components of N_ϵ and $g(N_\epsilon)$ is the sum of the genus of the connected components of N_ϵ. We have $\mu(\hat{b}_\epsilon) = 6$, $e(b_\epsilon) = 0$ and $\mu(N_\epsilon) \leq 6$, so N_ϵ is made out of 6 disks. Denote these disks by $D_{1,\epsilon}, \ldots, D_{6,\epsilon}$ in such a way that $\partial D_{i,\epsilon} = L_{i,\epsilon}$. Define also $\overline{D_{i,\epsilon}} = conj(D_{i,\epsilon})$. When ϵ tends to 0, these 12 disks glue together along S_C, and C is the result of this gluing. Moreover, $C \setminus \mathbb{R}C$ is the result of the gluing of these 12 disks along $S_C \setminus \mathbb{R}C$. Hence, to find the type of C, we just have to study how the 12 disks glue along $S_C \setminus \mathbb{R}C$.

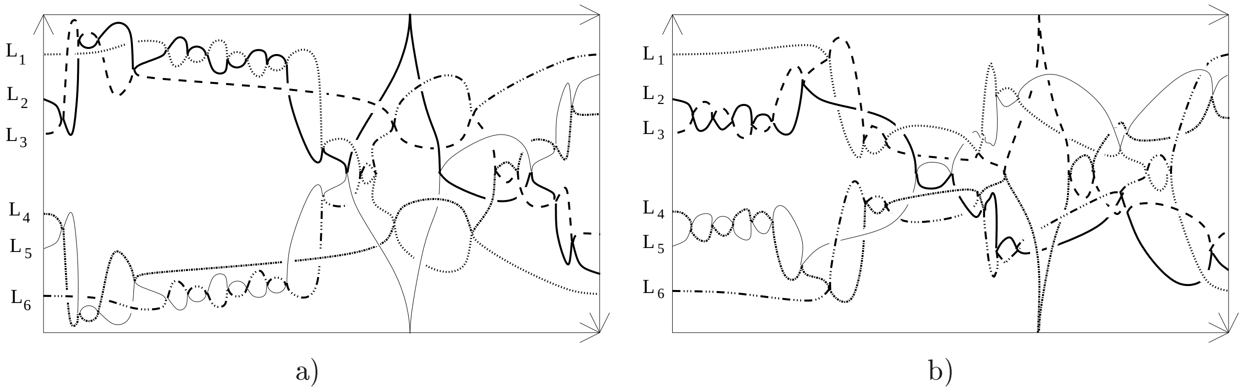


Figure 5.11:

Denote by $D_{i,\epsilon}||D_{j,\epsilon}$ the relation "$D_{i,\epsilon}$ and $D_{j,\epsilon}$ are glued along a connected component of $S_C \setminus \mathbb{R}C$ as ϵ tends to 0". Using the fact that each connected component of $S_C \setminus \mathbb{R}C$ is globally invariant under the complex conjugation, we have (see Figure 5.11a)) :

$$\begin{array}{lllll} D_{1,\epsilon}||\overline{D_{2,\epsilon}}, & D_{1,\epsilon}||\overline{D_{6,\epsilon}}, & D_{1,\epsilon}||\overline{D_{4,\epsilon}}, & D_{2,\epsilon}||\overline{D_{3,\epsilon}}, & D_{2,\epsilon}||\overline{D_{5,\epsilon}}, \\ D_{3,\epsilon}||\overline{D_{6,\epsilon}}, & D_{3,\epsilon}||\overline{D_{4,\epsilon}}, & D_{4,\epsilon}||\overline{D_{5,\epsilon}}, & D_{5,\epsilon}||\overline{D_{6,\epsilon}}, & D_{i,\epsilon}||\overline{D_{j,\epsilon}} \Longrightarrow D_{j,\epsilon}||\overline{D_{i,\epsilon}}. \end{array}$$

The curve C is a dividing curve if and only if there exist two equivalence classes for $||$. Here the equivalence classes are $\{D_{1,\epsilon}, D_{3,\epsilon}, D_{5,\epsilon}, \overline{D_{2,\epsilon}}, \overline{D_{4,\epsilon}}, \overline{D_{6,\epsilon}}\}$ and $\{D_{2,\epsilon}, D_{4,\epsilon}, D_{6,\epsilon}, \overline{D_{1,\epsilon}}, \overline{D_{3,\epsilon}}, \overline{D_{5,\epsilon}}\}$, Hence, C is a dividing curve and the proposition is proved. □

Proposition 5.27 *The real pseudoholomorphic curve C' constructed above is a dividing curve.*

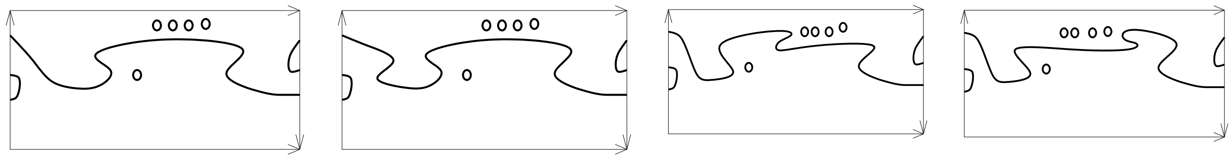
Proof. We keep the same notations as in Proposition 5.26. As in this proposition, the closure of the braid b_ϵ has 6 components, the surface N_ϵ is composed by 6 disks, and the two equivalence classes for the relation $||$ are $\{D_{1,\epsilon}, D_{2,\epsilon}, D_{4,\epsilon}, \overline{D_{3,\epsilon}}, \overline{D_{5,\epsilon}}, \overline{D_{6,\epsilon}}\}$ and $\{D_{3,\epsilon}, D_{5,\epsilon}, D_{6,\epsilon}, \overline{D_{1,\epsilon}}, \overline{D_{2,\epsilon}}, \overline{D_{4,\epsilon}}\}$ (see Figure 5.11b)). Hence, C is a dividing curve and the proposition is proved. □

Corollary 5.28 *The complex schemes $\langle J \amalg 4_+ \amalg 4_- \amalg 1_+ \langle 2_+ \amalg 2_- \rangle\rangle_I$ and $\langle J \amalg 2_+ \amalg 2_- \amalg 1_+ \langle 4_+ \amalg 4_- \rangle\rangle_I$ are realizable by nonsingular symmetric real pseudoholomorphic curves of degree 7 in $\mathbb{R}P^2$.*

5.5 Algebraic statements

5.5.1 Prohibitions

We use here the comb theoretical method exposed in section 3.3.2. In particular, we use the algorithm to compute the multiplicity of a weighted comb, that is to say to check whether a trigonal $\mathcal{L}$-scheme is algebraically realizable or not. This algorithm turned out to be very efficient for our purpose and all the needed calculations have been done on a computer in a short time.



a) b) c) d)

Figure 5.12:

Lemma 5.29 *The $\mathcal{L}$-schemes depicted in Figures 5.12b), c) and d) are not realizable by nonsingular real trigonal pseudoholomorphic curves in Σ_3.*

Proof. Compute the braids associated to these $\mathcal{L}$-schemes :

$$b^{14} = \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_1\sigma_2^{-4}\sigma_1^{-1}\Delta_3^3, \qquad b^{15} = \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-5}\sigma_1^{-1}\Delta_3^3,$$
$$b^{16} = \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-4}\sigma_1^{-1}\sigma_2^{-1}\Delta_3^3.$$

These braids verify $e(b) = 0$, so they are quasipositive if and only if they are trivial. Computing their Garside normal form, we find

$$b^{14} = \sigma_2^3\sigma_1^2\sigma_2^2\sigma_1^2\Delta_3^{-3}, \quad b^{15} = \sigma_1\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^2\Delta_3^{-3}, \quad b^{16} = \sigma_1\sigma_2^3\sigma_1^2\sigma_2^2\sigma_1\Delta_3^{-3}.$$

Thus, according to Lemma 2.75, no one of these braids is quasipositive. □

Lemma 5.30 *The $\mathcal{L}$-scheme depicted in Figure 5.12a) is not realizable by nonsingular real algebraic trigonal curves in Σ_3.*

Proof. The weighted comb associated to this $\mathcal{L}$-scheme is

$$w_1 = (g_3g_6g_1g_4g_1g_6g_5g_2g_3g_6g_1g_4g_1g_6(g_3g_2)^3g_3g_6g_1g_4g_1g_6g_5g_2, 1, 2, 0).$$

We have $\mu(w_1) = 0$ so, according to Proposition 3.15 the lemma is proved.

Proposition 5.31 *The real scheme $\langle J\amalg 8\amalg 1\langle 4\rangle\rangle$ is not realizable by nonsingular symmetric dividing real algebraic curves of degree 7 in $\mathbb{R}P^2$.*

Proof. Suppose that there exists a dividing symmetric curve which contradicts the lemma. Let us denote by X its quotient curve. Blow up Σ_2 at the intersection point of X and the exceptional section and blow down of the strict transform of the fiber. Then the strict transform of X has a double point with non-real tangents. Smooth this double point in order to obtain an oval. Then, according to Propositions 5.23 and 3.15, the only possible $\mathcal{L}$-schemes which can be obtained are depicted in Figures 5.12a), b), c), and d). However, according to Lemmas 5.29 and 5.30, these $\mathcal{L}$-schemes are not algebraically realizable, so there is a contradiction. ☐

Lemma 5.32 *None of the $\mathcal{L}$-schemes depicted in Figures 5.13a), ..., f) is realizable by nonsingular real pseudoholomorphic trigonal curves in Σ_5.*

Proof. Compute the braids associated to these $\mathcal{L}$-schemes :

$$\begin{array}{ll} b^{17} = \sigma_2^{-3}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}c, & b^{18} = \sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{-1}\sigma_2\sigma_1^{-1}c, \\ b^{19} = \sigma_2^{-2}\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^{-1}c, & b^{20} = \sigma_2^{-4}\sigma_1^{-2}\sigma_2\sigma_1^{-1}c, \\ b^{21} = \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}\sigma_2\sigma_1^{-1}c, & b^{22} = \sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-1}\sigma_1\sigma_2^{-2}\sigma_1^{-1}\sigma_2\sigma_1^{-1}c, \end{array}$$
$$\text{where } c = \sigma_1^{-3}\sigma_2^{-1}\sigma_1\sigma_2^{-4}\sigma_1^{-1}\sigma_2\Delta_3^5.$$

The braids b^{17} and b^{21} verify $e(b) = 2$, and the computation of their determinant gives 301 and 805, respectively, and these numbers are not squares in $\mathbb{Z}$. So according to Proposition 2.81, the two braids are not quasipositive.
The braids b^{18}, b^{19}, b^{20}, and b^{22} verify $e(b) = 1$, and the computation of their Alexander polynomials gives respectively

$$\begin{array}{l} p^{18} = (-1+t)(t^2+1)(t^2-t+1), \\ p^{19} = (-1+t)(t^2-t+1)(t^2+1)(t^6-t^5+2t^4-3t^3+2t^2-t+1), \\ p^{20} = (-1+t)(t^4-t^3+2t^2-2t+1)(t^4-2t^3+2t^2-t+1)(t^2+1)^2, \\ p^{22} = (-1+t)(t^4-t^3+t^2-t+1)(t^6-2t^5+4t^4-5t^3+4t^2-2t+1)(t^2-t+1)(t^2+1). \end{array}$$

According to Proposition 2.80, these four braids are not quasipositive. ☐

Lemma 5.33 *The $\mathcal{L}$-schemes depicted in Figures 5.13g), ..., l) are not realizable by nonsingular real algebraic trigonal curves in Σ_5.*

Proof. The weighted combs associated to these $\mathcal{L}$-schemes are

$$\begin{array}{ll} w_2 = (ag_3bg_5g_2ag_3bag_3bg_5g_2ag_3ba^3g_3b, 1, 4, 0), & w_5 = (g_3bg_5g_2a^2g_3b(a^3g_3b)^2, 2, 5, 1), \\ w_3 = (ag_3bg_5g_2ag_3ba^3g_5bg_3ba^3g_3b, 1, 4, 0), & w_6 = (ag_3bg_5g_2ag_3b(a^3g_3b)^2, 2, 5, 1), \\ w_4 = (g_3bg_5g_2a^2g_3bg_3bg_5g_2a^2g_3ba^3g_3b, 1, 4, 0), & w_7 = ((a^3g_3b)^3, 3, 6, 2)\ , \end{array}$$
$$\text{where } a = g_3g_2 \text{ and } b = g_6g_1g_4g_1g_6.$$

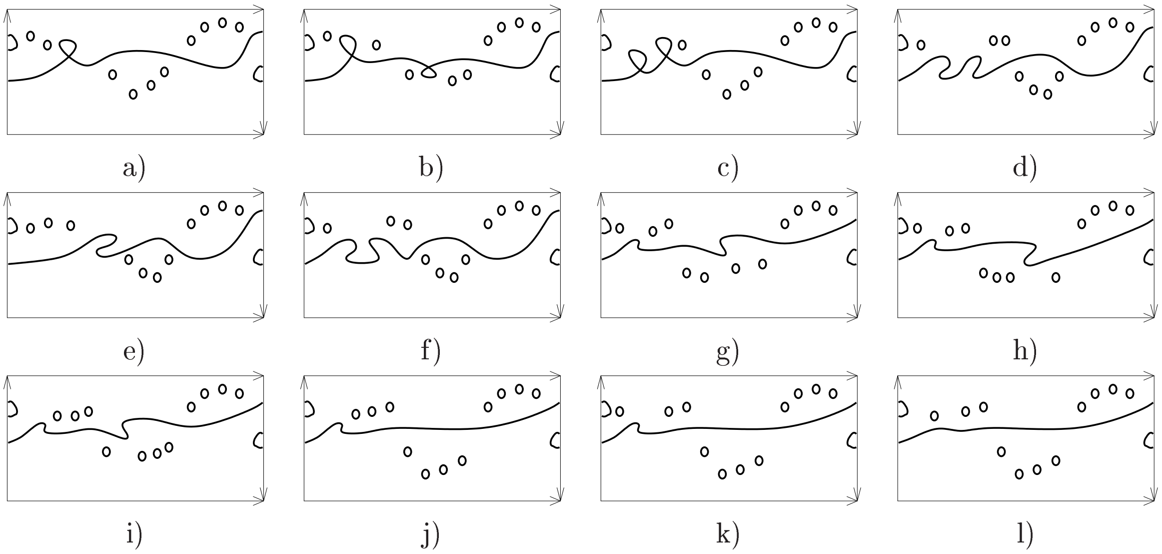


Figure 5.13:

The multiplicity of all these weighted combs is 0. Thus, according to Proposition 3.15, none of these $\mathcal{L}$-schemes is realizable by nonsingular real algebraic trigonal curves in Σ_5. $\square$

Definition 5.34 *An $\mathcal{L}$-scheme is minimal if no operation $\subset_j \supset_{j\pm1} \rightarrow \emptyset$ is possible on it.*

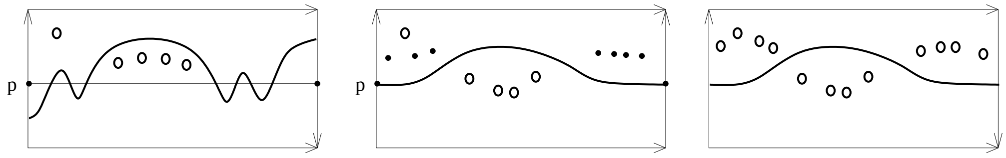
Proposition 5.35 *The real scheme $\langle J \amalg 4 \amalg 1\langle 8\rangle\rangle$ is not realizable by nonsingular symmetric dividing real algebraic curves of degree 7 in $\mathbb{R}P^2$.*

Proof. Suppose that there exists a dividing symmetric curve which contradicts the lemma. Let us denote by X its quotient curve. Then, according to Proposition 5.23, the only possible minimal $\mathcal{L}$-scheme for X with respect to a base is depicted in Figure 5.7f). Blow up Σ_2 at the intersection point of X and the exceptional section and blow down the strict transform of the fiber. Then the strict transform of X, denoted by X' is a trigonal curve in Σ_3 with a double point with non-real tangents at p. Moreover, X' is arranged with respect to a base of Σ_3 as depicted in Figure 5.14a). Then the cubic resolvent of the union of the base and X' is a real algebraic trigonal curve on Σ_6 depicted in Figure 5.14b). The point p is a triple point and the other singular points are double points with non-real tangents. Smoothing all the double points in order to obtain ovals, blowing up Σ_6 at p and blowing down the strict transform of the fiber, we obtain a nonsingular algebraic trigonal curve C in Σ_5 realizing the minimal $\mathcal{L}$-scheme depicted in Figure 5.14c). According to Proposition 2.79 and Lemma 5.32, the only possibilities for the $\mathcal{L}$-scheme realized by C which do not contradict obviously Proposition 3.15 are depicted in Figures 5.13g), ..., l). It has been proved in Lemma 5.33 that all these $\mathcal{L}$-schemes are not algebraically realizable, so there is a contradiction. $\square$

5.5.2 Perturbation of a reducible symmetric curve

The standard method to construct a lot of different isotopy types of nonsingular algebraic curves is to perturb a singular curve in many ways. So the first idea to construct symmetric algebraic curves is to perturb in many symmetric ways a singular symmetric algebraic curve.

Proposition 5.36 *All the real schemes listed in the following table are realizable by nonsingular symmetric real algebraic curves of degree 7 in $\mathbb{R}P^2$. Moreover, those marked with the symbol **



a) b) c)

Figure 5.14:

are realized by a dividing symmetric curve and those marked with the symbol $^{\circ}$ *are realized by a non-dividing curve.*

$\langle J\rangle^{\circ}$	$\langle J\amalg 10\amalg 1\langle 1\rangle\rangle^{\circ}$	$\langle J\amalg 8\amalg 1\langle 3\rangle\rangle^{\circ}$	$\langle J\amalg 1\langle 6\rangle\rangle^{\circ}$	$\langle J\amalg 1\langle 9\rangle\rangle^{\circ}$
$\langle J\amalg 1\rangle^{\circ}$	$\langle J\amalg 11\amalg 1\langle 1\rangle\rangle^{\circ,*}$	$\langle J\amalg 9\amalg 1\langle 3\rangle\rangle^{\circ,*}$	$\langle J\amalg 1\amalg 1\langle 6\rangle\rangle^{\circ}$	$\langle J\amalg 1\amalg 1\langle 9\rangle\rangle^{\circ,*}$
$\langle J\amalg 2\rangle^{\circ}$	$\langle J\amalg 12\amalg 1\langle 1\rangle\rangle^{\circ}$	$\langle J\amalg 10\amalg 1\langle 3\rangle\rangle^{\circ}$	$\langle J\amalg 2\amalg 1\langle 6\rangle\rangle^{\circ}$	$\langle J\amalg 2\amalg 1\langle 9\rangle\rangle^{\circ}$
$\langle J\amalg 3\rangle^{\circ}$	$\langle J\amalg 13\amalg 1\langle 1\rangle\rangle^{*}$	$\langle J\amalg 11\amalg 1\langle 3\rangle\rangle^{*}$	$\langle J\amalg 3\amalg 1\langle 6\rangle\rangle^{\circ}$	$\langle J\amalg 3\amalg 1\langle 9\rangle\rangle^{\circ,*}$
$\langle J\amalg 4\rangle^{\circ}$			$\langle J\amalg 4\amalg 1\langle 6\rangle\rangle^{\circ,*}$	
$\langle J\amalg 5\rangle^{\circ}$	$\langle J\amalg 1\langle 2\rangle\rangle^{\circ}$	$\langle J\amalg 4\langle 1\rangle\rangle^{\circ}$		
$\langle J\amalg 6\rangle^{\circ}$	$\langle J\amalg 1\amalg 1\langle 2\rangle\rangle^{\circ}$	$\langle J\amalg 1\amalg 1\langle 4\rangle\rangle^{\circ}$	$\langle J\amalg 6\amalg 1\langle 6\rangle\rangle^{\circ}$	
$\langle J\amalg 7\rangle^{\circ,*}$	$\langle J\amalg 2\amalg 1\langle 2\rangle\rangle^{\circ,*}$	$\langle J\amalg 2\amalg 1\langle 4\rangle\rangle^{\circ,*}$		$\langle J\amalg 1\langle 10\rangle\rangle^{\circ,*}$
$\langle J\amalg 8\rangle^{\circ}$	$\langle J\amalg 3\amalg 1\langle 2\rangle\rangle^{\circ}$	$\langle J\amalg 3\amalg 1\langle 4\rangle\rangle^{\circ}$		$\langle J\amalg 1\amalg 1\langle 10\rangle\rangle^{\circ}$
$\langle J\amalg 9\rangle^{\circ,*}$	$\langle J\amalg 4\amalg 1\langle 2\rangle\rangle^{\circ,*}$	$\langle J\amalg 4\amalg 1\langle 4\rangle\rangle^{\circ}$		$\langle J\amalg 2\amalg 1\langle 10\rangle\rangle^{\circ}$
$\langle J\amalg 10\rangle^{\circ}$	$\langle J\amalg 5\amalg 1\langle 2\rangle\rangle^{\circ}$	$\langle J\amalg 5\amalg 1\langle 4\rangle\rangle^{\circ}$	$\langle J\amalg 1\langle 7\rangle\rangle^{\circ}$	$\langle J\amalg 3\amalg 1\langle 10\rangle\rangle^{\circ}$
$\langle J\amalg 11\rangle^{\circ,*}$	$\langle J\amalg 6\amalg 1\langle 2\rangle\rangle^{\circ}$	$\langle J\amalg 6\amalg 1\langle 4\rangle\rangle^{\circ,*}$	$\langle J\amalg 1\amalg 1\langle 7\rangle\rangle^{\circ,*}$	$\langle J\amalg 4\amalg 1\langle 10\rangle\rangle^{*}$
$\langle J\amalg 12\rangle^{\circ}$	$\langle J\amalg 7\amalg 1\langle 2\rangle\rangle^{\circ}$		$\langle J\amalg 2\amalg 1\langle 7\rangle\rangle^{\circ}$	
$\langle J\amalg 13\rangle^{\circ,*}$	$\langle J\amalg 8\amalg 1\langle 2\rangle\rangle^{\circ,*}$	$\langle J\amalg 8\amalg 1\langle 4\rangle\rangle^{\circ}$	$\langle J\amalg 3\amalg 1\langle 7\rangle\rangle^{\circ,*}$	$\langle J\amalg 1\langle 11\rangle\rangle^{\circ}$
$\langle J\amalg 14\rangle^{\circ}$	$\langle J\amalg 9\amalg 1\langle 2\rangle\rangle^{\circ}$			$\langle J\amalg 1\amalg 1\langle 11\rangle\rangle^{\circ,*}$
$\langle J\amalg 15\rangle^{*}$	$\langle J\amalg 10\amalg 1\langle 2\rangle\rangle^{\circ}$	$\langle J\amalg 10\amalg 1\langle 4\rangle\rangle^{*}$	$\langle J\amalg 5\amalg 1\langle 7\rangle\rangle^{*}$	$\langle J\amalg 2\amalg 1\langle 11\rangle\rangle^{\circ}$
	$\langle J\amalg 11\amalg 1\langle 2\rangle\rangle^{\circ}$			$\langle J\amalg 3\amalg 1\langle 11\rangle\rangle^{*}$
$\langle J\amalg 1\langle 1\rangle\rangle^{\circ}$	$\langle J\amalg 12\amalg 1\langle 2\rangle\rangle^{*}$	$\langle J\amalg 1\langle 5\rangle\rangle^{\circ}$		
$\langle J\amalg 1\amalg 1\langle 1\rangle\rangle^{\circ}$		$\langle J\amalg 1\amalg 1\langle 5\rangle\rangle^{\circ}$		$\langle J\amalg 1\langle 12\rangle\rangle^{\circ}$
$\langle J\amalg 2\amalg 1\langle 1\rangle\rangle^{\circ}$	$\langle J\amalg 1\langle 3\rangle\rangle^{\circ}$	$\langle J\amalg 2\amalg 1\langle 5\rangle\rangle^{\circ}$	$\langle J\amalg 1\langle 8\rangle\rangle^{\circ}$	
$\langle J\amalg 3\amalg 1\langle 1\rangle\rangle^{\circ,*}$	$\langle J\amalg 1\amalg 1\langle 3\rangle\rangle^{\circ}$	$\langle J\amalg 3\amalg 1\langle 5\rangle\rangle^{\circ,*}$	$\langle J\amalg 1\amalg 1\langle 8\rangle\rangle^{\circ}$	$\langle J\amalg 2\amalg 1\langle 12\rangle\rangle^{*}$
$\langle J\amalg 4\amalg 1\langle 1\rangle\rangle^{\circ}$	$\langle J\amalg 2\amalg 1\langle 3\rangle\rangle^{\circ}$	$\langle J\amalg 4\amalg 1\langle 5\rangle\rangle^{\circ}$	$\langle J\amalg 2\amalg 1\langle 8\rangle\rangle^{\circ,*}$	
$\langle J\amalg 5\amalg 1\langle 1\rangle\rangle^{\circ}$	$\langle J\amalg 3\amalg 1\langle 3\rangle\rangle^{\circ,*}$	$\langle J\amalg 5\amalg 1\langle 5\rangle\rangle^{\circ,*}$	$\langle J\amalg 3\amalg 1\langle 8\rangle\rangle^{\circ}$	$\langle J\amalg 1\langle 13\rangle\rangle^{\circ}$
$\langle J\amalg 6\amalg 1\langle 1\rangle\rangle^{\circ}$	$\langle J\amalg 4\amalg 1\langle 3\rangle\rangle^{\circ}$	$\langle J\amalg 6\amalg 1\langle 5\rangle\rangle^{\circ}$	$\langle J\amalg 4\amalg 1\langle 8\rangle\rangle^{\circ}$	$\langle J\amalg 1\amalg 1\langle 13\rangle\rangle^{*}$
$\langle J\amalg 7\amalg 1\langle 1\rangle\rangle^{\circ,*}$	$\langle J\amalg 5\amalg 1\langle 3\rangle\rangle^{\circ,*}$	$\langle J\amalg 7\amalg 1\langle 5\rangle\rangle^{\circ,*}$		
$\langle J\amalg 8\amalg 1\langle 1\rangle\rangle^{\circ}$	$\langle J\amalg 6\amalg 1\langle 3\rangle\rangle^{\circ}$	$\langle J\amalg 8\amalg 1\langle 5\rangle\rangle^{\circ}$		$\langle J\amalg 1\amalg 1\langle 1\langle 1\rangle\rangle\rangle^{*}$
$\langle J\amalg 9\amalg 1\langle 1\rangle\rangle^{\circ,*}$	$\langle J\amalg 7\amalg 1\langle 3\rangle\rangle^{\circ,*}$	$\langle J\amalg 9\amalg 1\langle 5\rangle\rangle^{*}$		

Proof. In order to apply the Viro method without any change of coordinates, we consider here the symmetry with respect to the line $\{Y - Z = 0\}$. Consider the union of the line $\{X = 0\}$ and three symmetric conics in $\mathbb{R}P^2$ tangent to each other into the two symmetric points $[0 : 0 : 1]$ and $[0 : 1 : 0]$. Using the Viro method as explained in section 2.2.1 and the classification, up to isotopy, of the curves of degree 7 in $\mathbb{R}P^2$ with only one singular point Z_{15} established by Korchagin in [Kor88], we perturb these reducible symmetric curves. In order to obtain nonsingular symmetric curves, we have to perturb symmetrically the two singular points. That is to say, if we perturb the singular point at $[0 : 0 : 1]$ gluing the chart of a polynomial $P(X, Y)$, we have to perturb the singular point at $[0:1:0]$ gluing the chart of the polynomial $Y^7P(\frac{X}{Y}, \frac{1}{Y})$. $\square$

Remark. Using this method, we have constructed nonsingular symmetric algebraic curves of degree 7 in $\mathbb{R}P^2$ realizing the complex schemes $\langle J \amalg 4_+ \amalg 5_- \amalg 1_+\langle 1_+\rangle\rangle_I$ and $\langle J \amalg 3_+ \amalg 6_- \amalg 1_-\langle 1_+\rangle\rangle_I$. Consequently, unlike in the M-curves case, the real scheme of a nonsingular symmetric curves of degree 7 in $\mathbb{R}P^2$ does not determine its complex scheme.

Proposition 5.37 *The complex schemes*

$$\langle J \amalg 1_-\langle 1_+ \amalg 3_-\rangle\rangle_I,\ \langle J \amalg 2_+ \amalg 3_- \amalg 1_-\langle 1_-\rangle\rangle_I \text{ and } \langle J \amalg 1_- \amalg 1_-\langle 2_+ \amalg 3_-\rangle\rangle_I$$

are realizable by nonsingular symmetric real algebraic dividing curves of degree 7 in $\mathbb{R}P^2$.

Proof. In [II01], symmetric sextics realizing the complex schemes $\langle 1_-\langle 1_+ \amalg 3_-\rangle\rangle_I$, $\langle 5 \amalg 1_-\langle 1_-\rangle\rangle_I$ and $\langle 1 \amalg 1_-\langle 2_+ \amalg 3_-\rangle\rangle_I$ are constructed. Consider the union of each of these curves and a real line oriented and disposed in $\mathbb{R}P^2$ such that the (symmetric) perturbations according to the orientations satisfies the Rokhlin-Mischachev orientation formula. So, according to Theorem 4.8 in [Vir84c], the obtained real algebraic symmetric curves of degree 7 in $\mathbb{R}P^2$ are of type I and realize the announced complex schemes. $\square$

5.5.3 Parametrization of a rational curve

Here we apply the method used in [Ore98a] and [Ore98b]. Namely, we construct a singular rational curve and perturb it using Shustin's results on the independent perturbations of generalized Newton nondegenerate singular points of a curve keeping the same Newton polygon.

Proposition 5.38 *There exists a rational algebraic curve of degree 4 in $\mathbb{R}P^2$ situated with respect to the lines $\{X = 0\}$, $\{Y = 0\}$, $\{Z = 0\}$, and $\{Y = -Z\}$ as depicted in Figure 5.15a), with a singular point of type A_4 at $[0:0:1]$, a singular point of type A_1 at p and a tangency point of order 2 with the line $\{X = 0\}$ at $[0:1:0]$.*

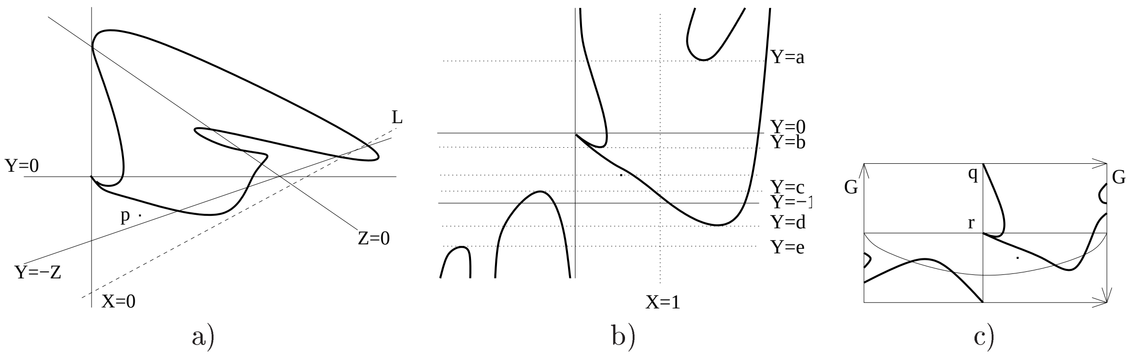


Figure 5.15:

Proof. Consider the map from $\mathbb{C}$ to $\mathbb{C}P^2$ given by $t \mapsto [x(t) : y(t) : z(t)]$, where

$$\begin{cases} x(t) = t^2 \\ y(t) = \alpha t^2(t-\gamma)(t+\gamma) \\ z(t) = -((t-1)(\epsilon t - 1)(\delta t - 1)) \end{cases} \qquad \text{with } \alpha = \frac{1}{11},\ \gamma = \delta = \frac{9}{10} \text{ and } \epsilon = \frac{99}{100}.$$

The curve defined by this map has a singular point of type A_4 at $[0:0:1]$, as we can see using the following identity :

$$y(t)z(t) + \alpha\gamma^2 x(t)z(t) - \alpha x(t)^2 = -\frac{81}{1000}t^7 + \frac{2781}{11000}t^6 - \frac{289}{1100}t^5.$$

Moreover, it is clear from the equations that the curve has a tangency point of order 2 with the line $\{x = 0\}$ at $[0:1:0]$. This map defines an algebraic curve of degree 4 in $\mathbb{C}P^2$, $C = Res_t(x(t)Y - y(t)X, x(t)Z - z(t)X)/X^3$. Considering C in the affine plane $\{Z = 1\}$, we obtain

$$\begin{aligned} C = \quad & -\frac{1}{11}Y^2 - \frac{13468021579}{1331000000000000}X^4 + \frac{1666467523}{1210000000000}YX^3 + \frac{275261}{3025000}X^2Y - \frac{65485017}{1100000000}X^2Y^2 \\ & + \frac{2781}{5500}XY^2 - \frac{81}{6050}XY + \frac{793881}{1000000}XY^3 + \frac{26346141}{6655000000}X^3 - \frac{6561}{13310000}X^2. \end{aligned}$$

In order to show that C is the expected curve, we study its position with respect to the pencil of lines from the point $[1:0:0]$. First, we compute the discriminant of C with respect to the variable X and we obtain

$$\begin{aligned} D = \quad & \frac{361}{3452271214393100000000000000000000000000000000}Y^5(10356281643411Y^5 + 7639877699379882 0Y^4 \\ & -17692852212979722900Y^3 - 35412592529600440000Y^2 - 17689562158356000000Y \\ & -6887475360000000)(368625411Y + 54936010)^2. \end{aligned}$$

The root of order 5 at 0 corresponds to the singular point A_4 at the origin, and the root of order 2 corresponds to a double point of C. We have

$$C(X, -\tfrac{54936010}{368625411}) = -\frac{41257520691820587X^2 - 15271344975233127000X + 50334951883300000000)(714609X - 289000)^2}{2082161160567862626483000000000000}$$

so the curve C has a double point of at $(\frac{289000}{714609}, -\frac{54936010}{368625411}) \simeq (0.4044169609, -0.1490293625)$. By the genus formula, C has no other singular point in $\mathbb{C}P^2$. The simple roots of D correspond to the horizontal lines which are tangent to C. So, when y ranges from $-\infty$ to ∞ and passes a simple root of D, the number of real roots of $D(X, y)$ is changed by ± 2. Using the Budan-Fourier Theorem (see [BPR03]), we see that all simple roots of D are real and lie in the following intervals :

- $[\frac{-7784171}{1024}; \frac{-3892085}{512}] \simeq [-7601.729492; -7601.728516]$,
- $[\frac{-527}{512}; \frac{-1053}{1024}] \simeq [-1.029296875; -1.028320312]$,
- $[\frac{-979}{1024}; \frac{-489}{512}] \simeq [-0.9560546875; -0.9550781250]$,
- $[\frac{-5}{1024}; \frac{-1}{256}] \simeq [-0.004882812500; -0.003906250000]$,
- $[\frac{116055}{512}; \frac{232111}{1024}] \simeq [226.6699219; 226.6708984]$.

We denote these roots by $a > b > c > d > e$. Using again the Budan-Fourier Theorem, we see that for $y = 230$ the polynomial $C(X, y)$ has 4 real roots and that each of the following intervals contains exactly one of these roots :

$$[0; \frac{1}{1024}], [\frac{7555467}{1024}; \frac{1888867}{256}], [\frac{8093529}{1024}; \frac{4046765}{512}], [\frac{8403971}{512}; \frac{16807943}{1024}].$$

As the number of real roots of the polynomial $C(X,y)$ is constant on each connected component of the complement of the roots of D, $C(X,y)$ has exactly

- 4 real roots for $y > a$,
- 2 real roots for $a > y > 0$,
- 4 real roots for $0 > y > b$,
- 2 real roots for $b > y > c$ except for $y = -\frac{54936010}{368625411}$,
- 4 real roots for $c > y > d$,
- 2 real roots for $d > y > e$,
- 4 real roots for $e > y$.

Using the last time the Budan-Fourier Theorem, we see that the real polynomial $C(X,y)$ with $y = -\frac{54936010}{368625411}$ has exactly one simple real root in each of the following intervals

- $[\frac{3405}{1024}; \frac{1703}{512}] \simeq [3.325195312; 3.326171875]$,
- $[\frac{46953}{128}; \frac{375625}{1024}] \simeq [366.8203125; 366.8212891]$.

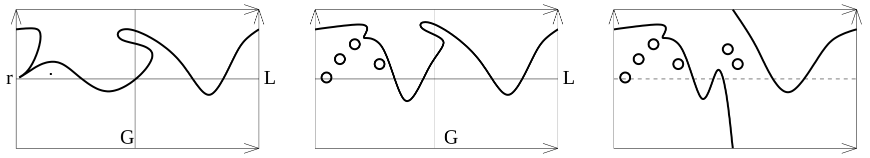
As we know that $C(X,Y)$ has a double root at $(\frac{289000}{714609}, -\frac{54936010}{368625411})$, we deduce that the double point of C has non-real tangents and is situated on the line $\{Y = -\frac{54936010}{368625411}\}$ as shown in figure 5.15b) with respect to the two other roots and 0.
Now it remains to determine which branches of C "glue" at each tangency point with an horizontal line. This can be done using Proposition 12.38 in [BPR03]. The topology of the curve C is depicted in Figure 5.15b) and it is clear that this is the desired curve. □

Corollary 5.39 *The complex scheme $\langle J \amalg 5 \amalg 1\langle 7\rangle\rangle_{II}$ is realizable by nonsingular non-dividing symmetric real algebraic curves of degree 7 in $\mathbb{R}P^2$.*

Proof. The strict transform of the curve constructed in Proposition 5.38 under the blow up of $\mathbb{C}P^2$ at the point $[0:1:0]$ is the rational algebraic curve of bidegree $(3,1)$ in Σ_1 depicted in Figure 5.15c). Blowing up the point q and blowing down the strict transform of the fiber, we obtain the rational trigonal curve in Σ_2 depicted in Figure 5.16a), with a singular point of type A_6 at the point r. Then according to Theorem 2.22, it is possible to smooth this curve as depicted in Figure 5.16b). Perturbing the union of this curve and the fiber G, we obtain the nonsingular curve of bidegree $(3,1)$ in Σ_2 arranged with the base $\{Y = 0\}$ as shown in Figure 5.16c). The corresponding symmetric curve realizes the real scheme $\langle J \amalg 5 \amalg 1\langle 7\rangle\rangle$ and according to Proposition 5.9 this is a non-dividing symmetric curve. □

Corollary 5.40 *The real schemes $\langle J \amalg 5 \amalg 1\langle 8\rangle\rangle$ and $\langle J \amalg 4 \amalg 1\langle 7\rangle\rangle$ are realizable by nonsingular symmetric real algebraic curves of degree 7 in $\mathbb{R}P^2$.*

Proof. One obtains these two curves modifying slightly the previous construction. To obtain the real scheme $\langle J \amalg 5 \amalg 1\langle 8\rangle\rangle$, one can keep all the ovals above the base depicted in Figure 5.16b). To obtain the real scheme $\langle J \amalg 4 \amalg 1\langle 7\rangle\rangle$, one can consider the line L (dashed in Figure 5.15a)) instead of the line $\{Y + Z = 0\}$ in Figure 5.15a). □



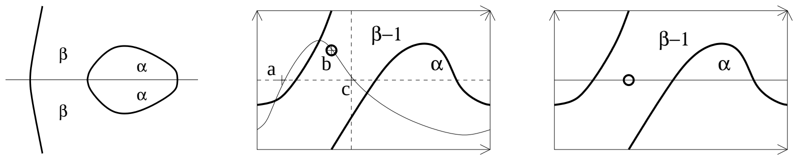
a) b) c)

Figure 5.16:

5.5.4 Change of coordinates in Σ_2

Proposition 5.41 *The real schemes $\langle J \amalg 7 \amalg 1\langle 4\rangle\rangle$ and $\langle J \amalg 5 \amalg 1\langle 6\rangle\rangle$ are realizable by nonsingular symmetric real algebraic curves of degree 7 in $\mathbb{R}P^2$.*

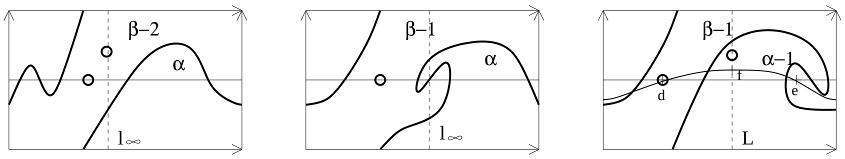
Proof. In section 5.5.2, we have constructed the symmetric curves in $\mathbb{R}P^2$ shown in Figure 5.17a). According to Lemma 2.49, their quotient curve X is depicted in Figure 5.17b).



a) b) c)

$(\alpha, \beta) = (2, 4)$ or $(3, 3)$

Figure 5.17:



a) b) c)

$(\alpha, \beta) = (2, 4)$ or $(3, 3)$

Figure 5.18:

Let H be the base which passes through the points a, b and c. All possible mutual arrangements for H and the quotient curves which do not contradict the Bézout theorem and Lemma 2.51 are depicted in Figures 5.17c) and 5.18.

First, we prohibit pseudoholomorphically the $\mathcal{L}$-schemes realized by the union of X and H in Figure 5.18a) and by X in Figure 5.18b). Choose l_∞ as depicted, and the braid corresponding to these $\mathcal{L}$-schemes are :

$$b^{23}_{\alpha,\beta} = \sigma_3^{-(\beta-1)}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_2^{-\alpha}\sigma_3^{-1}\sigma_2\sigma_1^{-4}\sigma_2^{-1}\sigma_3^2\sigma_2\sigma_1\sigma_2^{-2}\sigma_3^{-1}\Delta_4^2 \text{ with } (\alpha,\beta) = (3,3), (2,4),$$
$$b^{24} = \sigma_1^{-7}\sigma_2\sigma_1\Delta_3^2.$$

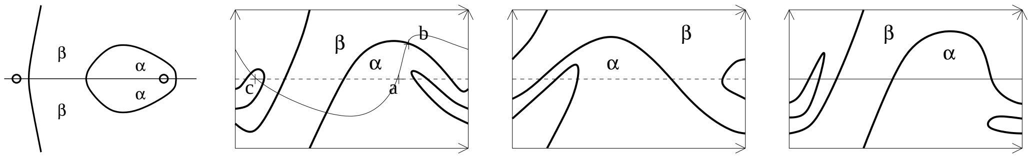
The braid $b^{23}_{3,3}$ was already shown to be not quasipositive in section 5.4.2. The computation of the Alexander polynomials of the remaining braids gives

$$p^{24} = (-1+t)(t^4 - t^3 + t^2 - t + 1), \quad p^{23}_{4,2} = (t^2 - t + 1)(-1+t)^3.$$

Since $e(b^{24}) = 1$ and $e(b^{23}_{\alpha,\beta}) = 2$, according to Proposition 2.80, none of these braids is quasipositive. Thus, the two remaining possibilities for the mutual arrangement of X and H are depicted in Figures 5.17c) and 5.18c).
In the first case, let $H' = H$.
In the second case, consider the base G passing through the points d, e, and f, where f is some point on the fiber L. For some f, the base G passes through an oval of X. Since G cannot have more than 7 common points with X, there exists f for which the mutual arrangement of G and X is as shown in Figure 5.17c). Let H' be the base corresponding to such an f.
The symmetric curves of degree 7 in $\mathbb{R}P^2$ corresponding to the mutual arrangement of H' and X realizes the real schemes $\langle J \amalg 7 \amalg 1\langle 4\rangle\rangle$ and $\langle J \amalg 5 \amalg 1\langle 6\rangle\rangle$. □

Proposition 5.42 *The real schemes $\langle J \amalg 1 \amalg 1\langle 12\rangle\rangle$ and $\langle J \amalg 9 \amalg 1\langle 4\rangle\rangle$ are realizable by nonsingular symmetric real algebraic curves of degree 7 in $\mathbb{R}P^2$.*

Proof. In the section 5.5.2, we have constructed the symmetric curves of degree 7 in $\mathbb{R}P^2$ depicted in Figure 5.19a). These are M-curves, so according to Theorem 5.6, the $\mathcal{L}$-scheme realized by their quotient curve X is one of these depicted in Figures 5.19b) and c). The braids associated to the



a) b) c) d)
$(\alpha, \beta) = (6, 0)$ or $(2, 4)$

Figure 5.19:

$\mathcal{L}$-schemes shown in Figure 5.19c) are

$$b^{25}_{\alpha,\beta} = \sigma_2\sigma_1^{-\alpha}\sigma_2^{-1}\sigma_1\sigma_2^{-\beta}\Delta_3^2 \text{ with } (\alpha, \beta) = (6, 0) \text{ or } (2, 4).$$

The computation of the corresponding Alexander polynomials gives

$$p^{25}_{6,0} = (-1+t)(t^4 - t^3 + t^2 - t + 1), \quad p^{25}_{2,4} = (-1+t)(t^2 - t + 1).$$

Since $e(b^{25}_{\alpha,\beta}) = 1$, according to Proposition 2.80, these two braids are not quasipositive.
Hence, X realizes the $\mathcal{L}$-scheme shown in Figure 5.19b). Let H be the base which passes through the points a, b, and c in Figure 5.19b). The only possible mutual arrangement for H and X which does not contradict the Bézout theorem, Theorem 5.6 and Lemma 2.51 is depicted in Figure 5.19d). The corresponding symmetric curves realize the real schemes $\langle J \amalg 1 \amalg 1\langle 12\rangle\rangle$ and $\langle J \amalg 9 \amalg 1\langle 4\rangle\rangle$. □

5.5.5 Construction of auxiliary curves

Lemma 5.43 *For any real positive numbers α, β, γ, there exist real curves of degree 3 in $\mathbb{R}P^2$ having the charts and the arrangement with respect to the axis $\{Y = 0\}$ shown in Figures 5.20a) and b) with truncation on the segment $[(0,3);(3,0)]$ equal to $(X - \alpha Y)(X - \beta Y)(X - \gamma Y)$.*

Proof. Consider the points $A = [\alpha : 1 : 0], B = [\beta : 1 : 0], C = [\gamma : 1 : 0]$ and four lines L_1, L_2, L_3, L_4 as shown in Figure 5.20c). For t small enough and of suitable sign, the curve $YZL_1 + tL_2L_3L_4$ is arranged with respect to the coordinate axis and the lines L_1, L_2, L_3, L_4 as shown in Figure 5.20c).

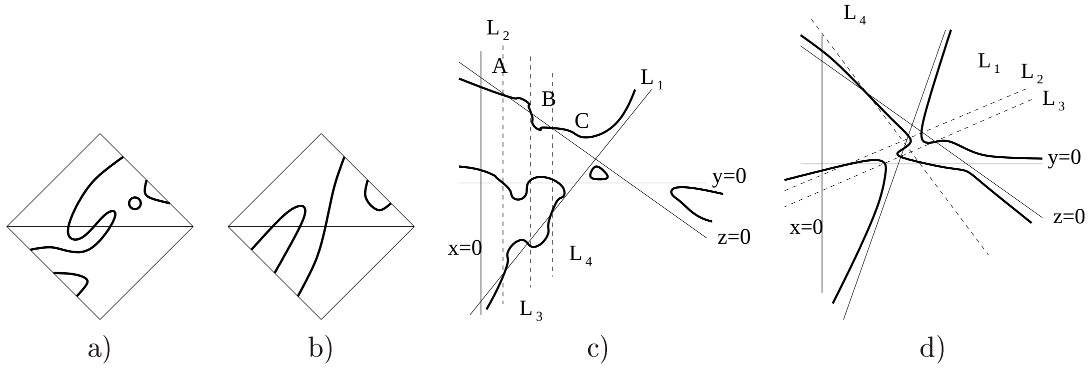


Figure 5.20:

To construct the curve with the chart depicted in Figure 5.20b), we perturb the third degree curve YZL_1 as shown in Figure 5.20d). $\square$

Corollary 5.44 *For any real positive numbers α, β, γ, there exist real symmetric dividing curves of degree 6 in $\mathbb{R}P^2$ with a singular point of type J_{10} at $[1 : 0 : 0]$ having the charts, the arrangement with respect to the axis $\{Y = 0\}$ and the complex orientations shown in Figures 5.21a), b), and c) with truncation on the segment $[(0, 3); (6, 0)]$ equal to $(X - \alpha Y^2)(X - \beta Y^2)(X - \gamma Y^2)$.*

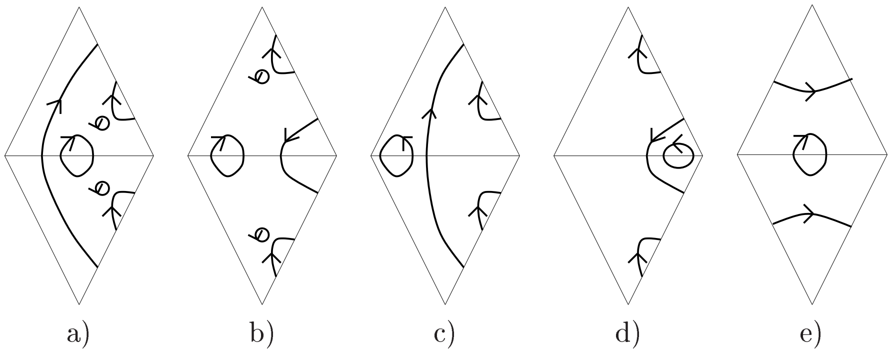


Figure 5.21:

Proof. The Newton polygon of the third degree curves constructed in Lemma 5.43 lies inside the triangle with vertices $(0, 3), (0, 0)$ and $(6, 0)$, so these curves can be seen as a (singular) trigonal curve in Σ_2. The corresponding symmetric curves are of degree 6 and has the chart and the arrangement with respect to the axis $\{Y = 0\}$ shown in Figures 5.21a) and c). Moreover, it is well known that such curves are of type I, and we deduce their complex orientations from their quotient curve.
If we perform the coordinate change $(X, Y) \mapsto (-X + \delta Y^2, Y)$ with $\delta \in \mathbb{R}$ to the curves with chart depicted in Figure 5.21a), we obtain curves with the chart depicted in Figure 5.21b). $\square$

The following lemma can be proved using the same technique.

Lemma 5.45 *For any real positive numbers α and β, there exist real symmetric dividing curves of degree 4 in $\mathbb{R}P^2$ with a singular point of type A_3 at $[1 : 0 : 0]$ having the charts, the arrangement with respect to the axis $\{Y = 0\}$ and the complex orientations shown in Figure 5.21d) with truncation on the segment $[(0, 2); (4, 0)]$ equal to $(X - \alpha Y^2)(X - \beta Y^2)$.* $\square$

5.5.6 Perturbation of irreducible singular symmetric curves

Proposition 5.46 *The complex schemes* $\langle J \amalg 2_- \amalg 1_+\langle 4_+ \amalg 2_-\rangle\rangle_I$ *and* $\langle J \amalg 2_+ \amalg 4_- \amalg 1_+\langle 2_+\rangle\rangle_I$ *are realizable by nonsingular symmetric real algebraic curves of degree* 7 *in* $\mathbb{R}P^2$.

Proof. First, we construct the symmetric singular dividing curve of degree 7 with two singular points J_{10} depicted in Figure 5.22d). In order to do this, we use the Hilbert method (as in [Vir84a]) : we symmetrically perturb the union of a symmetric conic and symmetric line disjoint from the conic, (Figure 5.22a)) keeping the tangency points with the conic. Then, we symmetrically perturb the union of the third degree curve obtained and the same conic (Figure 5.22b)) keeping the tangency points with the conic of order 4. Finally, we symmetrically perturb the union of the curve of degree 5 obtained and the same conic (Figure 5.22c)) keeping the tangency points with the conic of order 6, and we obtained the expected curve.

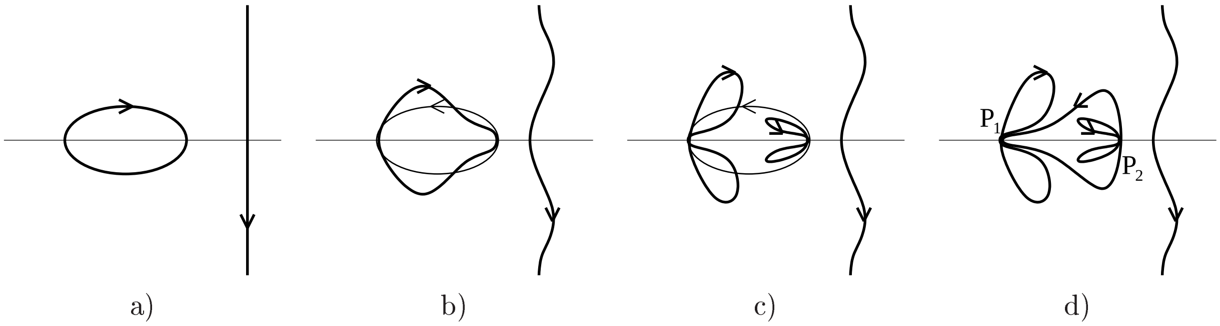


Figure 5.22:

Now we symmetrically perturb the singular points using the chart shown in Figure 5.21b) (resp., d)) in P_1 and 5.21c) (resp., a)) in P_2 and obtain the desired curves. □

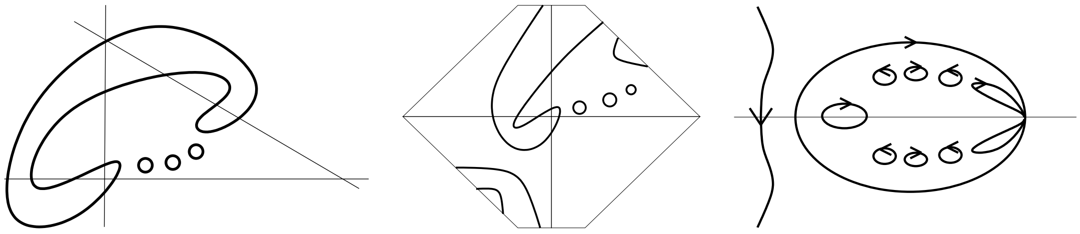
Proposition 5.47 *The complex scheme* $\langle J \amalg 1_+\langle 6_+ \amalg 6_-\rangle\rangle_I$ *is realizable by nonsingular symmetric real algebraic curves of degree* 7 *in* $\mathbb{R}P^2$.

Proof. Consider the curve of degree 4 with a C-shaped oval constructed in [Kor88] and the coordinate system depicted in Figure 5.23a). In this coordinate system, the Newton polygon of the curve is the trapeze with vertices $(0,0), (0,3), (1,3)$, and $(4,0)$, and its chart is depicted in Figure 5.23b). Thus, we can see this curve as a singular curve of bidegree $(3,1)$ in the surface Σ_2. The corresponding symmetric curve of degree 7 has a singular point J_{10} at $[1:0:0]$ and is depicted in Figure 5.23c). According to Proposition 2.40, this curve is of type I. Looking at its quotient curve, we see that the complex orientations of the symmetric curve are those represented in Figure 5.23c). Finally, we symmetrically smooth the singular point using the chart depicted in Figure 5.21a) and obtain the expected curve. □

Denote by f_P the real birational transformation of $\mathbb{C}P^2$ given by $(X,Y) \mapsto (X, Y - P(X))$ in the affine chart $\{Z = 1\}$, where P is a polynomial of degree 2. Some details about such a birational transformation are given in appendix.

Proposition 5.48 *The complex scheme* $\langle J \amalg 6_+ \amalg 4_- \amalg 1_+\langle 1_+ \amalg 1_-\rangle\rangle_I$ *is realizable by nonsingular symmetric real algebraic curves of degree* 7 *in* $\mathbb{R}P^2$.

Proof. Consider the nodal curve of degree 3 depicted in Figure 5.24a) with a contact of order 3 at the point $[0:1:0]$ with the line $\{Z = 0\}$. Then, there exists a unique polynomial P of degree

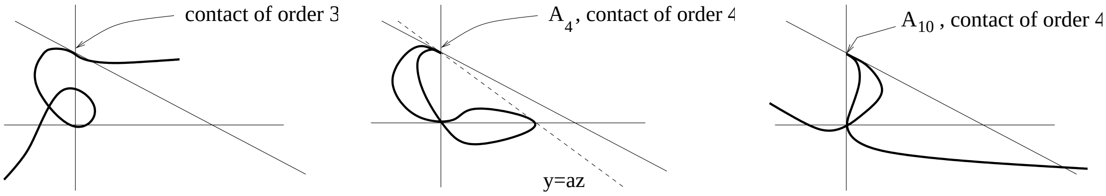


a) b) c)

Figure 5.23:

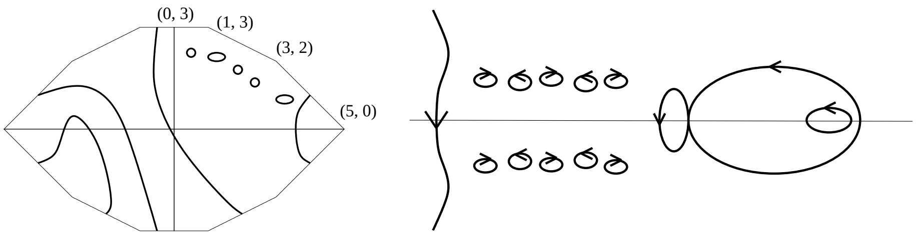
2 such that the image of the cubic under f_P is the curve of degree 4 depicted in Figure 5.24b), with a singular point of type A_4 at $[0:1:0]$ and a contact of order 4 at this point with the line $\{Z=0\}$. Moreover, the line $\{Y=0\}$ intersects the quartic in two points. One of them is the node, and this line is tangent at one of the local branches at the node. At the second intersection point, a line $\{Y=aZ\}$ is tangent at the curve of degree 4. Perform the change of coordinates of $\mathbb{C}P^2$ $[X:Y:Z] \mapsto [Y:Y:Y-aZ]$. For this new coordinate system, there exists a polynomial Q of degree 2 such that the image of the quartic under f_Q is the curve of degree 5 depicted in Figure 5.24c), with a singular point of type A_{10} at $[0:1:0]$ and a contact of order 4 at this point with the line $\{Z=0\}$. Applying the change of coordinates $[X:Y:Z] \mapsto [Y:Z:X]$ and using Theorem 2.22, we can smooth the singular point A_{10} in order to obtain a curve with the chart depicted in Figure 5.25a).

Hence, we can see this curve as a singular curve of bidegree $(3,1)$ in the surface Σ_2. The corresponding symmetric curve of degree 7 has a singular point A_3 at $[1:0:0]$ and is depicted in Figure 5.25b). This curve is maximal according to Proposition 2.40, so of type I, and we can deduce its complex orientations from its quotient curve. Finally, we symmetrically smooth the singular point using the chart depicted in Figure 5.21e) and obtain the desired curve. □



a) b) c)

Figure 5.24:



a) b)

Figure 5.25:

Appendix : A birational transformation of $\mathbb{C}P^2$

Here we detail the action of the birational transformation we use in the proof of proposition 5.48. We also illustrate this explanation by giving all the steps in the first use of the transformation in the proof. We recall that f_P is the real birational transformation of $\mathbb{C}P^2$ which is given by $(X,Y) \mapsto (X, Y - P(X))$ in the affine chart $\{Z = 1\}$, where P is a polynomial of degree 2.
The action of f_P on $\mathbb{C}P^2$ can be seen as explained below.

- Blow up the projective plane at $[0 : 1 : 0]$. The surface obtained is Σ_1. Denote by r the intersection point of the exceptional divisor and the strict transform of the line $\{Z = 0\}$ (see figure 5.26c)).

- Blow up Σ_1 at r and blow down the strict transform of the fiber. The surface obtained is Σ_2. Denote by F the fiber that appeared during the blowing up (see figure 5.26c)).

- Perform the change of coordinate $X' = X$ and $Y' = Y - P(X)$ in Σ_2. Denote by s the intersection point of F and $\{Y' = 0\}$ (see figure 5.26d)).

- Blow up Σ_2 at s and blow down the strict transform of the fiber. The surface obtained is Σ_1 (see figure 5.26e)).

- Blow down the exceptional divisor (see figure 5.26f)).

Let us follow each step of this action on the curve C of degree 3 used in the proof of proposition 5.48. Let us denote by p_1 the point of C which has a vertical tangent, by p_2 the double points of C, and by Q_1 and Q_2 the two local branches of C at p_2 as depicted in Figure 5.26a). Then, there exists a unique polynomial P of degree 2 such that the curve $Y - P(X)$ passes through p_1 and p_2 and is tangent to Q_1.
All the steps described above and concerning the action of f_P on $\mathbb{C}P^2$ and C are depicted in Figure 5.26.

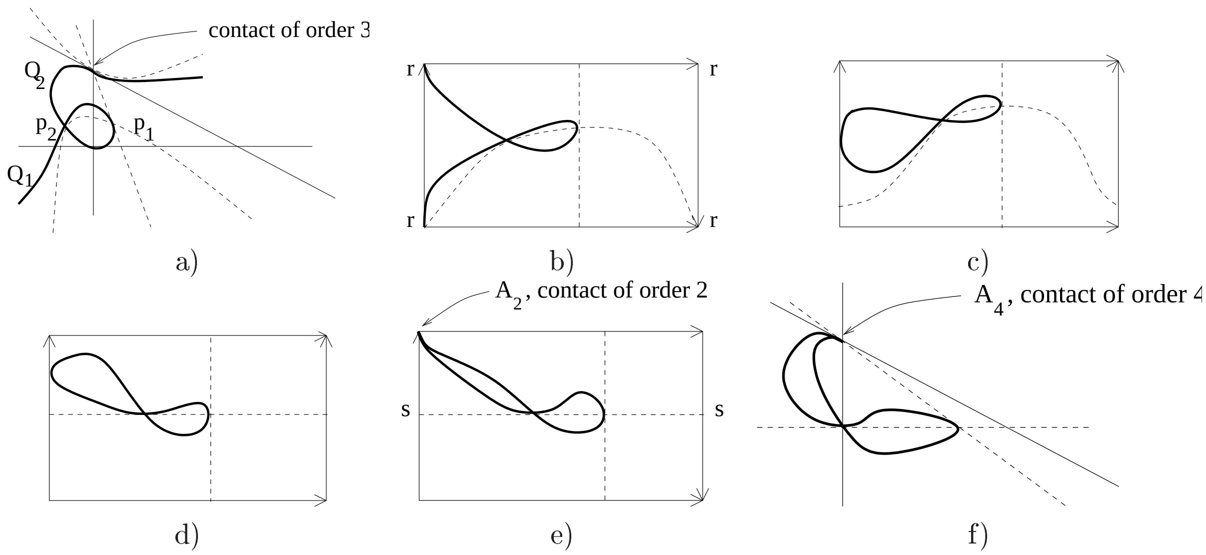

contact of order 3
Q2
p2
p1
Q1
a)
r
r
r
r
b)
c)
d)
A2, contact of order 2
s
s
e)
A4, contact of order 4
f)

Figure 5.26:

Bibliography

[AGZV85] V. I. Arnol'd, S. M. Gusei'n-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Vol. I*, volume 82 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985. The classification of critical points, caustics and wave fronts.

[Arn71] V. I. Arnol'd. The situation of ovals of real plane algebraic curves, the involutions of four-dimensional smooth manifolds, and the arithmetic of integral quadratic forms. *Funkcional. Anal. i Priložen.*, 5(3):1–9, 1971.

[Bea83] A. Beauville. *Complex algebraic surfaces*, volume 68 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1983.

[Ber] B. Bertrand. Hypersurfaces et intersections complètes maximales dans les variétés toriques. Thèse doctorale, 2002.

[Bih] F. Bihan. Asymptotiques de nombres de Betti d'hypersurfaces projectives réelles. Preprint arXiv: math.AG/0312259, (French).

[BPR03] S. Basu, R. Pollack, and M. F. Roy. *Algorithms in real algebraic geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 2003.

[Bru21] L. Brusotti. Sulla "piccola variazione" di una curva piana algebrica reali. *Rend. Rom. Ac. Lincei*, 30:375–379, 1921. (Italian).

[CF] D. Cimasoni and V. Florens. Signature of colored links and Seifert forms. In preparation.

[Che02] B. Chevallier. Four M-curves of degree 8. *Funct. Anal. Appl.*, 36(1):76–78, 2002.

[DK00] A. I. Degtyarev and V. M. Kharlamov. Topological properties of real algebraic varieties: Rokhlin's way. *Russian Math. Surveys*, 55(4):735–814, 2000.

[Fie] T. Fiedler. Unpublished.

[Fie83] T. Fiedler. Pencils of lines and the topology of real algebraic curves. *Math. USSR Izvestia*, 2(1), 1983.

[Flo] V. Florens. Signature of colored links with an application to real algebraic curves. To appear in Journal of Knot Theory and its Ram.

[FLT] S. Fiedler-Le Touzé. Orientations complexes des courbes algébriques réelles. Thèse doctorale, 2000.

[FLT97] S. Fiedler-Le Touzé. Orientations complexes des courbes algébriques réelles, 1997. Mémoire de DEA (French).

[FLTO02] S. Fiedler-Le Touzé and S. Yu. Orevkov. A flexible affine M-sextic which is algebraically unrealizable. *J. Algebraic Geom.*, 11(2):293–310, 2002.

[Ful93] W. Fulton. *Introduction to toric varieties*, volume 131 of *Ann. Math. Studies*. Princeton Univ. Press., 1993.

[Gar69] F. A. Garside. The braid group and other groups. *Quart. J. Math. Oxford Ser. (2)*, 20:235–254, 1969.

[Gil81] P. Gilmer. Configurations of surfaces in 4-manifolds. *Trans. Amer. Math. Soc.*, 264(2):353–380, 1981.

[Gro85] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[Gud69] D. A. Gudkov. Complete topological classification of the disposition of ovals of a sixth order curve in the projective plane. *Gor'kov. Gos. Univ. Učen. Zap. Vyp.*, 87:118–153, 1969.

[Haa] B. Haas. Real algebraic curves and combinatorial constructions. Thèse doctorale, 1997.

[Haa95] B. Haas. Les multilucarnes: nouveaux contre-exemples à la conjecture de Ragsdale. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(12):1507–1512, 1995. (French).

[Har76] A. Harnack. Über vieltheiligkeit der ebenen algebraischen curven. *Math. Ann.*, 10:189–199, 1876. (German).

[Hil01] D. Hilbert. Mathematische probleme. *Arch. Math. Phys.*, 1:43–63, 1901. (German).

[II01] V. Itenberg and I. Itenberg. Symmetric sextics on a real projective plane, and auxiliary conics. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 279(Geom. i Topol. 6):154–167, 248–249, 2001.

[IS02] I. Itenberg and E. Shustin. Combinatorial patchworking of real pseudo-holomorphic curves. *Turkish J. Math.*, 26(1):27–51, 2002.

[IS03] I. Itenberg and E. Shustin. Viro theorem and topology of real and complex combinatorial hypersurfaces. *Israel J. Math.*, 133:189–238, 2003.

[Ite93] I. Itenberg. Contre-exemples à la conjecture de Ragsdale. *C. R. Acad. Sci. Paris Sér. I Math.*, 317(3):277–282, 1993. (French).

[Ite95] I. Itenberg. Groups of monodromy of non-singular curves of degree 6. In *Real analytic and algebraic geometry (Trento, 1992)*, pages 161–168. de Gruyter, Berlin, 1995.

[Ite01] I. Itenberg. On the number of even ovals of a nonsingular curve of even degree in $\mathbb{R}P^2$. In *Topology, ergodic theory, real algebraic geometry*, volume 202 of *Amer. Math. Soc. Transl. Ser. 2*, pages 121–129. Amer. Math. Soc., Providence, RI, 2001.

[IV] I. Itenberg and O. Ya. Viro. Maximal real algebraic hypersurfaces of projective spaces. in preparation.

[Jac90] A. Jacquemard. About the effective classification of conjugacy classes of braids. *J. Pure Appl. Algebra*, 63(2):161–169, 1990.

[Kha78] V. M. Kharlamov. Isotopic types of non-singular surfaces of fourth degree in $\mathbb{R}P^3$. *Funct. Anal. Appl.*, 12:68–69, 1978.

[Kha81] V. M. Kharlamov. Rigid classification up to isotopy of real plane curves of degree 5. *Funktsional. Anal. i Prilozhen.*, 15(1):88–89, 1981. (Russian).

[Kha84] V. M. Kharlamov. Classification of non-singular surfaces of degree 4 in $\mathbb{R}P^3$ with respect to rigid isotopies. *Funct. Anal. Appl.*, 18(1):49–56, 1984.

[KI96] V. Kharlamov and I. Itenberg. Towards the maximal number of components of a non-singular surface of degree 5 in $\mathbf{R}\mathrm{P}^3$. In *Topology of real algebraic varieties and related topics*, volume 173 of *Amer. Math. Soc. Transl. Ser. 2*, pages 111–118. Amer. Math. Soc., Providence, RI, 1996.

[Kle22] F. Klein. Gesammelte mathematische abhandlungen. *Berlin*, 2, 1922. (German).

[Kor88] A. B. Korchagin. Isotopy classification of plane seventh degree curves with the only singular point Z_{15}. In *Topology and geometry—Rohlin Seminar*, volume 1346 of *Lecture Notes in Math.*, pages 407–426. Springer, Berlin, 1988.

[Kor89] A. B. Korchagin. The new M-curves of degrees 8 and 9. *Soviet Math. Doklady*, 39:569–572, 1989.

[KT02] S. Kaplan and M. Teicher. Identifying half-twists using randomized algorithm methods. *J. Symbolic Comput.*, 34(2):91–103, 2002.

[LdM] L. Lopez de Medrano. Courbure totale des variétés algébriques réelles projectives. Thèse doctorale, 2004, (French).

[Lic97] R. Lickorish. *An introduction to knot theory*, volume 175 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997.

[McD94] D. McDuff. Singularities and positivity of intersections of J-holomorphic curves. In *Holomorphic curves in symplectic geometry*, volume 117 of *Progr. Math.*, pages 191–215. Birkhäuser, Basel, 1994.

[Mik] G. Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. arXiv:math.AG/0312530.

[Mil68] J. Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J., 1968.

[Mis75] N. M. Mischachev. Complex orientations of plane M-curves of odd degree. *Funktional Anal. i Prilozhen*, 9:77–78, 1975. (Russian).

[MS98] D. McDuff and D. Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[Mur65] K. Murasugi. On a certain numerical invariant of link types. *Trans. of the American Math. Society*, 117:387–422, 1965.

[Nik79] V. V. Nikulin. Integer symmetric bilinear forms and some of their geometric applications. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(1):111–177, 238, 1979.

[NSV02] S. Natanzon, B. Shapiro, and A. Vainshtein. Topological classification of generic real rational functions. *J. Knot Theory Ramifications*, 11(7):1063–1075, 2002.

[Ore] S. Yu. Orevkov. Constructions of arrangements of a M-quartic and a M-cubic with maximally intersecting oval and odd branch. (Russian). To appear. English translation available at http://picard.ups-tlse.fr/˜orevkov/.

[Ore98a] S. Yu. Orevkov. A new affine M-sextic. *Funct. Anal. and Appl.*, 32(2):141–143, 1998.

[Ore98b] S. Yu. Orevkov. A new affine M-sextic. II. *Russian Math. Surveys*, 53(5):1099–1101, 1998.

[Ore99] S. Yu. Orevkov. Link theory and oval arrangements of real algebraic curves. *Topology*, 38(4):779–810, 1999.

[Ore00] S. Yu. Orevkov. Link theory and new restrictions for M-curves of degree 9. *Funktsional. Anal. i Prilozhen.*, 34(3):84–87, 2000.

[Ore01a] S. Yu. Orevkov. Complex orientations of M-curves of degree 7. In *Topology, ergodic theory, real algebraic geometry*, volume 202 of *Amer. Math. Soc. Transl. Ser. 2*, pages 215–227. Amer. Math. Soc., Providence, RI, 2001.

[Ore01b] S. Yu. Orevkov. Quasipositivity test via unitary representations of braid groups and its applications to real algebraic curves. *J. Knot Theory Ramifications*, 10(7):1005–1023, 2001.

[Ore02a] S. Yu. Orevkov. Classification of flexible M-curves of degree 8 up to isotopy. *Geom. Funct. Anal.*, 12(4):723–755, 2002.

[Ore02b] S. Yu. Orevkov. Cours donné a l'université rennes 1, 2002.

[Ore03] S. Yu. Orevkov. Riemann existence theorem and construction of real algebraic curves. *Annales de la Faculté des Sciences de Toulouse*, 12(4):517–531, 2003.

[Ore04] S. Yu. Orevkov. Quasipositivity problem for 3-braids. *Turkish J. Math.*, 28(1):89–93, 2004.

[OS] S. Yu. Orevkov and E. Shustin. Pseudoholomorphic, algebraically unrealizable curves. to appear in Moscow Math. J.

[OS02] S. Yu. Orevkov and E. I. Shustin. Flexible, algebraically unrealizable curves: rehabilitation of Hilbert-Rohn-Gudkov approach. *J. Reine Angew. Math.*, 551:145–172, 2002.

[Pet33] I. G. Petrovsky. Sur la topologie des courbes réelles algébriques. *C. R. Acad. Sci. Paris Sér. I Math.*, 197:1270–1272, 1933.

[PS97] V. V. Prasolov and A. B. Sossinsky. *Knots, links, braids and 3-manifolds, an introduction to the new invariants in low-dimensional topology*, volume 154 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 1997.

[Rag06] V. Ragsdale. On the arrangement of the real branches of plane algebraic curves. *Am. J. Math.*, 28:377–404, 1906.

[Ris92] J. J. Risler. Construction d'hypersurfaces réelles (d'après Viro). *Séminaire Bourbaki*, 763, 1992. (French).

[Rok74] V. A. Rokhlin. Complex orientations of real algebraic curves. *Funktsional Anal. i Prilozhen*, 8:71–75, 1974. (Russian).

[Rud83] L. Rudolph. Algebraic functions and closed braids. *Topology*, 22(2):191–202, 1983.

[San] F. Santos. Unpublished.

[Sha77] I. R. Shafarevitch. *Basic algebraic geometry*. Springer Verlag, 1977.

[She] V. V. Shevchishin. Pseudoholomorphic curves and the symplectic isotopy problem. arXiv:math.SG/0010262.

[Shu87] E. Shustin. A new M-curve of degree height. *Maths. Notes of Acad. Sci. USSR*, 42:606–610, 1987.

[Shu88] E. Shustin. New M and $M-1$ curves of degree 8. In *Rokhlin Seminar*, volume 1346, pages 487–493. Springer Lectures Notes in Math., 1988.

[Shu91] E. Shustin. New restrictions on the topology of real curves of degree a multiple of 8. *Math. USSR-Izvestiya*, 37:421–443, 1991.

[Shu98] E. Shustin. Gluing of singular and critical points. *Topology*, 37(1):195–217, 1998.

[Shu99] E. Shustin. Lower deformations of isolated hypersurface singularities. *Algebra i Analiz*, 11(5):221–249, 1999.

[Shu05] E. Shustin. Patchworking singular algebraic curves, non-Archimedean amoebas and enumerative geometry. *Algebra i Analiz*, 17:170–214, 2005.

[Sou01] M. L. Soum. Construction de courbes algébriques réelles de degre 7, 2001. Mémoire de DEA, (French).

[SV03] B. Shapiro and A. Vainshtein. Counting real rational functions with all real critical values. *Mosc. Math. J.*, 3(2):647–659, 745, 2003.

[Tri69] A. G. Tristram. Some cobordism invariant for links. *Proc. of th Cambridge Phil. Society*, 66:251–264, 1969.

[Tri01] S. Trille. Symétrie et entrelacs de courbes réelles algébriques, 2001. Thèse doctorale, (French).

[Vira] O. Ya. Viro. Unpublished.

[Virb] O. Ya. Viro. Patchworking real algebraic varieties. http://www.math.uu.se/∼oleg.

[Vir73] O. Ya. Viro. Branched coverings of manifolds with boundary, and invariants of links. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 37:1241–1258, 1973.

[Vir80] O. Ya. Viro. The curves of degree 7, the curves of degree 8, and Ragsdale conjecture. *Doklady AN USSR*, 254(6):1135–1150, 1980.

[Vir84a] O. Ya. Viro. Gluing of plane real algebraic curves and constructions of curves of degrees 6 and 7. In *Topology (Leningrad, 1982)*, volume 1060 of *Lecture Notes in Math.*, pages 187–200. Springer, Berlin, 1984.

[Vir84b] O. Ya. Viro. Plane real curves of degree 7 and 8: new restrictions. *Math. USSR-Izvestia*, 23:409–422, 1984.

[Vir84c] O. Ya. Viro. Progress in the topology of real algebraic varieties over the last six years. *Russian Math. Surveys*, 41:55–82, 1984.

[Vir89] O. Ya. Viro. Real plane algebraic curves: constructions with controlled topology. *Leningrad Math. J.*, 1(5):1059–1134, 1989.

[Wel02] J. Y. Welschinger. Courbes algébriques réelles et courbes flexibles sur les surfaces réglées de base $\mathbb{C}P^1$. *Proc. London Math. Soc. (3)*, 85(2):367–392, 2002. (French).

[Wil78] G. Wilson. Hilbert's sixteenth problem. *Topology*, 17(1):53–73, 1978.

[Zvo] A. Zvonkin. Unpublished.

Index of symbols

$\Delta(F)$	Newton polyhedron of the polynomial F
F^δ	truncation of the polynomial F along the face δ of $\Delta(F)$
μ_Δ	moment map associated to the polyhedron Δ
$\mathbb{C}\mu_\Delta$	complexification of μ_Δ
$\mathbb{C}\Delta$	complexification of Δ
$\mathbb{R}\Delta$	real part of $\mathbb{C}\Delta$
$Tor_{\mathbb{C}}(\Delta)$	toric variety associated to the convex polyhedron Δ
$\mathbb{C}Ch_\Delta(F)$	complex chart of the polynomial F in Δ
$\mathbb{C}Ch(F)$	complex chart of the polynomial F in $\Delta(F)$
$\mathbb{R}Ch_\Delta(F)$	real chart of the polynomial F in Δ
$\mathbb{R}Ch(F)$	real chart of the polynomial F in $\Delta(F)$
$\Gamma(C)$	Newton diagram at the origin of the polynomial C
$Sing(C)$	set of singular points of the curve C
Σ_n	the n^{th} rational geometrically ruled surfaces
E	the exceptional section of Σ_n
$\mathcal{L}$	pencil of lines in Σ_n
(X, ω)	a symplectic manifold
Π_+	number of positive injective pairs of ovals of a curve
Π_-	number of negative injective pairs of ovals of a curve
Λ_+	number of positive ovals of a curve of odd degree
Λ_-	number of negative ovals of a curve of odd degree
$lk(L_1, L_2)$	linking number of the oriented links L_1 and L_2
$det(L)$	determinant of the link L
B_m	braid of m strings group
σ_i	i^{th} standard generator of B_m
$e(b)$	exponent sum of the braid b
Δ_m	Garside element in B_m
p	number of even ovals of a curve of even degree
n	number of odd ovals of a curve of even degree

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Résumé

Cette thèse est motivée par l'étude des courbes algébriques réelles dans $\mathbb{R}P^2$ et dans les surfaces rationnelles géométriquement réglées, munis de leur structure réelle standard. Deux problèmes ont particulièrement retenus notre attention.
Les ovales d'une courbe non singulière dans $\mathbb{R}P^2$ de degré pair sont naturellement divisés en deux ensembles disjoints : les *ovales pairs*, contenus dans un nombre pair d'ovales, et les *ovales impairs*. La combinaison des inégalités de Harnack et de Petrovsky permet d'obtenir une borne supérieure pour le nombre d'ovales pairs et le nombre d'ovales impairs en fonction du degré de la courbe. Généralisant une construction antérieure d'I. Itenberg, nous montrons que cette borne est *asymptotiquement optimale*.
La majorité des restrictions connues sur la topologie des courbes algébriques réelles sont aussi valables pour une classe plus vaste d'objets, les *courbes pseudoholomorphes réelles*. Un problème ouvert est celui de l'existence d'un *schéma réel* réalisable par une courbe pseudoholomorphe réelle non singulière, mais pas par une courbe algébrique réelle non singulière de même degré. Nous étudions dans cette thèse les courbes réelles non singulières *symétriques* de degré 7 dans $\mathbb{R}P^2$, algébriques et pseudoholomorphes. Nous obtenons en particulier plusieurs classifications, et exhibons deux schémas réels réalisables par des courbes pseudoholomorphes réelles séparantes symétriques non singulières de degré 7 mais pas par de telles courbes algébriques.
Certains des résultats de cette thèse sont basés sur l'utilisation des *dessins d'enfants*. En géométrie algébrique réelle, ces objets ont été utilisés la première fois par S. Yu. Orevkov. Ils permettent en particulier de répondre à la question suivante : Existe-t-il deux polynômes réels P et Q de degré n tels que les racines réelles de P, Q et $P+Q$ réalisent un arrangement donné dans $\mathbb{R}$? Suivant Orevkov, nous donnons une condition nécessaire et suffisante à l'existence de deux tels polynômes, formulée en terme de dessins d'enfants. Nous donnons aussi un algorithme permettant d'établir si un $\mathcal{L}$*-schéma* donné est réalisable par une *courbe algébrique réelle trigonale*.

Abstract

This thesis is motivated by the study of real algebraic curves in $\mathbb{R}P^2$ and in rational geometrically ruled surfaces equipped with their standard real structure. We were especially interested in two particular problems.
The ovals of a nonsingular curves in $\mathbb{R}P^2$ of even degree are naturally divided in two disjoint sets : *even ovals*, contained in an even number of ovals, and *odd ovals*. Combining the Harnack and Petrovsky inequalities, one obtains an upper bound on the number of even ovals, and on the number of odd ovals with respect to the degree of the curve. We generalize here a previous construction of I. Itenberg, and show that this upper bound is *asymptotically sharp*.
Almost all known prohibitions on the topology of real algebraic curves are still valid for a wider class of objects, *real pseudoholomorphic curves*. An open problem is the existence of a *real scheme* realizable by a nonsingular real pseudoholomorphic curve but not realizable by a nonsingular real algebraic curve of the same degree. In this thesis, we study real nonsingular algebraic and pseudoholomorphic *symmetric* curves of degree 7 in $\mathbb{R}P^2$. In particular, we give several classifications and exhibit two real schemes realizable by dividing nonsingular real symmetric pseudoholomorphic curves of degree 7, but not realizable by such algebraic curves.
Some results of this thesis are based on the techniques of *dessins d'enfants*. In real algebraic geometry these objects were first used by S. Yu. Orevkov. In particular, they allow one to answer the following question : does there exist two real polynomials P and Q of degree n such that the real roots of P, Q, and $P+Q$ realize a given arrangement in $\mathbb{R}$? Following Orevkov, we give a necessary and sufficient condition for the existence of two such polynomials, formulated in terms of *dessins d'enfants*. We also give an algorithm which gives a possibility to check whether a given $\mathcal{L}$*-scheme* is realizable by a *real trigonal algebraic curve*.