DEFORMATION OF TROPICAL HIRZEBRUCH SURFACES

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ABSTRACT. I describe in the tropical setting the Kodaira deformation of Hirzebruch surfaces, as well as their deformation to the normal bundle of a smooth rational curve. In particular, we will see that the second operation is a blow-up of the first one.

1. Complex Hirzebruch surfaces

1.1. **Definition.** A *Hirzebruch surface* S, also called a rational geometrically ruled surface, is a compact complex surface which admits a holomorphic fibration $\pi : S \to \mathbb{C}P^1$ with fiber $\mathbb{C}P^1$. The two simplest examples are $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2$ blown up in a point. In the latter case, the fibration is given by the extension to the blowing up of the projection from p to a line which does not pass through p.

The classification of Hirzebruch surfaces up to biholomorphism is well known (see for example [Bea83]). If S is biholomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, let us denote by E any line $\mathbb{C}P^1 \times \{x\}$. Otherwise, there exists a unique nonsingular algebraic section E on S, called the *exceptional section*, which has a negative self-intersection. Two Hirzebruch surfaces are biholomorphic if and only if their exceptional sections have the same self-intersection. When $E \circ E = -n$, we call this surface the Hirzebruch surface of degree n and we denote by Σ_n. From this description, one sees that Σ_1 is the blow-up of $\mathbb{C}P^2$ in a point.

The surface Σ_n is a projective toric surface which can be obtained by taking two copies of $\mathbb{C} \times \mathbb{C}P^1$ glued by the biholomorphism

$$\begin{array}{ccc} \mathbb{C}^* \times \mathbb{C}P^1 & \longrightarrow & \mathbb{C}^* \times \mathbb{C}P^1 \\ (z_1, w_1) & \longmapsto & (\frac{1}{z_1}, \frac{w_1}{z_1^n}) \end{array}$$

The coordinate system (z_1, w_1) in the first chart is called *standard*. If $n \geq 1$, the exceptional section is given by the equation $\{w_1 = +\infty\}$. The surface Σ_n is the toric surface defined by the polygon depicted in Figure 1a (the numbers labeling the edges correspond to the integer lengh of the corresponding edge).

Let us denote by B (resp. F) the curve given by the equation $\{z_1 = 0\}$ (resp., $\{w_1 = 0\}$). The group $Pic(\Sigma_n) = H_2(\Sigma_n, \mathbb{Z})$ is isomorphic to $\mathbb{Z}\times\mathbb{Z}$ and is generated by the classes of B and F. Note that we have

$$B \circ B = n, \quad F \circ F = 0, \quad \text{and} \quad E = B - nF \quad \text{in } H_2(\Sigma_n, \mathbb{Z}).$$

An algebraic curve C in Σ_n is said to be of *bidegree* (a, b) if it realizes the homology class $aB+bF$ in $H_2(\Sigma_n, \mathbb{Z})$. If E is not a component of C, then the Newton polygon

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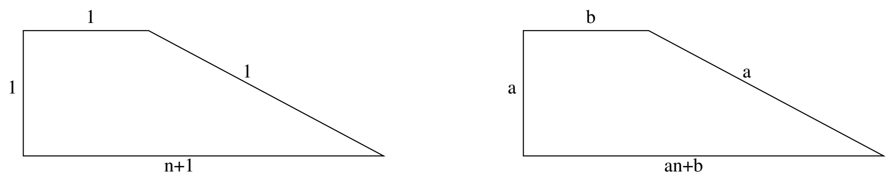


a) A polytope defining Σ_n

b) The Newton polygon of a curve of bidegree (a, b) in Σ_n

FIGURE 1

of such a curve in a standard coordinate system lies inside the trapeze with vertices $(0, 0)$, $(0, a)$, (b, a), and $(an + b, 0)$ (see Figure 1b).

1.2. **Kodaira deformation of Hirzebruch surfaces.** As we have seen, two Hirzebruch surfaces Σ_n and $\Sigma_{n'}$ are not biholomorphic if $n \neq n'$. However, if n and n' have the same parity, one can deform one of the two surfaces to the other one.

Theorem 1.1 (Kodaira, see [Kod86])**.** *Let $n, k \geq 0$ be two integer numbers. There exists a complex manifold $X_{n,k}$ of dimension 3 equipped with a submersion $\phi_{n,k} : X_{n,k} \to \mathbb{C}$ such that*

$$\forall t \neq 0,\ \phi_{n,k}^{-1}(t) = \Sigma_n, \quad \textit{and} \quad \phi_{n,k}^{-1}(0) = \Sigma_{n+2k}.$$

Note that this implies that Σ_n and Σ_{n+2k} are diffeomorphic. We will prove theorem 1.1 in section 2 in the tropical language. However, our proof translates litteraly in the complex setting.

1.3. **Kodaira deformation and deformation to the normal cone.** Kodaira deformation of Hirzebruch surfaces can actually be reduced to a standard procedure in both complex (deformation to the normal cone, see [Ful84]) and symplectic (stretching the neck, see [EGH00]) geometries.

Theorem 1.2 (see [Ful84])**.** *Let S be a complex surface, and C be a smooth algebraic curve in S. There exists a complex manifold $X'_{(S,C)}$ of dimension 3 equipped with a holomorphic map $\phi'_{(S,C)} : X'_{(S,C)} \to \mathbb{C}$ which is a submersion over $\mathbb{C}^*$, such that*

$$\forall t \neq 0,\ \phi'^{-1}_{(S,C)}(t) = S,$$

and $\phi'^{-1}_{(S,C)}(0)$ is the union of S and $\overline{\mathcal{N}(C)}$, the compactification of the normal bundle of C in S, intersecting along C.

The first Chern class of $\mathcal{N}(C)$ is the self-intersection of C in S. In particular, if C is rational of self-intersection m, then $\overline{\mathcal{N}(C)} = \Sigma_{|m|}$.

Suppose now that $S = \Sigma_n$ and C is a smooth rational curve of bidegree $(1, k)$. Since C has self-intersection $n + 2k$ in Σ_n, we get from Theorem 1.2 that

$$\phi'^{-1}_{(\Sigma_n,C)}(0) = \Sigma_n \cup \Sigma_{n+2k}.$$

In this case, it turns out that the two complex 3-folds $X_{n,k}$, from Theorem 1.1, and $X'_{\Sigma_n,k}$, from Theorem 1.2, are related by a blow-up: one can contract the copy of Σ_n in $\phi'^{-1}_{(\Sigma_n,C)}(0)$ to obtain $X_{n,k}$.

Proposition 1.3. *The complex manifold $X'_{\Sigma_n,k}$ is the blow-up Π of $X_{n,k}$ along the exceptional curve of Σ_{n+2k}, and*

$$\phi'_{(\Sigma_n,C)} = \phi_{n,k} \circ \Pi.$$

We prove Proposition 1.3 in section 2. Once again, our tropical proof translates litteraly in the complex setting.

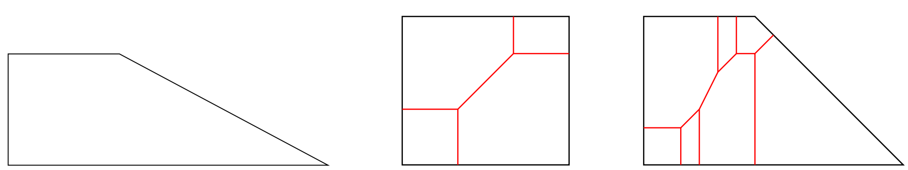
2. Tropical Hirzebruch surfaces

2.1. **Toric construction.** The construction of any non-singular toric variety can be performed exactly in the same way in tropical and classical geometry. In particular, the tropical Hirzebruch surface of degree $n \geq 0$, denoted by $\mathbb{T}\Sigma_n$ is constructed by taking two copies of $\mathbb{T} \times \mathbb{T}P^1$ glued along $\mathbb{T}^* \times \mathbb{T}P^1$ via the tropical isomorphism

$$\begin{array}{cccll} \psi: & \mathbb{T}^* \times \mathbb{T}P^1 & \longrightarrow & \mathbb{T}^* \times \mathbb{T}P^1 & \\ & (x_1, y_1) & \longmapsto & (\text{“}\frac{1}{x_1}\text{”}, \text{“}\frac{y_1}{x_1^n}\text{”}) & = (-x_1, y_1 - nx_1) \end{array}$$

As in the complex setting, $\mathbb{T}\Sigma_0 = \mathbb{T}P^1 \times \mathbb{T}P^1$, and $\mathbb{T}\Sigma_1$ is $\mathbb{T}P^2$ blown up at $[-\infty : 0 : -\infty]$.

The map ψ maps the vector $(1, n)$ to the vector $(-1, 0)$. For this reason, the tropical surface $\mathbb{T}\Sigma_n$ is usually represented by a quadrangle with two horizontal edges, one vertical edge, and one edge of slope $-\frac{1}{n}$, see Figure 2a. More generally, once a linear system is fixed, the tropical moment map provides an homeomorphism from any non-singular toric tropical variety to its Newton polygon. This homeomorphism is also given by the Veronese embedding corresponding to the chosen linear system.



a) $\mathbb{T}\Sigma_n$

b) A curve of bidegree $(1, 1)$ in $\mathbb{T}\Sigma_0$

c) A curve of bidegree $(1, 2)$ in $\mathbb{T}\Sigma_1$

FIGURE 2

Let us denote by B the tropical curve in $\mathbb{T}\Sigma_n$ defined by the tropical polynomial y_1, and by F the curve defined by the polynomial x_1 (in the coordinate system defined above on $\mathbb{T}\Sigma_n$). That is to say, the curve B is the lowest horizontal edge, and F is the left vertical edge. Note that $B \circ B = n$ and $F \circ F = 0$. The Picard group of $\mathbb{T}\Sigma_n$ is the free abelian group of rank two generated by B and F, and a tropical 1-cycle C in $\mathbb{T}\Sigma_n$ is said to have bidegree (a, b) if it is linearly equivalent to $aB + bF$. Equivalently, C is of bidegree (a, b) if and only if

$$C \circ B = an + b \quad \text{and} \quad C \circ F = a.$$

Example 2.1. We depicted in Figure 2b a tropical curve of bidegree $(1,1)$ in $\mathbb{T}\Sigma_0$, and a tropical curve of bidegree $(1,2)$ in $\mathbb{T}\Sigma_1$ in Figure 2c.

The exceptional divisor of $\mathbb{T}\Sigma_n$ is the upper horizontal edge, defined by the rational function "$\frac{1}{y_1}$", and represents the class $B-nF$ in $Pic(\mathbb{T}\Sigma_n)$. In particular, one has $E^2=-n$.

2.2. **Deformation of tropical Hirzebruch surfaces.** Let C be a non-singular tropical curve in $\mathbb{T}\Sigma_n$ of bidegree $(1,k)$. By the genus formula C is rational. Moreover, we have $C^2=n+2k$.

We start by describing the deformation of $\mathbb{T}\Sigma_n$ to the normal cone of C. Recall that the sign "=" between two tropical varieties means "isomorphic up to tropical modifications"

Theorem 2.2. *There exists a non-singular tropical variety $\mathbb{T}X'$ of dimension 3 and a tropical morphism $\Phi':\mathbb{T}X'\to\mathbb{T}P^1$ such that*

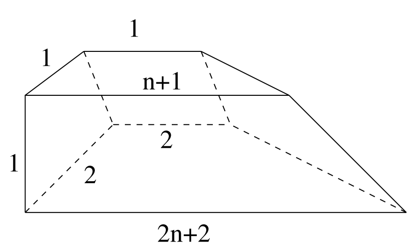
$$\forall t\neq-\infty,\ \Phi'^{-1}(t)=\mathbb{T}\Sigma_n,\quad \textit{and}\quad \Phi'^{-1}(-\infty)=\mathbb{T}\Sigma_n\cup\mathbb{T}\Sigma_{n+2k}.$$

Moreover, the intersection curve of the two latter surfaces is C in $\mathbb{T}\Sigma_n$, and the exceptional section E in $\mathbb{T}\Sigma_{n+2k}$.

Proof. Let Δ' be the polytope in $\mathbb{R}^3$ with vertices (see Figure 3)

$$(0,0,0),\ (2n+2,0,0),\ (2,2,0),\ (0,2,0),\ (0,0,1),\ (n+1,0,1),\ (1,1,1),\ (0,1,1).$$

The polytope Δ' defines a non-singular tropical toric variety of dimension 3, denoted by $Tor(\Delta')$. If (x,y,z) are the coordinates in the dense orbit of $Tor(\Delta')$, then the map $(x,y,z)\to(x,y)$ induces a tropical morphism $\pi: Tor(\Delta')\to\mathbb{T}\Sigma_n$ whose firbers are $\mathbb{T}P^1$.



The polytope Δ'

FIGURE 3

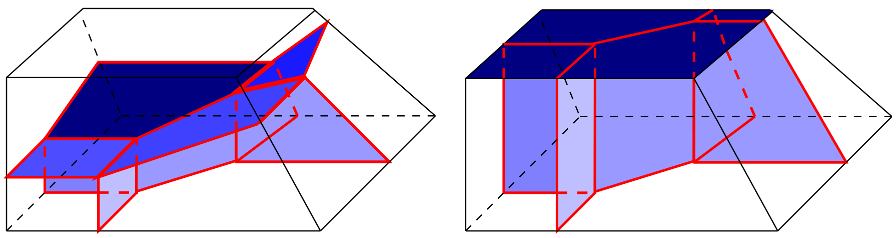
We fix a tropical polynomial $P(x,y)$ defining the curve C in $\mathbb{T}\Sigma_n$, and we define $\mathbb{T}X'$ as the hypersurface in $\mathbb{T}P^1\times Tor(\Delta')$ defined by the tropical polynomial

$$\text{“}tz+P(x,y)\text{”}$$

where t is the coordinate in $\mathbb{T}P^1$.

The tropical variety $\mathbb{T}X'$ is non-singular, and there is a natural tropical morphism $\Phi':\mathbb{T}X'\to\mathbb{T}P^1$ whose fiber over t_0 is the tropical hypersurface in $Tor(\Delta')$ defined by the tropical polynomial "$t_0z+P(x,y)$" (see Figure 4a in the case $n=0$ and

$k = 1$). If $t_0 \neq -\infty$, then $\Phi'^{-1}(t_0)$ is non singular, and the morphisms $\pi_{|\Phi'^{-1}(t_0)}$: $\Phi'^{-1}(t_0) \to \mathbb{T}\Sigma_n$ is a tropical modification of $\mathbb{T}\Sigma_n$ along C. In particular $\Phi'^{-1}(t_0) = \mathbb{T}\Sigma_n$. The hypersurface $\Phi'^{-1}(-\infty)$ is the union of the tropical surface S_0 defined by $P(x, y)$ with the surface S_1 in $Tor(\Delta')$ defined by the rational function "$\frac{1}{z}$" (i.e. the upper horizontal face of $Tor(\Delta')$), see Figure 4b.



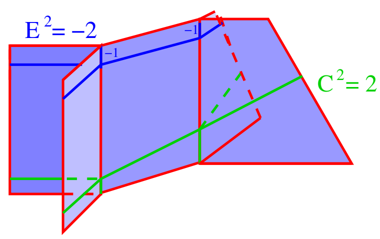
a) $\Phi'^{-1}(t_0)$ with $t_0 \neq -\infty$

b) $\Phi'^{-1}(-\infty)$

FIGURE 4

It is clear from Δ' that $S_1 = \mathbb{T}\Sigma_n$, so it remains to prove that $S_0 = \mathbb{T}\Sigma_{n+2k}$. First, the morphisms π restricts on a tropical morphism $S_0 \to C$ whose fibers are $\mathbb{T}P^1$. Next, by adding a vertical edge to each tropical intersection points of C in $\mathbb{T}\Sigma_n$, we see that the self-intersection of C in $\mathbb{T}\Sigma_n$ and S_0 are the same, i.e. $S_0 = \mathbb{T}\Sigma_{n+2k}$ (see Figure 5).

Note that for the same reasons, we have $E^2 = -n - 2k$ in S_0. $\square$



Self intersection of C and E in S_0

FIGURE 5

Example 2.3. We depicted in Figure 4 this degeneration process when $n = 0$, $k = 1$, and C is the tropical curve depicted in Figure 2b.

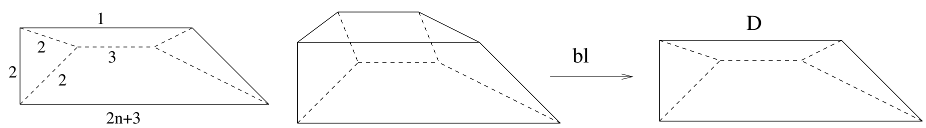
In the next lemma, we use notation used in the proof of Theorem 2.2.

Lemma 2.4. *One can blow down S_1 to E in $\mathbb{T}X'$, i.e. there exists a non-singular tropical variety $\mathbb{T}X$ of dimension 3 and a tropical blow down $bl : \mathbb{T}X' \to \mathbb{T}X$ which contracts the surface S_1 to a curve isomorphic (up to tropical modifications) to E.*

Proof. This is an immediate consequence of the fact that one can blow down the surface S_1 in $Tor(\Delta')$. Indeed, let Δ be the polytope with vertices (see Figure 6a)

$$(0, 0, 0),\ (2n + 3, 0, 0),\ (3, 2, 0),\ (0, 2, 0),\ (0, 0, 1),\ (1, 0, 1).$$

The tropical 3-fold $Tor(\Delta)$ is non-singular, and the existence of the desired blow-up $Tor(\Delta') \to Tor(\Delta)$ can be read directly on the polytopes Δ and Δ' (see Figure 6b). □



a) Δ b) Toric blow up of the curve D

FIGURE 6

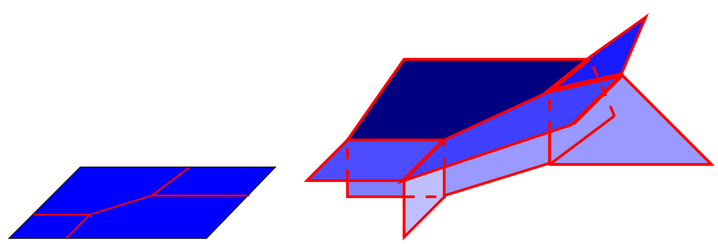
As an immediate corollary, we obtain the tropical version of Kodaira deformation of Hirzebruch surfaces.

Corollary 2.5. *There exists a non-singular tropical variety $\mathbb{T}X$ of dimension 3 and a tropical morphism $\Phi : \mathbb{T}X \to \mathbb{T}P^1$ such that*

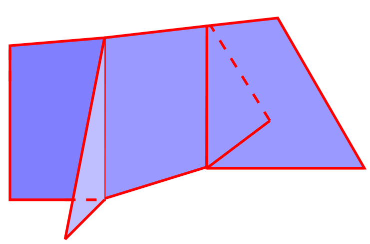
$$\forall t \neq -\infty,\ \Phi^{-1}(t) = \mathbb{T}\Sigma_n, \quad \textit{and} \quad \Phi^{-1}(-\infty) = \mathbb{T}\Sigma_{n+2k}.$$

Proof. Take $\mathbb{T}X$ as in Lemma 2.4, and $\Phi : \mathbb{T}X \to \mathbb{T}P^1$ such that $\Phi' = \Phi \circ bl$. □

Example 2.6. We depicted in Figure 7 this deformation when $n = 0$, $k = 1$, and C is the tropical curve depicted in Figure 2b.



a) $\Phi^{-1}(+\infty)$ b) $\Phi^{-1}(t_0)$ with $t_0 \neq \pm\infty$



$\Phi^{-1}(-\infty)$

FIGURE 7

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