TROPICAL CURVES

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Abstract. These are rough notes of a five hours series of introductary lectures given in July 2013 at Max Planck Institute for Mathematics in Bonn.

The goal of these lectures is to give a basic introduction to tropical geometry, via tropical curves. Instead of going to a general theory, I focused on some particular aspects of tropical geometry for sake of simplicity. As a result, in many places a more conceptual presentation of the subject could have been given, and would definitely be preferable for further developpements. Hopefully, the present text can be a source of motivation to turn to more conceptual expositions.

Tropical geometry is intimately linked to classical algebraic geometry, and I tried as much as possible to illustrate this aspect by justifying any appearing tropical notion by its relation to classical geometry.

An extended version of the first two sections, with sometimes more details, can be found in the introductary texts [Bru09, Bru12][1]. These two texts also contain a tropical version of the combinatorial Patchworking construction. However the presentation I adopted here in Section 3 is a bit different.

The material presented here is not new, and I do not claim any orginality in the exposition. Other introductions to tropical geometry exist, for example [BPS08] (in French), [RGST05], [IM12], [Vir08], [Vir11] or [Gat06]. A more advanced reader may refer to [Mik06], [Mik04], [IMS07], and references therein.

1. Tropical algebra

1.1. **Tropical semi-field.** The set of *tropical numbers* is defined as $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$, that we endow with the following operations on $\mathbb{R}$, called *tropical addition* and *multiplication*, in the following way:

$$“x + y” = \max\{x, y\} \qquad “x \times y” = x + y$$

with the usual convention that

$$\forall x \in \mathbb{T}, \quad “x + (-\infty)” = \max(x, -\infty) = x \quad \text{and} \quad “x \times (-\infty)” = x + (-\infty) = -\infty.$$

In the entire text, tropical operations will be placed under quotation marks. Just as in classical algebra we often abbreviate “$x \times y$” to “xy”. Tropical numbers form a semi-field, i.e. it satisfies all the axioms of a field except the existence of an inverse for the law “ + ”.

To familiarise ourselves with these two operations, let us do some simple calculations:

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[1]Many thanks to Kristin Shaw for her authorization to use some parts of her translation of the original French version of [Bru09].

$$\text{“}1+1\text{”} = 1, \quad \text{“}1+2\text{”} = 2, \quad \text{“}1+2+3\text{”} = 3, \quad \text{“}1\times 2\text{”} = 3, \quad \text{“}1\times(2+(-1))\text{”} = 3,$$
$$\text{“}1\times(-2)\text{”} = -1, \quad \text{“}(5+3)^2\text{”} = 10, \quad \text{“}(x+y)^n\text{”} = \text{“}x^n+y^n\text{”}.$$

Be sure to be careful when writing tropical formulas! As, “$2x$” $\neq$ “$x+x$” but “$2x$” $= x+2$, similarly “$1x$” $\neq x$ but “$1x$” $= x+1$, and again “$0x$” $= x$ and “$(-1)x$” $= x-1$.

A very important feature of the tropical semi-ring is that it is *idempotent*, this means that “$x+x$” $= x$ for all x in $\mathbb{T}$. This implies in particular that one cannot solve the problem of non-existence of tropical substraction by adding more elements to $\mathbb{T}$ to try to cook up additive inverses (see Exercise (1)). Our only choice is to get used to the lack of tropical additive inverses.

1.2. **Maslov Dequantization.** Let us explain how the tropical semi-field arises naturally as the limit of some classical semi-fields. This procedure, studied by Victor Maslov and his collaborators beginning in the 90's, is known as *dequantisation of the non-negative real numbers*.

A well-known semi-field is $(\mathbb{R}_{\geq}, +, \times)$, the set of non-negative real numbers together with the usual addition and multiplication. If t is a strictly positive real number, then the logarithm of base t provides a bijection between the sets $\mathbb{R}$ and $\mathbb{T}$. This bijection induces a semi-field structure on $\mathbb{T}$ with the operations denoted by “ $+_t$ ” and “ $\times_t$ ”, and given by:

$$\text{“}x +_t y\text{”} = \log_t(t^x + t^y) \quad \text{and} \quad \text{“}x \times_t y\text{”} = \log_t(t^x t^y) = x+y.$$

The equation on the right-hand side already shows classical addition appearing as an exotic kind of multiplication on $\mathbb{T}$. Notice that by construction, all of the semi-fields $(\mathbb{T}, \text{“}+_t\text{”}, \text{“}\times_t\text{”})$ are isomorphic to $(\mathbb{R}_+, +, \times)$. The trivial inequality $\max(x,y) \leq x+y \leq 2\max(x,y)$ on $\mathbb{R}_+$ together with the fact that the logarithm is an increasing function gives us the following bounds for “ $+_t$ ”:

$$\forall t>0, \quad \max(x,y) \leq \text{“}x +_t y\text{”} \leq \max(x,y) + \log_t 2.$$

If we let t tend to infinity, $\log_t 2$ tends to 0, and the operation “ $+_t$ ” therefore tends to the tropical addition “ $+$ ”! Hence the tropical semi-field comes naturally from degenerating the classical semi-field $(\mathbb{R}_+, +, \times)$. From an alternative perspective, we can view the classical semi-field $(\mathbb{R}_+, +, \times)$ as a deformation of the tropical semi-field. This explains the use of the term “dequantisation”.

1.3. **Tropical polynomials.** As in classical algebra, a tropical polynomial $P(x) = \text{“}\sum_{i=0}^d a_i x^i\text{”}$ induces a tropical polynomial function, still denoted by P, on $\mathbb{T}$:

$$\begin{array}{rccc} P: & \mathbb{T} & \longrightarrow & \mathbb{T} \\ & x & \longmapsto & \text{“}\sum_{i=0}^d a_i x^i\text{”} = \max_{i=1}^d (a_i + ix) \end{array}$$

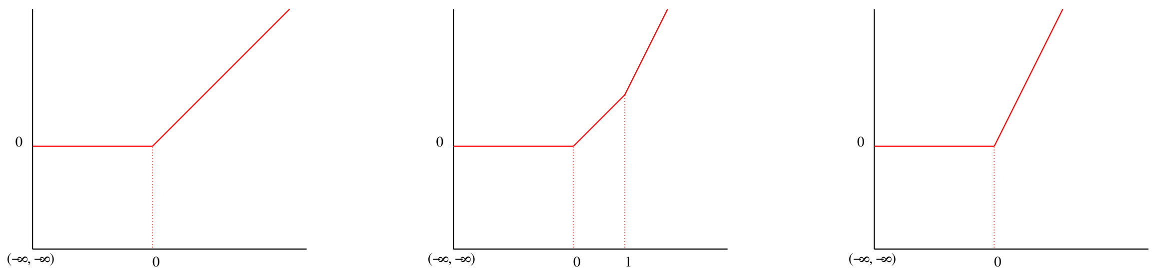
Note that the map wich associate its tropical polynomial function to a tropical polynomial is surjective, by definition, but is *not* injective. In the whole text, tropical polynomials have to be understood as *tropical polynomial functions*!

Let us look at some examples of tropical polynomials:

$$\text{“}x\text{”} = x, \quad \text{“}1+x\text{”} = \max(1,x), \quad \text{“}1+x+3x^2\text{”} = \max(1,x,2x+3),$$
$$\text{“}1+x+3x^2+(-2)x^3\text{”} = \max(1,x,2x+3,3x-2).$$

Now let us define the the roots of a tropical polynomial. For this, let us take a geometric point of view of the problem. A tropical polynomial is a convex piecewise affine function and each piece has an integer slope (see Figure 1). We call *tropical roots* of the polynomial $P(x)$ all points x_0 of $\mathbb{T}$

for which the graph of $P(x)$ has a corner at x_0. Moreover, the difference in the slopes of the two pieces adjacent to a corner gives the *order* of the corresponding root.



a) $P(x) = $ "$0 + x$" b) $P(x) = $ "$0 + x + (-1)x^2$" b) $P(x) = $ "$0 + x^2$"

Figure 1. The graphs of some tropical polynomials

Definition 1.1. *The roots of a tropical polynomial* $P(x) = $ "$\sum_{i=0}^{d} a_i x^i$" *are the tropical numbers* x_0 *for which there exists a pair* $i \neq j$ *such that* $P(x_0) = $ "$a_i x_0^i$" $=$ "$a_j x_0^j$".

The order of the root x_0 *is the maximum of* $|i - j|$ *for all possible such pairs* i, j.

Thus, the polynomial "$0 + x$" has a simple root at $x_0 = 0$, the polynomial "$0 + x + (-1)x^2$ has simple roots 0 and 1 and the polynomial "$0 + x^2$ has a double root at 0.

Proposition 1.2. *The tropical semi-field is algebraically closed. In other words, every tropical polynomial of degree* d *has exactly* d *roots when counted with multiplicities.*

For example, one may check that we have the following factorisations[2]:

$$\text{“}0 + x + (-1)x^2\text{”} = \text{“}(-1)(x + 0)(x + 1)\text{”} \quad \textit{and} \quad \text{“}0 + x^2\text{”} = \text{“}(x + 0)^2\text{”}$$

1.4. **Relation to classical algebra.** Let $P_t(z) = \sum \alpha_i(t) z^i$ be a family complex polynomials indexed by t a large enough positive real number. We make the assumption that

$$\forall i, \quad \exists a_i \in \mathbb{T}, \quad \exists \beta_i \in \mathbb{C}, \quad \alpha_i \sim_{t \to \infty} \beta_i t^{a_i}.$$

Then we define the tropical polynomial, called the *tropicalization* of the family P_t, by

$$P_{trop}(x) = \text{“}\sum a_i x^i\text{”}.$$

We also define the map

$$\begin{array}{rccc} \mathrm{Log}_t : & \mathbb{C} & \longrightarrow & \mathbb{T} \\ & z & \longmapsto & \log_t(|z|) \end{array}.$$

The following theorem can be seen a dual version of Newton-Puiseux method. This actual formulation is a particular case of a more general result by Mikhalkin about approximations of tropical hypersurfaces by amoebas (see also Seciton 2.4). A analogous non-archimedean version has also been proved by Kapranov.

[2] Once again the equalities hold in terms of polynomial functions not on the level of the polynomials. Just as, "$0 + x^2$" and "$(0 + x)^2$" are equal as polynomial functions but not as polynomials.

Theorem 1.3. *One has*

$$\mathrm{Log}_t\left(\{\text{roots of } P_t\}\right) \to_{t\to\infty} \{\text{roots of } P_{trop}\}.$$

Moreover, the order of x_0 is exactly the number of roots of P_t whose logarithm converge to x_0.

1.5. **Exercises.**

(1) Why does the idempotent property of tropical addition prevent the existence of inverses for this operation?
(2) Find two distinct tropical polynomials defining the same tropical polynomtal functions.
(3) Draw the graphs of the tropical polynomials $P(x) =$ "$x^3 + 2x^2 + 3x + (-1)$" and $Q(x) =$ "$x^3 + (-2)x^2 + 2x + (-1)$", and determine their tropical roots.
(4) Prove that the root of the tropical polynomial $P(x) =$ "x" is $-\infty$.
(5) Prove that x_0 is a root of order k of a tropical polynomial if there exists a tropical polynomial $Q(x)$ such that $P(x) =$ "$(x + x_0)^k Q(x)$" and x_0 is not a root of $Q(x)$.

Note that the factor $x - x_0$ in classical algebra gets transformed to the factor "$x + x_0$", since the root of the polynomial "$x + x_0$" is x_0 and not $-x_0$.
(6) Prove Proposition 1.2.
(7) Let $a \in \mathbb{R}$ and $b, c, d \in \mathbb{T}$. Determine the roots of the polynomials "$ax^2 + bx + c$" and "$ax^3 + bx^3 + cx + d$".

2. Tropical curves in $\mathbb{R}^2$

Let us now extend the preceeding notions to the case of tropical polynomials in two variables. Since this makes all definitions, statements and drawings simpler, we restrict ourselves to tropical curves in $\mathbb{R}^2$ instead of $\mathbb{T}^2$. However this does not affect at all the generality of what will be discussed here.

2.1. **Definition.** A tropical polynomial in two variables is written $P(x, y) =$ "$\sum_{i,j} a_{i,j} x^i y^j$", or better yet $P(x, y) = \max_{i,j}(a_{i,j} + ix + jy)$ in classical notation. In this way our tropical polynomial is again a convex piecewise affine function, and we denote by $\widetilde{V}(P)$ the corner locus of this function. That is to say,

$$\widetilde{V}(P) = \left\{(x_0, y_0) \in \mathbb{R}^2 \mid \exists (i,j) \neq (k,l), \quad P(x_0, y_0) = \text{“}a_{i,j} x_0^i y_0^j\text{”} = \text{“}a_{k,l} x_0^k y_0^l\text{”}\right\}.$$

Example 2.1. Let us look at the tropical line defined by the polynomial $P(x, y) =$ "$x + y + 0$". We must find the points (x_0, y_0) in $\mathbb{R}^2$ that satisfy one of the following three systems of equations:

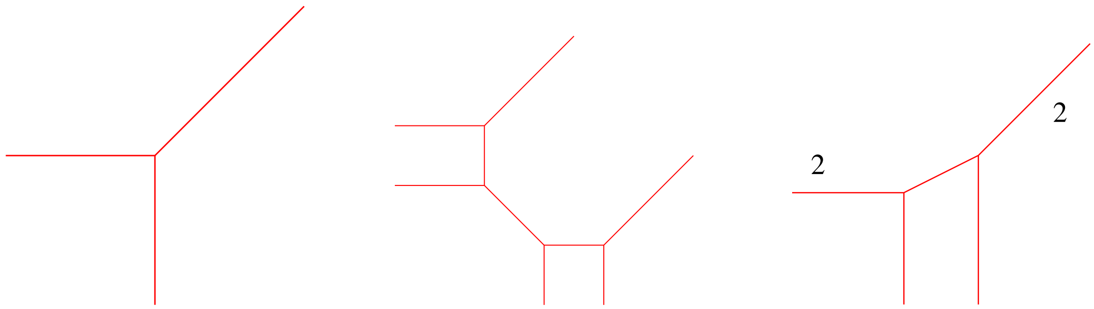
$$x_0 = 0 \geq y_0, \qquad y_0 = 0 \geq x_0, \qquad x_0 = y_0 \geq 0$$

We see that the set $\widetilde{V}(P)$ is made up of three standard half-lines

$$\{(0, y) \mid y \leq 0\},\ \{(x, 0) \mid x \leq 0\},\ \text{and}\ \{(x, x) \mid x \geq 0\}$$

(see Figure 2a).

We are still missing one bit of information to properly define a tropical curve. The set $\widetilde{V}(P)$ is a piecewise linear graph in $\mathbb{R}^2$ (careful: from now on the word "graph" has to be understood in its graph theoretical sense). Just as in the case of polynomials in one variable, for each edge of a tropical curve, we must take into account the difference in the slope of $P(x, y)$ on the two sides of the edge. Doing this we arrive at the following formal definition of a tropical curve.



a) "$x + y + 0$" b) "$3 + 2x + 2y + 3xy + y^2 + x^2$" c) "$0 + x + y^2 + (-1)x^2$"

Figure 2. A tropical line and two tropical conics.

Definition 2.2. *The weight of an edge of $\widetilde{V}(P)$ is defined as the maximum of the greatest common divisor (gcd) of the numbers $|i - k|$ and $|j - l|$ for all pairs (i, j) and (k, l) such that the value of $P(x, y)$ on this edge is given by the corresponding monomials.*

The tropical curve defined by $P(x, y)$ is the set $\widetilde{V}(P)$ equipped with this weight function on its edges.

We will see soon how to see this weight function geometrically on the dual subdivision of the tropical curve.

Example 2.3. In pictures of tropical curves, the weight of an edge is only indicated if this latter is at least two. For example, in the case of the tropical line, all edges are of weight 1. Thus, Figure 2a represents fully the tropical line. Two examples of tropical curves of degree 2 are shown in Figure 2b and c. The tropical conic in Figure 2c has two edges of weight 2.

2.2. **Dual subdivision.** Let $P(x, y) =$ "$\sum_{i,j} a_{i,j} x^i y^j$" be a tropical polynomial. The *Newton polygon* of $P(x, y)$, denoted by $\Delta(P)$, is defined by[3]

$$\Delta(P) = Conv\{(i, j) \mid a_{i,j} \neq -\infty\}.$$

A polynomial $P(x, y)$ over any field or semi-field always comes with its Newton polygon. A tropical polynomial comes in addition with a subdivision of $\Delta(P)$, called its dual subdivision.

Given $(x_0, y_0) \in \mathbb{R}^2$, we define

$$\Delta_{(x_0,y_0)} = Conv\{(i, j) \mid P(x_0, y_0) = \text{“}a_{i,j} x_0^i y_0^j\text{”}\}.$$

The tropical curve defined by $P(x, y)$ induces a polyhedral decomposition of $\mathbb{R}^2$, and it is easy to see that the polygon $\Delta_{(x_0,y_0)}$ only depends on the cell F of this decomposition which contains (x_0, y_0). Hence we write Δ_F rather than $\Delta_{(x_0,y_0)}$.

Example 2.4. Let us go back to the tropical line L defined by the equation $P(x, y) =$ "$x + y + 0$" (see Figure 2a). On the 2-cell $F_1 = \{x, y < 0\}$, the value of $P(x, y)$ is given by the monomial 0, and so $\Delta_{F_1} = \{(0, 0)\}$. Similarly, we have $\Delta_{F_2} = \{(1, 0)\}$ and $\Delta_{F_3} = \{(0, 1)\}$ for the cells $F_2 = \{x > y, 0\}$ and $F_3 = \{y > x, 0\}$.

[3]in classical algebra, one should replace $-\infty$ by 0 in the definition of $\Delta(P)$.

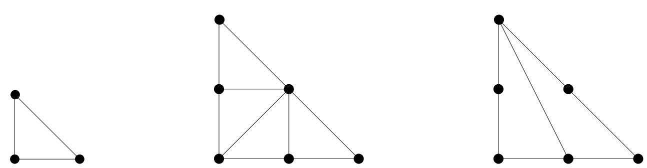
Along the horizontal edge e_1 of L the value of $P(x,y)$ is given by the monomials 0 and y, and so Δ_{e_1} is the the vertical edge of $\Delta(P)$. In the same way Δ_{e_2} is the the horizontal edge of $\Delta(P)$ for the vertical edge e_2 of L, and Δ_{e_3} is the the edge of $\Delta(P)$ with endpoints $(1,0)$ and $(0,1)$ for the edge e_3 of L with slope 1.

The point $(0,0)$ is the vertex v of the line C. This is where the three monomials 0, x and y take the same value, and so $\Delta_v = \Delta(P)$. (see Figure 3a).

More generally, all polyhedra Δ_F form a subdivision of the Newton polygon $\Delta(P)$ dual to the polyhedral subdivision of $\mathbb{R}^2$ induced by the tropical curves C in a sense precised in next proposition. The subdivision of $\Delta(P)$ is said to be *dual* to the tropical curve defined by $P(x,y)$.

Proposition 2.5. *One has*
- $\Delta(P) = \cup_F \Delta_F$, *where F range over all cells of the polyhedral subdivision of $\mathbb{R}^2$ induced by the tropical curve defined by $P(x,y)$;*
- $\dim F = \operatorname{codim} \Delta_F$*;*
- Δ_F *and F are orthogonal;*
- $\Delta_F \subset \Delta_{F'} \Leftrightarrow F' \subset F$*; moreover in this case Δ_F is a face of $\Delta_{F'}$;*
- $\Delta_F \subset \partial\Delta(P) \Leftrightarrow F$ *is unbounded.*

Example 2.6. The dual subdivisions of the tropical curves in Figure 2 are drawn in Figure 3 (the black points represent the points of $\mathbb{R}^2$ with integer coordinates, notice they are not necessarily the vertices of the dual subdivision).



a) b) c)

FIGURE 3. Some dual subdivisions

The weight of an edge of a tropical curve may be read off directly from its dual subdivision.

Proposition 2.7. *An edge e of a tropical curve has weight w if and only if the* integer length *of Δ_e is w, i.e. $Card(\Delta_e \cap \mathbb{Z}^2) - 1 = w$.*

2.3. **Balanced graphs and tropical curves.** The first consequence of this duality is that a certain relation, know as the *balancing condition*, is satisfied at each vertex of a tropical curve. Suppose v is a vertex of a tropical curve C adjacent to the edges $e_1, \ldots, e_k$ with respective weights $w_1, \ldots, w_k$. Recall that every edge e_i is supported on a line (in the usual sense) defined by an equation with integer coefficients. Because of this there exists a unique integer vector $v_i = (\alpha, \beta)$ in the direction of e_i such that $\gcd(\alpha, \beta) = 1$ (see Figure 4a). Following the previous section, the polygon Δ_v dual to v yields immediately the vectors $w_1 v_1, \ldots, w_k v_k$: if we orient the boundary of Δ_v in the counter-clockwise direction, so that each edge δ_{e_i} of Δ_v dual to e_i is obtained from a vector $w_i v_i$ by rotating by an angle of exactly $\pi/2$ (see Figure 4b).

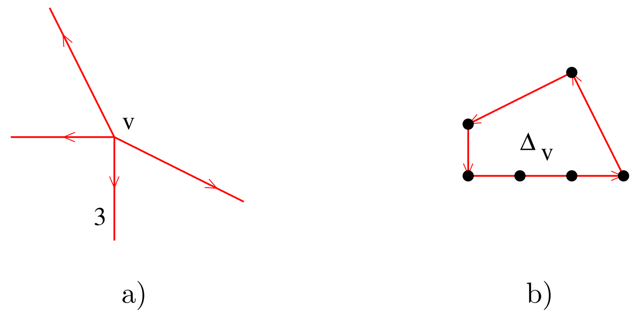


FIGURE 4. Balancing condition.

The fact that the polygon Δ_v is closed immediately implies the following balancing condition:

$$\sum_{i=1}^{k} w_i v_i = 0$$

A graph in $\mathbb{R}^2$ whose edges have rational slopes and are equipped with positive integer weights is a *balanced graph* if it satisfies the balancing condition at each one of its vertices. We have just seen that every tropical curve is a balanced graph. In fact, the converse is also true.

Theorem 2.8 (Mikhalkin)**.** *Tropical curves in $\mathbb{R}^2$ correspond exactly to balanced graphs in $\mathbb{R}^2$.*

Thus, this theorem affirms that there exist tropical polynomials of degree 3 whose tropical curves are the weighted graphs in figure 5. We have also drawn for each curve, the associated dual subdivision of their Newton polygon.

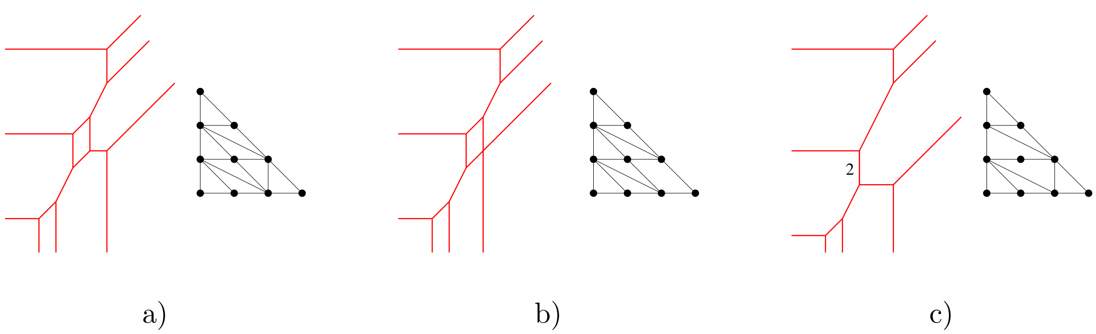


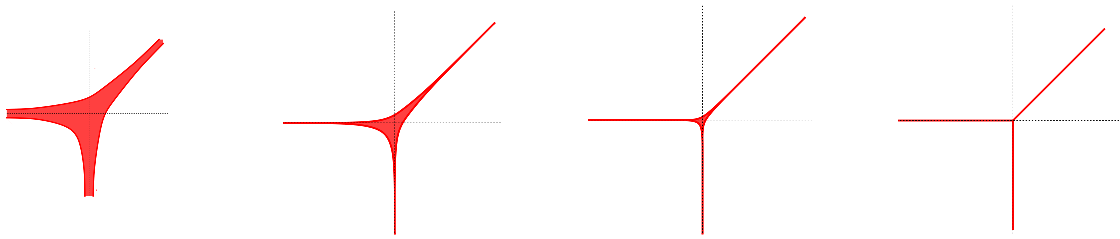
FIGURE 5.

2.4. **Tropical curves as limits of amoebas.** As in the case of polynomials in one variable, tropical curves can be approximated, via the logarithm map, by algebraic curves in $(\mathbb{C}^*)^2$. For this, we need the following map

$$\begin{array}{cccc} \mathrm{Log}_t : & (\mathbb{C}^*)^2 & \longrightarrow & \mathbb{R}^2 \\ & (z,w) & \longmapsto & (\log_t |z|, \log_t |w|) \end{array}.$$

Definition 2.9 (Gelfand-Kapranov-Zelevinsky)**.** *The* amoeba *(in base t) of an algebraic curve in $(\mathbb{C}^*)^2$ is its image under the map Log_t.*

Let us look more closely at these amoebas with the help of a concrete example, namely the line $\mathcal{L}$ with equation $z + w + 1 = 0$ in $(\mathbb{C}^*)^2$. One can compute by hand that the amoeba of $\mathcal{L}$ is as depicted in Figure 6a. In particular, we see that it has three asymptotic directions: $(-1, 0)$, $(0, -1)$, and $(1, 1)$. By definition of $\log_t$, the amoeba of $\mathcal{L}$ in base t is a contraction by a factor $\log t$ of the



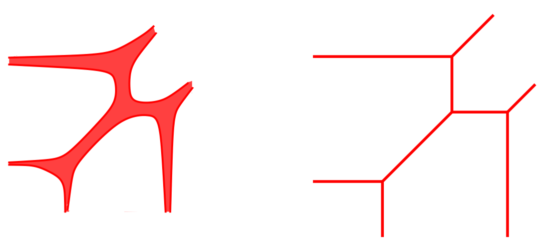
a) $\mathrm{Log}(\mathcal{L})$ b) $\mathrm{Log}_{t_1}(\mathcal{L})$ c) $\mathrm{Log}_{t_2}(\mathcal{L})$ d) $\lim_{t\to\infty} \mathrm{Log}_t(\mathcal{L})$

Figure 6. Dequantization of a line ($e < t_1 < t_2$)

amoeba of L in base e (see Figures 6b and c). Hence when t goes to $+\infty$, the whole amoeba is contracted to the origin, only the three asymptotic directions are remaining. In other words, what we see at the limit in Figure 6d is a tropical line!

Of course, the same strategy applied to any classical curve will produce a similar picture at the limit: the origin from which the asymptotic directions of the amoeba emerge. To get a more interesting limit, one should look at the family of amoebas $(\mathrm{Log}_t(\mathcal{C}_t))_{t>0}$ where $(\mathcal{C}_t)_{t>0}$ is a family of complex curves. If we do so, then the limit becomes much richer.

Example 2.10. We depicted in Figure 7 the shape of the amoeba of the curve with equation $1 - z - w + t^2z^2 - tzw + t^2y^2 = 0$ for t large enough, and its limit which is... a tropical conic.



$\mathrm{Log}_t(\mathcal{C}_t)$ $\lim_{t\to\infty} \mathrm{Log}_t(\mathcal{C}_t)$

Figure 7. $\mathcal{C}_t : 1 - z - w + t^2z^2 - tzw + t^2y^2 = 0$

Theorem 2.11 (Mikhalkin, Rullgård)**.** *Let $P_t(z, w) = \sum_{i,j} \alpha_{i,j}(t) z^i w^j$ be a polynomial whose coefficients are functions $\alpha_{i,j} : \mathbb{R} \to \mathbb{C}$, and suppose that $\alpha_{i,j}(t) \sim \gamma_{i,j} t^{a_{i,j}}$ when t goes to $+\infty$ with $\gamma_{i,j} \in \mathbb{C}^*$ and $a_{i,j} \in \mathbb{T}$.*

If $\mathcal{C}_t$ denotes the curve in $(\mathbb{C}^*)^2$ defined by the polynomial $P_t(z,w)$, then the amoeba $Log_t(\mathcal{C}_t)$ converges to the tropical curve defined by the tropical polynomial $P_{trop}(x,y) =$ "$\sum_{i,j} a_{i,j}x^iy^j$".

It remains us to explain the relation between amoebas and weights of a tropical curve. Let $P_t(z,w)$ and $P'_t(z,w)$ be two families of complex polynomials, defining two families of complex algebraic curves $(\mathcal{C}_t)_{t>0}$ and $(\mathcal{C}'_t)_{t>0}$ respectively. As in Theorem 2.11, these two families of polynomials induce two tropical polynomials $P_{trop}(x,y)$ and $P'_{trop}(x,y)$, which in their turn define two tropical curves C and C'.

Proposition 2.12 (Mikhalkin)**.** *Let $p \in C \cap C'$ which is a vertex neither of C nor of C'. Then the number of intersection points of $\mathcal{C}_t$ and $\mathcal{C}'_t$ whose image under Log_t converges to p is exactly equal to twice the euclidean area of the polygon Δ_p dual to p in the subdivision dual to $C \cup C'$.*

This above number is called the *multiplicity* of the intersection point p of C and C'. It is worth remarking that the number of intersection points which converge to p *only depends on C and C'*, that is to say only on the order at infinity of the coefficients of $P_t(z,w)$ and $P'_t(z,w)$.

Example 2.13. We depicted in Figures 8a and c different mutual positions of a tropical line and a tropical conic. The corresponding dual subdivision of the union of the twi curves is depicted in Figures 8b and d.

In Figure 8a the tropical line intersects the tropical conic in two points of multiplicity 1, and in one point of multiplicity 2 in 8c.

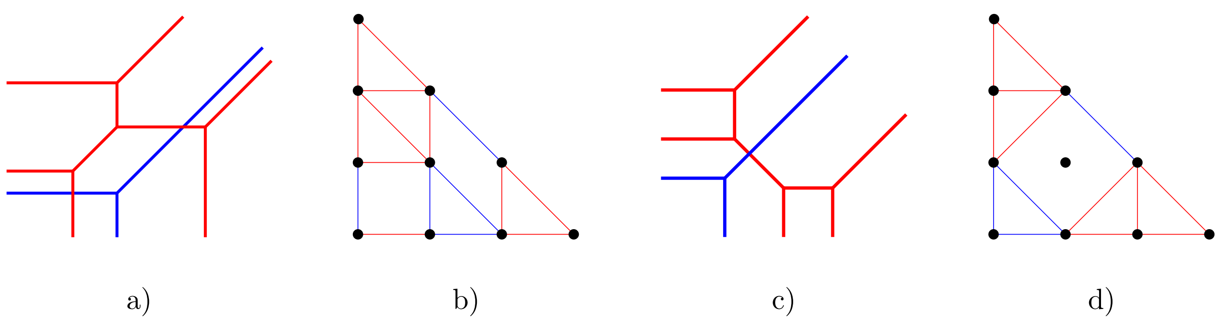


FIGURE 8. Tropical intersections

The combination of Theorem 2.11 and Proposition 2.12 allows one to deduce Bernstein Theorem in classical algebraic geometry from the tropical Bernstein Theorem.

2.5. **Tropical varieties of higher (co)dimension.** We focused so far on plane tropical curves. What about tropical varieties of higher dimension and codimension?

We have seen three equivalent definitions of a tropical curve:

(1) an algebraic one via tropical polynomials;
(2) a combinatorial one via balanced graphs;
(3) a geometric one via limits of amoebas.

All these three definitions can be generalized to arbitrary dimensions.

In the case of tropical hypersurfaces of $\mathbb{R}^n$, all these three definitions remain equivalent, and proofs of all statement given in this section do actually not depend on the dimension of the ambient space.

However these three definitions produce different objects in higher codimension. For example, the definition of a balanced graph we gave above still makes sense for any piecewise linear graph in $\mathbb{R}^n$ with rational slopes. It is known in classical geometry that any cubic curve of genus 1 in $\mathbb{C}P^3$ is contained in a plane. On the other hand, Mikhalkin constructed a tropical cubics of genus 1 in the 3-space which is not contained in any tropical hyperplane[4]: start with the plane tropical cubic C of genus 1 depicted in figure 5a, and draw it in the plane with equation $z = 0$ in $\mathbb{R}^3$; choose a point on each edge unbounded in the direction $(1,1,0)$ in such a way that these three points are not contained in a tropical line in $z = 0$; at those points, replace the unbounded part of C in the direction $(1,1,0)$ by two unbounded edges, one in the direction $(0,0,-1)$ and one in the direction $(1,1,1)$ (see Figure 9). The choice we made of the three points ensure that the resulting tropical

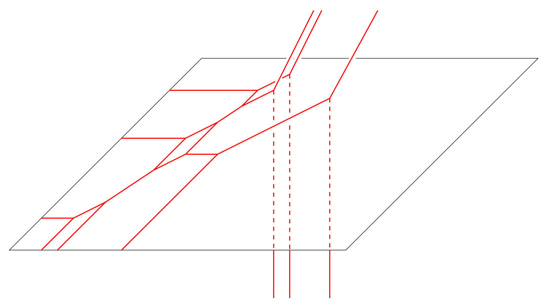


FIGURE 9.

cubic is not contained in any tropical hyperplane. Since it has genus 1, such a pathological tropical curve cannot be a limit of amoebas of any family of spatial complex cubic curves.

The problem of determining which balanced polyhedral complexes are limit of amoebas is very important in tropical geometry, and still widely open.

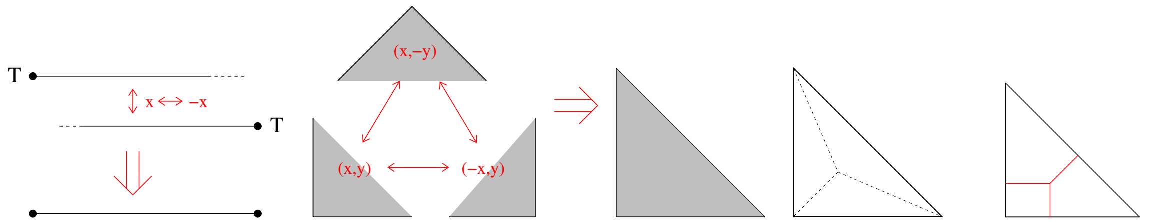
2.6. **Digression: tropical toric varieties.** The logarithm transforms multiplications to additions. As a consequence, any operation performed in complex algebraic geometry using only monomial maps translates mutatis mutandis in the tropical setting. In other words, tropical toric varieties can be constructed exactly as in complex geometry. Let us illustrate this with a classical construction: projective spaces.

The projective line $\mathbb{C}P^1$ may be obtained by taking two copies of $\mathbb{C}$, with coordinates z_1 and z_2, and gluing them along $\mathbb{C}^*$ via the identification $z_2 = z_1^{-1}$. Similarly, the projective plane $\mathbb{C}P^2$ can be constructed by taking three copies of $\mathbb{C}^2$, with coordinates (z_1, w_1), (z_2, w_2), and (z_3, w_3), and gluing them along $(\mathbb{C}^*)^2$ via the identifications

$$(z_2, w_2) = (z_1^{-1}, w_1) \quad \text{and} \quad (z_3, w_3) = (z_1, w_1^{-1}).$$

Since "x^{-1}" $= -x$, the above constructions also yield the projective tropical line $\mathbb{T}P^1$ and plane $\mathbb{T}P^2$. In particular, we see that $\mathbb{T}P^1$ is a segment (Figure 10a), and $\mathbb{T}P^2$ is a triangle (Figure 10b). More generally, the projective space $\mathbb{T}P^n$ is a simplex of dimension n, each of its faces corresponding to a coordinate hyperplane. For example, the tropical 3-space $\mathbb{T}P^3$ is a tetrahedron (see Figure 10c). Note that tropical toric varieties carry much more than just a topological structure: since all gluing

[4]A tropical hyperplane in $\mathbb{R}^3$ is defined by a tropical polynomial of degree 1 in 3 variables.



a) $\mathbb{T}P^1$ b) $\mathbb{T}P^2$ c) $\mathbb{T}P^3$ d) a line in $\mathbb{T}P^2$

FIGURE 10. Tropical projective spaces

maps are classical linear maps with integer coefficients, each open face of dimension p can be identify to $\mathbb{R}^p$ together with the lattice $\mathbb{Z}^p$ inside.

As usual, the space $\mathbb{R}^2 = (\mathbb{T}^*)^2$ embeds naturally into $\mathbb{T}P^2$, and any tropical curve in $\mathbb{R}^2$ has a closure in $\mathbb{T}P^2$. For example, we depicted in Figure 10d the closure in $\mathbb{T}P^2$ of a tropical line in $\mathbb{R}^2$.

2.7. Exercises.

(1) Draw the tropical curves defined by the tropical polynomials $P(x,y) =$ "$5+5x+5y+4xy+1y^2+x^2$" and $Q(x,y) =$ "$7+4x+y+4xy+3y^2+(-3)x^2$", as well as their dual subdivisions.
(2) Show that a tropical curve of degree d has at most d^2 vertices.
(3) Find an equation for each of the tropical curves in Figure 5. The following reminder might be helpful: if v is a vertex of a tropical curve defined by a tropical polynomial $P(x,y)$, then the value of $P(x,y)$ in a neighbourhood of v is given uniquely by the monomials corresponding to the polygon dual to v.
(4) Prove the tropical Bernstein Theorem: let C and C' be two tropical curves such that $C \cap C'$ does not contain any vertex neither of C nor of C'; then the sum of the multiplicity of all intersection points of C and C' is equal to

$$\mathcal{A}(\Delta(C \cup C') - \mathcal{A}(\Delta(C)) - \mathcal{A}(\Delta(C)).$$

Here $\mathcal{A}(\Delta(C))$ is the euclidean area of the Newton polygon of C.

Deduce the classical Bernstein Theorem from its tropical counterpart.
(5) Show that the spatial tropical cubic in Figure 9 constructed in section 2.5 is not contained in any tropical hyperplane.

3. PATCHWORKING

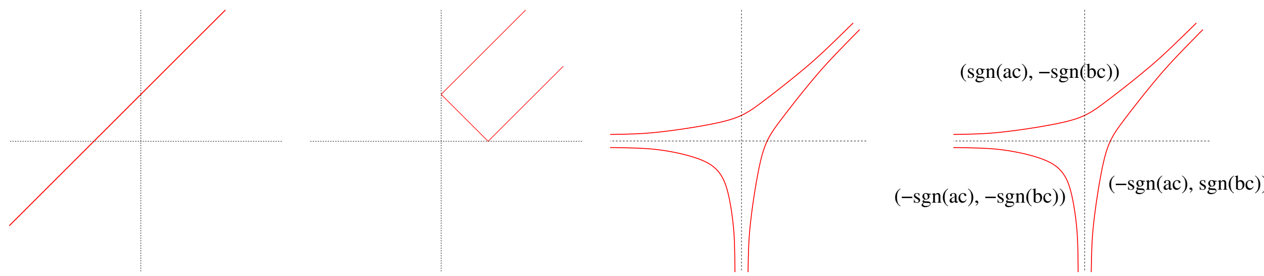
Here I present a first application of the material discussed above to real algebraic geometry. Patchworking technic invented by Oleg Viro at the end of the 70's constitutes one the roots of tropical geometry. At that time tropical geometry did not exist yet, and the original formulation of Patchworking is dual to the presentation I give here. Note that I only discuss combinatorial patchworking in these lectures. Although this is a particular case of general Patchworking Theorem, it turned out to be a powerful tool to construct real algebraic plane curves (and more generally real algebraic hypersurfaces of toric varieties).

The presentation I give here of combinatorial patchworking is in the spirit of Haas' thesis from the end of the 90's. However tropical geometry was still not existing at that moment, and Haas formulated all his statements in a language dual to the one adopted here. Tropical formulation of Patchworking became natural only with the clarification of the relation between amoebas and patchworking given by Viro and Mikhalkin in the early years of 2000 (see [Vir01] and [Mik00]).

3.1. **Patchworking of a line.** Let us start by looking more closely to the amoeba of the real algebraic line $\mathcal{L}$ with equation $az+bw+c=0$ with $a,b,c\in\mathbb{R}^*$. The whole amoeba $\mathcal{A}(\mathcal{L})$ is depicted in Figure 6a, and the amoeba of $\mathbb{R}\mathcal{L}$ is depicted in Figure 11c. Note that $\mathcal{A}(\mathcal{L})$ does not depend on a,b, and c up to a translation in $\mathbb{R}^2$, and that $\partial\mathcal{A}(\mathcal{L})=\mathcal{A}(\mathbb{R}\mathcal{L})$.

We may label each arc of $\mathcal{A}(\mathbb{R}\mathcal{L})$ by the pair of signs corresponding to the quadrant of $(\mathbb{R}^*)^2$ through which the corresponding arc of $\mathcal{L}\cap(\mathbb{R}^*)^2$ passes, see Figure 6d. This labbeling only depends on the signs of a,b, and c. Moreover, if two arcs of $\mathcal{A}(\mathbb{R}\mathcal{L})$ have an asymptotic direction (u,v) in common, then these pairs of signs differ by a factor $((-1)^u,(-1)^v)$.

In conclusion, up to symmetries $(z,w)\mapsto(\pm z,\pm w)$, the position of $\mathbb{R}\mathcal{L}$ in $\mathbb{R}^2$ with respect to the coordinate axis is entirely determined by the asymptotic directions of $\mathcal{A}(\mathcal{L})$, i.e. by $\lim_{t\to\infty}\mathcal{A}_t(\mathcal{L})$!



a) $\mathcal{L}: z-w+1=0$ b) $|\mathbb{R}\mathcal{L}|$ c) $\mathcal{A}(\mathbb{R}\mathcal{L})$ d) $\mathcal{A}(\mathbb{R}\mathcal{L})$ with signs

FIGURE 11. Amoeba of a real line

One can inverse this procedure, i.e. go from $\mathcal{A}(\mathbb{R}\mathcal{L})$ to the position of $\mathbb{R}\mathcal{L}$ in $\mathbb{R}^2$ with respect to the coordinate axis: assign some pair of signs to some arc of $\mathcal{A}(\mathbb{R}\mathcal{L})$ (Figure 12a); together with its asymptotic direction, this determines a pair of signs for the two other arcs of $\mathcal{A}(\mathbb{R}\mathcal{L})$ (Figure 12b); thinking of the plane $\mathbb{R}^2$ where $\mathcal{A}(\mathbb{R}\mathcal{L})$ sits as being the positive quadrant $(\mathbb{R}^*_{>0})^2$ of $\mathbb{R}^2$, draw the symmetric copy of each arc of $\mathcal{A}(\mathbb{R}\mathcal{L})$ in the corresponding quadrant of $\mathbb{R}^2$ (Figure 12c). It is true that this curve is not a honest line, but it still realises the same arrangement as a regular line in $\mathbb{R}^2$ (see Figure 11a)!

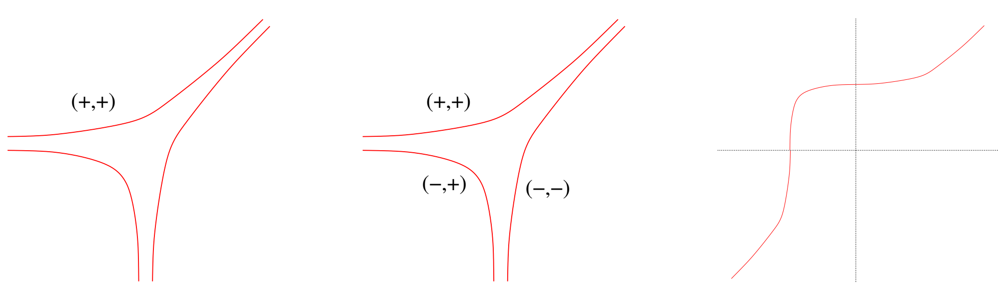


FIGURE 12. Patchworking of a line

3.2. **Patchworking of a non-singular tropical curves.** Viro's patchworking theorem is a generalisation of the previous observation, in the case of an approximation of a *non-singular tropical curve* by a family of amoebas of real algebraic curves.

Definition 3.1. *A tropical curve in* $\mathbb{R}^2$ *is* non-singular *if its dual subdvision only contains triangle of euclidean area* $\frac{1}{2}$.

Equivalently, a tropical curve is non-singular if and only if it has exactly d^2 vertices. Recall that a triangle $\Delta \subset \mathbb{R}^2$ with vertices in $\mathbb{Z}^2$ of euclidean area $\frac{1}{2}$ can be mapped via the composition of a translation and an element of $SL_2(\mathbb{Z})$ to the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$. In other words, an algebraic curve in $(\mathbb{C}^*)^2$ with Newton polygon Δ is nothing else but a line in suitable coordinates. This is the starting observation for what follows.

Let C be a non-singular tropical curve in $\mathbb{R}^2$. Let $(\mathcal{C}_t)$ be a family of real algebraic curves whose amoebas approximate C in the sense of Theorem 2.11. Then one can show that when t is large enough, the following holds:

- $\partial\mathcal{A}_t(\mathcal{C}_t) = \mathcal{A}_t(\mathbb{R}\mathcal{C}_t)$;
- for any vertex v of C, in a small neighborhood U of v the amoeba $\mathcal{A}_t(\mathbb{R}\mathcal{C}_t) \cap U$ is made of 3 arcs as depicted in Figure 13a, corresponding to 3 arcs on $\mathbb{R}\mathcal{C}_t$;
- for each bounded edge e of C adjacent to the vertices v and v', in a small neighborhood U of e the amoeba $\mathcal{A}_t(\mathbb{R}\mathcal{C}_t) \cap U$ is made of 4 arcs, corresponding to 4 arcs on $\mathbb{R}\mathcal{C}_t$, either as depicted in Figure 13b or c; moreover if e has primitive integer direction (u, v), then the two arcs of $\mathcal{A}_t(\mathbb{R}\mathcal{C}_t) \cap U$ converging to e correspond to arcs of $\mathbb{R}\mathcal{C}_t$ contained in quadrants of $\mathbb{R}^2$ whose corresponding pairs of signs differ by a factor $((-1)^u, (-1)^v)$.

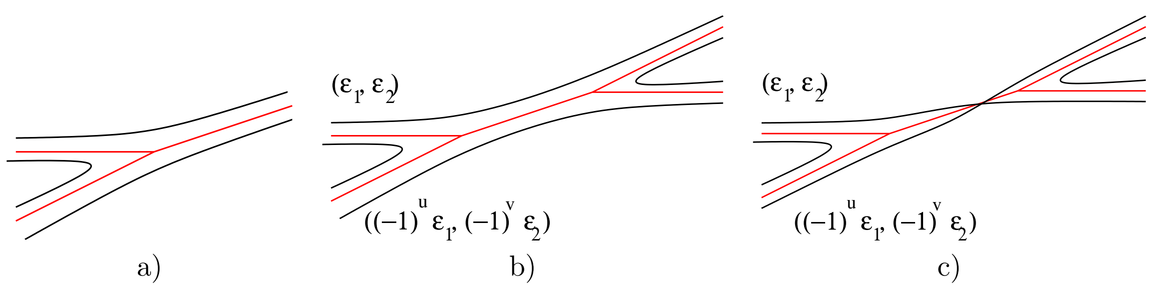


FIGURE 13. $\mathcal{A}_t(\mathbb{R}\mathcal{C}_t) \cap U$ for large t

The above last two properties can be reformulated as follows: the position of $\mathbb{R}\mathcal{C}_t$ in $\mathbb{R}^2$ with respect to the coordinate axis, up to the action of $(\mathbb{Z}/2\mathbb{Z})^2$ by axial symmetries, is entirely determined by the partition of the edges of C in edges depicted in Figure 13b and c.

Definition 3.2. *An edge of* C *as in Figure 13c is said to be* twisted.

It follows from what we just said that knowing all possible distributions of twisted edges arising from families $(\mathcal{C}_t)$ would allow us to construct in return many isotopy type of plane real algebraic curves. It turns out that such distributions are pretty easy to describe.

Definition 3.3. *A* patchworking *of a non-singular tropical curve* C *in* $\mathbb{R}^2$ *is a subset* T *of the set of bounded edges of* C *satisfying the following condition:*

for any cycle γ of C, if $e_1, \ldots, e_k$ are the edges in $\gamma \cap T$, and if (u_i, v_i) is a primitive integer vector of e_i, then

$$\sum_{i=1}^{k} (u_i, v_i) = 0 \quad mod\ 2.$$

Theorem 3.4 (Viro)**.** *For any patchworking T of a non-singular tropical curve C in $\mathbb{R}^2$, there exists an approximating family of real algebraic curves $(\mathcal{C}_t)$ in the sense of Theorem 2.11 whose set of twisted edges is precisely T.*

Remark 3.5. It is also possible to produce explicit families $(\mathcal{C}_t)$ for each Patchworking T.

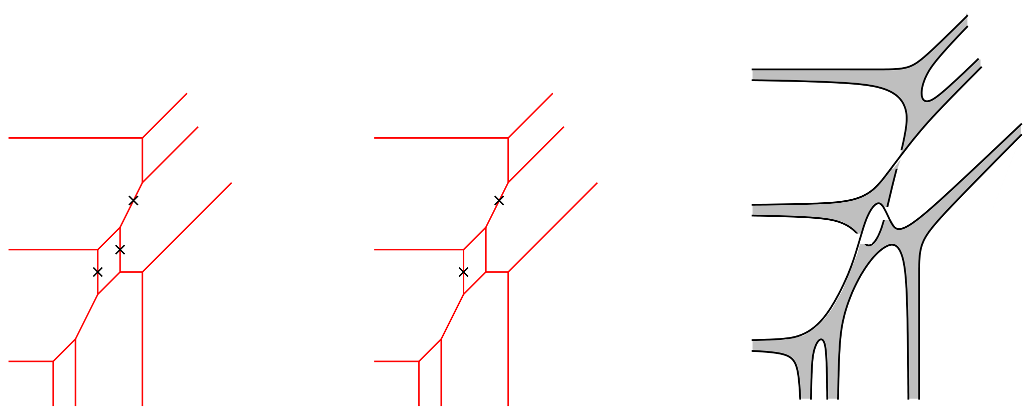
Remark 3.6. One can even go furhter, and determine the surface $\mathcal{C}_t/conj$ for large t, where $conj$ is the restriction on $\mathcal{C}_t$ of the complex conjugation on $(\mathbb{C}^*)^2$. Let's start with a small tubular neighborhood S of C in $\mathbb{R}^2$. For each twisted edge of C, cut S along a fiber at some point of e, perform a half twist and glue this segment back. In other words, one can start with the amoeba $\mathcal{A}_t(\mathcal{C}_t)$ and replace each double point of $\mathcal{A}_t(\mathbb{R}\mathcal{C}_t)$ by a twist.

Once all these twistings are performed, on obtains the surface $\mathcal{C}_t/conj$ for t large enough. For example, the surface $\mathcal{C}_t/conj$ corresponding to the patchwroking depicted in Figure 14a is depicted in Figure 14c (compare with Figure 16b).

Example 3.7. One may choose T to be empty. This construction corresponds to the construction of *simple Harnack curves* via Harnack distribution of signs (see [IV96]).

Example 3.8. Since a non-singular tropical conic is a tree, any set of twisting edges is possible.

Example 3.9. Let us consider the tropical cubic depicted in Figure 5a, and let us choose two subsets T of the set of edges of C (marked by a cross) as depicted in Figure 14a and b. The first one is a patchowrking of C, while the second is not.



a) A patchworking b) Not a patchworking c) The surface $\mathcal{C}_t/conj$

FIGURE 14.

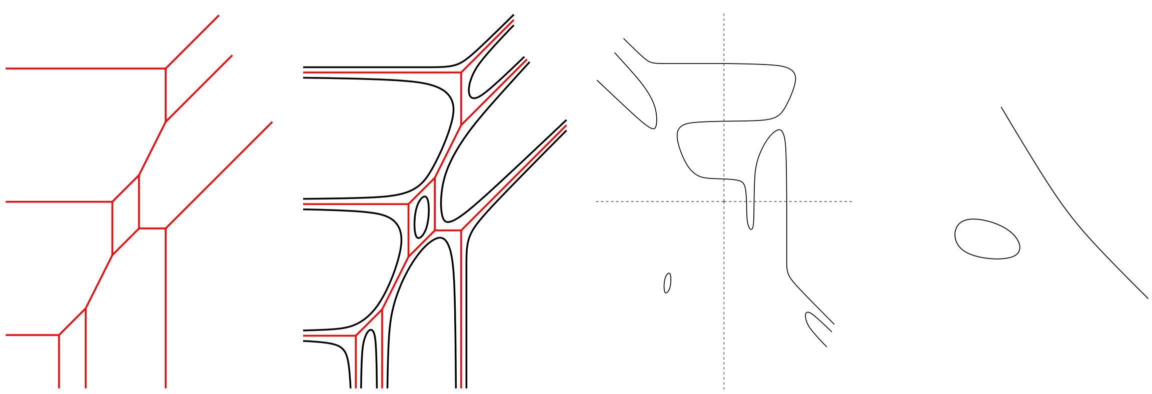
As explained above, one can recover the topology of $\mathbb{R}\mathcal{C}_t$ for large t out of the patchworking this family induces on $C = \lim_{t\to\infty} \mathcal{A}_t(\mathcal{C}_t)$. A patchworking of C will produce a real algebraic curve in the toric surface $Tor(\Delta(C))$ equipped with the real structure coming from $(\mathbb{C}^*)^2$. For simplicity, we assume that the Newton polygon of C is the triangle with vertices $(0,0)$, $(d,0)$, $(0,d)$. In this case $Tor(\Delta(C)) = \mathbb{C}P^2$ and everything becomes simpler to state.

Given T a pathworking of C, perform the following operations:

- at each vertex of C, draw 3 arcs as depicted in Figure 13a;
- at each bounded edge e adjacent to the vertices v and v', join the two corresponding arcs close to v to the corresponding ones for v'; if $e \notin T$, then join these arcs as depicted in Figure 13b; if $e \in T$, then join these arcs as depicted in Figure 13c; denote by $\mathcal{P}$ the obtained curve;
- assign a pair of signs to some arc of $\mathcal{P}$;
- extend this pair of signs to each arc of $\mathcal{P}$ using the following rule: given an edge e with primitive integer direction (u,v), the pairs of signs of the two arcs of $\mathcal{P}$ corresponding to e differ by a factor $((-1)^u, (-1)^v)$;
- thinking of the plane $\mathbb{R}^2$ where $\mathcal{P}$ sits as being the positive quadrant $(\mathbb{R}^*_{>0})^2$ of $\mathbb{R}^2$, draw the symmetric copy of each arc of $\mathcal{P}$ in the corresponding quadrant of $\mathbb{R}^2$.

Then the position of the resulting curve in $\mathbb{R}^2$ with respect to the coordinate axis, up to the action of $(\mathbb{Z}/2\mathbb{Z})^2$ by axial symmetries, is isotopic to $\mathbb{R}\mathcal{C}_t$ for t large enough! We should take a second to realise the depth and elegance of Viro's cominatorial Patchworking Theorem. A tropical curve and its Patchworkings are constructed by following the rules of a purely combinatorial game. It seems like magic to assert that there is a relationship between these combinatorial objects and actual real algebraic curves.

Example 3.10. We depicted the above procedure for several patchworkings in Figures 15, 16, 17, 18. In each case, the last picture is the isotopy type realized by the curve in $\mathbb{R}P^2$.

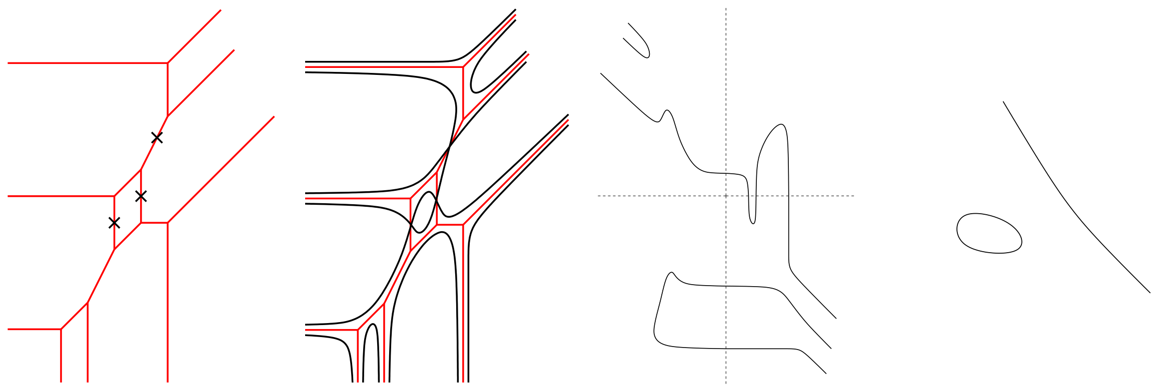


a) b) c) d)

Figure 15. A Harnack cubic

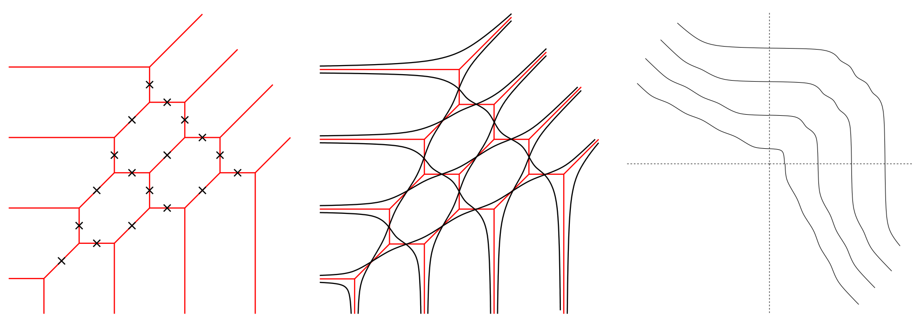
A real algebraic curve of degree 6 which realises the arrangement depicted in Figure 18d was first constructed by Gudkov in the 60's, by using much more complicated techniques. An interesting piece of trivia: Hilbert claimed in 1900 that such a curve could not exist!

3.3. **Haas Theorem.** Recall that a non-singular real algebraic curve $\mathcal{C}$ is said to be of *type I* if $\mathcal{C}/conj$ is orientable, and of *type II* otherwise.



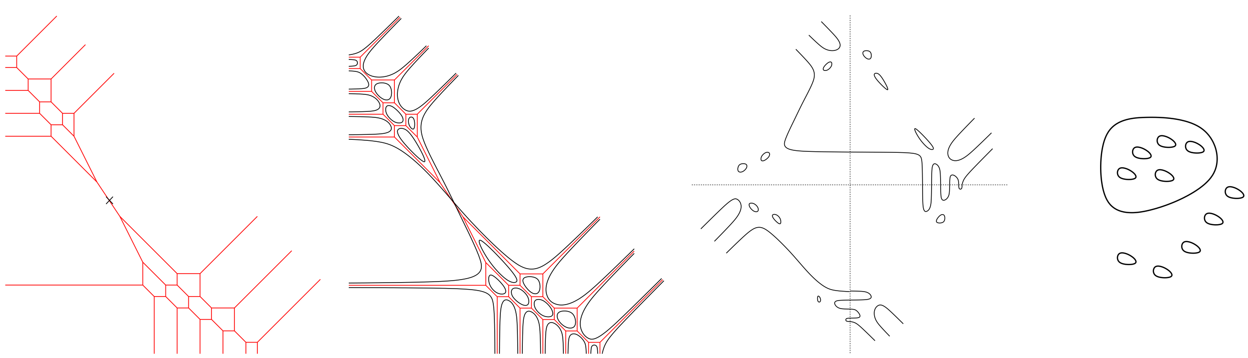
a) b) c) d)

FIGURE 16. Another patchworking of a cubic



a) b) c) d)

FIGURE 17. A hyperbolic quartic



a) b) c) d)

FIGURE 18. Gudkov's sextic

Let $\mathcal{C}$ be a real algebraic curve in $(\mathbb{C}^*)^2$, whose closure $\overline{\mathcal{C}}$ in $Tor(\Delta(\mathcal{C}))$ is non-singular. Recall that $\mathbb{R}\overline{\mathcal{C}}$ has at most $Card(\mathbb{Z}^2 \cap Int(\Delta(\mathcal{C}))) + 1$ connected components. When $\mathbb{R}\mathcal{C}$ has precisely this number of connected component, we say that $\mathbb{R}C$ is an *M-curve*.

There is a very nice criterion, due to Hass in his unpublished thesis [Haa], caracterizing all patchworkings producing M-curves: they are simple gluings of "Harnack type" pieces.

Definition 3.11. *Let C be a non singular tropical curve in $\mathbb{R}^2$, and T a patchworking of C.*

We say that T is of type I *if any cycle of C contains an even number of twisted edges.*

We say that T is maximal *if it is of type I, and if given any edge $e \in T$, either $C \setminus e$ is disconnected or there exists an edge $e' \in T$ suc that $C \setminus (e \cup e')$ is disconnected.*

Theorem 3.12 (Haas)**.** *Let C be a non singular tropical curve in $\mathbb{R}^2$, let T be a patchworking of C, and let $(\mathcal{C}_t)$ be a family of real algebraic curves corresponding to T. Then for t large enough, we have*

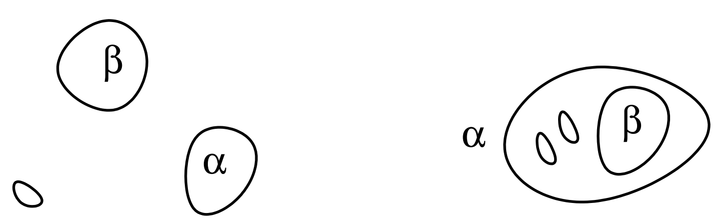
- *$\mathcal{C}_t$ is of type I if and only if T is of type I.*
- *$\mathcal{C}_t$ is an M-curve if and only if T is maximal.*

Example 3.13. One can verify Haas' Theorem in the construction performed above.

Example 3.14. Any patchworking T containing a bounded edge which is not adjacent to an unbounded connected component of $\mathbb{R}^2 \setminus C$ is not maximal (see Figure 17).

3.4. **Exercises.**

(1) Show that the first Betti number of a non-singular tropical curve in $\mathbb{R}^2$ is equal to the number of integer points contained in the interior of its Newton polygon.
(2) Let C be non-singular tropical curve, and let $(\mathcal{C}_t)$ be a family of real algebraic curves corresponding to the patchworking $T = \emptyset$. Prove that the isotopy type of $\mathcal{C}_t$ in $\mathbb{R}^2$ with respect to the coordinate axis, up to the action of $(\mathbb{Z}/2\mathbb{Z})^2$ by axial symmetries, only depends on $\Delta(C)$.
(3) Show that any patchworking of the tropical cubic depicted in Figure 5a will produce an M-cubic. Find a patchworkink of another tropical cubic which produces a real cubic whose real part is connected.
(4) Construct all isotopy types of M-curves in $\mathbb{R}P^2$ up to degree 6.
(5) It remains 6 isotopy types in $\mathbb{R}P^2$, depicted in Figure 19, whose realizability by a real algebraic curve of degree 8 are still open. Try to construct M-curves of degree 8 realizing those isotopy types.



$\alpha = 1, 4, 7, 9$ and $\alpha + \beta = 19$ $\qquad$ $\alpha = 4, 14$ and $\alpha + \beta = 18$

FIGURE 19. Isotopy types whose realizability by an M-curve of degree 8 is unknown (each number stands for a set of this cardinality of ovals, all of them lying outside each other)

4. Abstract tropical curves

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