RECURSIVE FORMULAS FOR WELSCHINGER INVARIANTS OF THE PROJECTIVE PLANE

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Abstract. Welschinger invariants of the real projective plane can be computed via the enumeration of enriched graphs, called marked floor diagrams. By a purely combinatorial study of these objects, we establish a Caporaso-Harris type formula which allows one to compute Welschinger invariants for configurations of points with any number of complex conjugated points.

1. Introduction

Welschinger invariants of symplectic 4-manifolds were introduced in [Wel03] (see also [Wel05a]), and since then they are attracting a lot of attention. One of the interests of these invariants is that they provide lower bounds in real enumerative geometry. As an example of application, it is proved in [IKS03] that through any configuration of $3d-1$ points in $\mathbb{R}P^2$ passes at least one real rational algebraic curve of degree d (and even a lot! see also [IKS04]).

Welschinger invariants can be seen as real analogs of genus 0 *Gromov-Witten invariants*, well known in complex enumerative geometry. Hence, studying relations between these two sequences of invariants is an important problem. Both invariants can be computed via the same kind of recursive formulas. In [KM94], Kontsevich deduced a recursive formula for all genus 0 Gromov-Witten invariants of $\mathbb{C}P^2$ in terms of those of lower degree from the so called *WDVV equation*. Recently, Solomon [Sol] designated suitable analogues of WDVV equation in *open Gromov-Witten theory* and obtained similar formulas involving Welschinger invariants. Another approach to compute Gromov-Witten invariants was proposed by Caporaso and Harris in [CH98]. They obtained a recursive formula by specializing one point after the other to lie on a given line in $\mathbb{C}P^2$. Moreover, this formula computes not only genus 0 Gromov-Witten invariants but *relative Gromov-Witten invariants* of any genus and degree. In this paper, we are interested in a similar formula involving Welschinger invariants.

Thanks to Mikhalkin's Correspondence Theorem (see [Mik05]), *tropical geometry* turned out to be a powerful tool to solve a lot of enumerative problems (see for example [Mik05], [Shu06],[IKS03], [BMb], [BM07]). In particular, it provides a quite combinatorial and very practical way to compute Gromov-Witten and Welschinger invariants.

In [GM07], Gathmann and Markwig gave a tropical version and a tropical proof of Caporaso-Harris formula. Then, Itenberg, Kharlamov and Shustin adapted in [IKS09] this tropical approach to a real setting and produced a Caporaso-Harris like recursive formula for Welschinger invariants in the totally real case. Their formula involves not only Welschinger invariants, but also other numbers which reveal to be tropical relative Welschinger invariants. This was quite surprising since in a non-tropical setting no relative Welschinger invariants are known.

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An even more combinatorial way to compute Gromov-Witten and Welschinger invariants has been proposed in [BMb] and [BM07], where tropical curves are replaced by *floor diagrams*. Gathmann-Markwig tropical formula in the complex case and Itenberg-Kharlamov-Shustin's one in the totally real case are easily translated into the language of floor diagrams. Moreover, the quite simple combinatoric of these objects allows one to generalize easily the formula in [IKS09] to Welschinger invariants for collections of points mixing real points and pairs of complex conjugated ones.

In this paper, we explain how to pass from marked floor diagrams to a Caporaso-Harris style formula which allows one to compute all Welschinger invariants of the projective plane. This work is based on [BMb, Theorem 4.16], which in its turn is based on the two Correspondence Theorems by Mikhalkin ([Mik05]) and Shustin ([Shu06]).

Our formula does not involve directly Welschinger invariants, but some other numbers, the sum of which gives Welschinger invariants. In regard to tropical relative Welschinger invariants for configurations of real points, it is natural to wonder about the invariance of the numbers involved in our formula. However, one sees easily that they do not correspond to any invariant, even tropical. Worse, we give an example in section 7.2 which shows that there is no hope to define as in [IKS09] tropical relative Welschinger invariants for real configurations of points containing some pairs of complex conjugated points.

Another approach for computing Welschinger invariants, based on the symplectic field theory, has been proposed by Welschinger in [Wel]. There he derived closed graph-combinatorial formulas in certain cases, in particular for the real projective plane and real configurations consisting of pairs of complex conjugated points together with up to two real points.

The structure of the paper is the following: In section 2 we fix notation and convention used thereafter. In section 3 we define Gromov-Witten and Welschinger invariants, and in section 4 we explain how to compute these enumerative invariants using floor diagrams. Section 5 is devoted to the statement of our formula, and the proof is given in section 6. Some final remarks and computations are given in the last section.

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2. Convention

We fix here notation and convention we use in this paper.

2.1. **Notation.** We denote by $\mathbb{N}$ the set of nonnegative integer numbers and by $\mathbb{N}^*$ the set of positive integer numbers, i.e. $\mathbb{N} = \{n \in \mathbb{Z} | \ n \geq 0\}$ and $\mathbb{N}^* = \{n \in \mathbb{Z} | \ n > 0\}$. The set of sequences of elements in $\mathbb{N}$ having only finitely many non zero terms is denoted by $\mathbb{N}^\infty$. We denote by e_i the vector in $\mathbb{N}^\infty$ whose all coordinates are 0 but the i^{th} which is equal to 1. If α is a vector in $\mathbb{N}^\infty$, we denote by $(\alpha)_i$ its i^{th} coordinate. Given two vectors α and β in $\mathbb{N}^\infty$, we write $\alpha \geq \beta$ if $(\alpha)_i \geq (\beta)_i$ for all i. Finally, we put

$$|\alpha| = \sum_{i=1}^{\infty} (\alpha)_i, \qquad I\alpha = \sum_{i=1}^{\infty} i(\alpha)_i, \qquad I^\alpha = \prod_{i=1}^{\infty} i^{(\alpha)_i}$$

If a and b are two integer numbers, $\left(\begin{array}{c} a \\ b \end{array}\right)$ denotes the binomial coefficient (i.e. is equal to $\frac{a!}{b!(a-b)!}$ if $0 \le b \le a$ and to 0 otherwise). If a and $b_1, b_2, \ldots, b_k$ are integer numbers then $\left(\begin{array}{c} a \\ b_1, \ldots, b_k \end{array}\right)$ denotes the multinomial coefficient, i.e.

$$\left(\begin{array}{c} a \\ b_1, \ldots, b_k \end{array}\right) = \prod_{i=1}^{k} \left(\begin{array}{c} a - \sum_{j=1}^{i-1} b_j \\ b_i \end{array}\right)$$

If $\alpha, \alpha_1, \ldots, \alpha_l$ are vectors in $\mathbb{N}^\infty$, then we put

$$\left(\begin{array}{c} \alpha \\ \alpha_1, \ldots, \alpha_l \end{array}\right) = \prod_{i=1}^{\infty} \left(\begin{array}{c} (\alpha)_i \\ (\alpha_1)_i, \ldots, (\alpha_l)_i \end{array}\right)$$

2.2. **Graphs.** An *oriented graph* Γ is a pair $(\overline{\text{Vert}}(\Gamma), \text{Edge}(\Gamma))$ of two finite sets $\overline{\text{Vert}}(\Gamma)$ and $\text{Edge}(\Gamma)$, where $\text{Edge}(\Gamma)$ is a list of elements of $\overline{\text{Vert}}(\Gamma) \times \overline{\text{Vert}}(\Gamma)$. Note that $\text{Edge}(\Gamma)$ might contain repeated elements. An element of $\overline{\text{Vert}}(\Gamma)$ is called a *vertex* of Γ. An element (v_1, v_2) of $\text{Edge}(\Gamma)$ is called an *edge* of Γ and is said to be oriented from v_1 to v_2. A vertex v and an edge e are said to be *adjacent* if $e = (v, v')$ or (v', v). A vertex of Γ such that all its adjacent edges are outgoing is called a *source*. We denote by $\text{Vert}^\infty(\Gamma)$ the set of sources of Γ, and we put $\text{Vert}(\Gamma) = \overline{\text{Vert}}(\Gamma) \setminus \text{Vert}^\infty(\Gamma)$. We also denote by $\text{Edge}^\infty(\Gamma)$ the set of edges adjacent to a source. A graph Γ is *connected* if for any two elements v_1 and v_2 of $\overline{\text{Vert}}(\Gamma)$, there exists a sequence $s_1 = v_1$, s_2, ..., s_{k-1}, $s_k = v_2$ of elements of $\overline{\text{Vert}}(\Gamma)$ such that $(s_i, s_{i+1}) \in \text{Edge}(\Gamma)$ or $(s_{i+1}, s_i) \in \text{Edge}(\Gamma)$ for all i. A *tree* is a connected graph with Euler characteristic equal to 1 (i.e. $\sharp\overline{\text{Vert}}(\Gamma) - \sharp\text{Edge}(\Gamma) = 1$). In this case, $\text{Edge}(\Gamma)$ is actually a subset of $\overline{\text{Vert}}(\Gamma) \times \overline{\text{Vert}}(\Gamma)$.

An oriented tree Γ is naturally enhanced with a partial ordering : an element (i.e. an edge or a vertex of Γ) a of Γ is greater than another element b if there exists a sequence $c_1 = a$, c_2, ..., c_{k-1}, $c_k = b$ of elements of Γ such that c_i is adjacent to c_{i+1} for all i, and if c_i (resp. c_{i+1}) is an edge of Γ then $c_i = (v, c_{i+1})$ (resp. $c_{i+1} = (c_i, v)$).

A *weighted graph* is a graph Γ enhanced with a function $\omega : \text{Edge}(\Gamma) \to \mathbb{N}^*$. The integer $\omega(e)$ is called the *weight* of the edge e. The weight allows one to define the *divergence* at the vertices. Namely, for a vertex $v \in \overline{\text{Vert}}(\Gamma)$ we define the divergence $\text{div}(v)$ to be the sum of the weights of all incoming edges minus the sum of the weights of all outgoing edges.

3. Complex and real enumerative problems

3.1. **Relative Gromov Witten invariants.** Fix $d \ge 1$ an integer number, $\omega = \{p_1, \ldots, p_{3d-1}\}$ a generic collection of $3d-1$ points in $\mathbb{C}P^2$ and denote by $\mathcal{C}(\omega)$ the set of all irreducible complex rational curves of degree d in $\mathbb{C}P^2$ containing ω. It is well known that the cardinal of $\mathcal{C}(\omega)$ does not depend on ω, and it is called the genus 0 *Gromov-Witten invariant* of degree d of $\mathbb{C}P^2$.

More generally, one can define *relative Gromov-Witten invariants.* Fix a line L in $\mathbb{C}P^2$, a degree $d \ge 1$ and two vectors α and β in $\mathbb{N}^\infty$ such that $I\alpha + I\beta = d$. Choose $\omega = \{p_1, \ldots, p_{2d-1+|\beta|}\}$ a collection of $2d-1+|\beta|$ points in $\mathbb{C}P^2$, and $\omega_L = \{p_1^1, \ldots, p^1_{(\alpha)_1}, p_1^2, \ldots, p^2_{(\alpha)_2}, \ldots, p_1^k, \ldots, p^k_{(\alpha)_k}, \ldots\}$ a collection of $|\alpha|$ points on L. Denote by $\mathcal{C}(\omega, \alpha, \beta)$ the set of all irreducible complex rational curves of degree d in $\mathbb{C}P^2$ containing ω, having no singular point on L, intersecting the line L at the point $p_i^j \in \omega_L$ with multiplicity j for all i and j, and intersecting L at $(\beta)_j$ additional points with multiplicity j for all j. Once again, the cardinal of $\mathcal{C}(\omega, \alpha, \beta)$ does not depend on ω and ω_L as long as these configurations are generic, and is denoted by $N^{\alpha,\beta}(d)$. The number $N^{0,de_1}(d)$ is the genus 0 Gromov-Witten invariant of degree d.

Note that one can define relative Gromov-Witten invariants of any genus, and that the Caporaso-Harris formula computes all these numbers. However, we are only interested in rational curves in this paper, and the formula we write in section 5.1 only deals with rational curves.

3.2. **Welschinger invariants.** Let us now consider real algebraic curves. Fix an integer number $d \geq 1$, and a generic collection $\omega = \{p_1, \ldots, p_{3d-1}\}$ of $3d-1$ points in $\mathbb{C}P^2$. Suppose that ω is real, i.e. $conj(\omega) = \omega$ where $conj$ is the standard complex conjugation on $\mathbb{C}P^2$. Consider the set $\mathbb{R}\mathcal{C}(\omega)$ of all irreducible real rational curves of degree d in $\mathbb{C}P^2$ containing ω. The cardinal of $\mathbb{R}\mathcal{C}(\omega)$ is no longer invariant with respect to ω, but Welschinger proved in [Wel05a] that if one counts the curves in $\mathbb{R}\mathcal{C}(\omega)$ with an appropriate sign, then one obtains an invariant.

If C is a real algebraic nodal curve in $\mathbb{C}P^2$, let us denote by $w(C)$ the number of isolated double points of C in $\mathbb{R}P^2$ (i.e. points where C has local equation $X^2 + Y^2 = 0$).

Theorem 3.1 (Welschinger). *The number*

$$\sum_{C \in \mathbb{R}\mathcal{C}(\omega)} (-1)^{w(C)}$$

only depends on the degree d and the number of pairs of complex conjugated points in ω.

These numbers are called *Welschinger invariants*, and we denoted them by $W_2(d, r)$ where r is the number of pairs of complex conjugated points in ω.

Of course, one can count with Welschinger signs real algebraic curves of any genus or with tangency conditions. However the number obtained is no longer an invariant (see [Wel05a] or [IKS09]). See section 7.2 for this discussion in the tropical setting.

4. Floor diagrams

Here we define floor diagrams and state their relation with Gromov-Witten and Welschinger invariants of $\mathbb{C}P^2$. The definitions of this paper differ slightly from those of [BMb] and [BM07]. The first reason is that here we are mainly interested in enumeration of plane curves, for which the floor diagrams we have to consider are much simpler than those required in arbitrary dimension. On the other hand, the floor diagrams which are needed to compute absolute invariants are not sufficient to deal with relative invariants, we have to allow edges in $\text{Edge}^\infty(\mathcal{D})$ with any positive weight.

Definition 4.1. *A connected weighted oriented tree $\mathcal{D}$ is called a floor diagram of genus 0 and degree d if the following conditions hold*

- *for any $v \in Vert(\mathcal{D})$, one has $div(v) = 1$,*
- *each source has a unique adjacent edge,*
- *one has $\sum_{v \in Vert^\infty(\mathcal{D})} div(v) = -d$.*

Note that Definition 4.1 implies that the set $\text{Vert}(\mathcal{D})$ has exactly d elements.

Here are the convention we use to depict floor diagrams : elements of $\text{Vert}^\infty(\mathcal{D})$ are represented by small horizontal segments, elements of $\text{Vert}(\mathcal{D})$ are represented by ellipses. Elements of $\text{Edge}(\mathcal{D})$ are represented by vertical lines, and the orientation is implicitly from down to up. The weight of an edge is indicated only when it is at least 2. All floor diagrams of degree 3 and genus 0 are depicted in Figure 1.

Let $\alpha, \beta, \gamma, \delta$ be four vectors in $\mathbb{N}^\infty$, and define $n = 2d - 1 + |\alpha + \beta + 2\gamma + 2\delta|$ and $\mathcal{P} = \{1, \ldots, n\}$. Let $\mathfrak{M} : \mathcal{P} \to \mathcal{D}$ be a map.

Definition 4.2. *The map $\mathfrak{M}$ is called a marking of $\mathcal{D}$ of type $(\alpha, \beta, \gamma, \delta)$ if the following conditions hold*

- *the floor diagram $\mathcal{D}$ is of genus 0 and of degree $I\alpha + I\beta + 2I\gamma + 2I\delta$,*

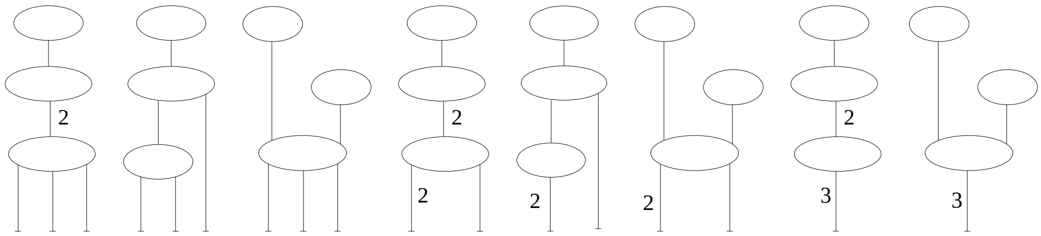


FIGURE 1. Floor diagrams of degree 3 and genus 0

- *the map* $\mathfrak{M}$ *is injective,*
- *if* $\mathfrak{M}(i) > \mathfrak{M}(j)$*, then* $i > j$*,*
- *if* $\sum_{j=1}^{k-1}(\alpha)_j + 1 \leq i \leq \sum_{j=1}^{k}(\alpha)_j$ *or* $|\alpha| + 2\sum_{j=1}^{k-1}(\gamma)_j + 1 \leq i \leq |\alpha| + 2\sum_{j=1}^{k}(\gamma)_j$*, then* $\mathfrak{M}(i)$ *is a source with divergence* $-k$*,*
- *for any* $k \geq 1$*, there are exactly* $(\beta)_k + 2(\delta)_k$ *elements of* $Edge^\infty(\mathcal{D})$ *with weight* k *in the image of* $\mathfrak{M}$*,*
- *for any source* v *of* $\mathcal{D}$ *adjacent to the edge* e*, exactly one of the two elements* v *or* e *is in the image of* $\mathfrak{M}$*.*

A floor diagram enhanced with a marking of type $(\alpha, \beta, \gamma, \delta)$ is called a *marked floor diagram of type* $(\alpha, \beta, \gamma, \delta)$ and is said to be marked by $\mathfrak{M}$. Two marked floor diagrams are called *equivalent* if they can be identified by a homeomorphism of oriented graphs.

A simple Euler characteristic computation shows that if $\mathfrak{M}$ is a marking of $\mathcal{D}$, then any vertex in $\text{Vert}(\mathcal{D})$ and any edge in $\text{Edge}(\mathcal{D}) \setminus \text{Edge}^\infty(\mathcal{D})$ is in the image of $\mathfrak{M}$. To any marked floor diagram, we assign a sequence of nonnegative numbers called *multiplicities* : a *complex* multiplicity, and some r*-real* multiplicities.

Definition 4.3. *The complex multiplicity of a marked floor diagram* $\mathcal{D}$ *of type* $(\alpha, \beta, \gamma, \delta)$*, denoted by* $\mu^{\mathbb{C}}(\mathcal{D})$*, is defined as*

$$\mu^{\mathbb{C}}(\mathcal{D}) = I^{-2\alpha-\beta-4\gamma-2\delta} \prod_{e \in Edge(\mathcal{D})} w(e)^2$$

Note that the complex multiplicity of a marked floor diagram depends only on the underlying floor diagram. Our formula mixes real and complex invariants (see section 5.2), hence we will need the following theorem in section 6. This theorem is proved in [BMb] (see also [BM07]) in the case $\alpha = 0$ and $\beta = de_1$. However, it is straightforward that the same proof works for any couple (α, β) (see also [GM07]).

Theorem 4.4 (Brugallé, Mikhalkin)**.** *For any* α *and* β *in* $\mathbb{N}^\infty$*, and* $d = I\alpha + I\beta$*, one has*

$$N^{\alpha,\beta}(d) = \sum \mu^{\mathbb{C}}(\mathcal{D})$$

where the sum is taken over all marked floor diagrams of degree d*, genus* 0 *and of type* $(\alpha, \beta, 0, 0)$*.*

Remark 4.5. Taking the sum over all marked floor diagrams of degree d, genus 0 and of type $(\alpha, \beta, \gamma, \delta)$, one obtains the number $N^{\alpha+2\gamma,\beta+2\delta}(d)$.

Let us fix $r \geq 0$ such that $2d - 1 + |\beta + 2\delta| - 2r \geq 0$, and $\mathcal{D}$ a floor diagram of type $(\alpha, \beta, \gamma, \delta)$ marked by $\mathfrak{M}$.

The set $\{i, i+1\}$ is a called r*-pair* if $i = |\alpha| + 2k - 1$ with $1 \leq k \leq |\gamma|$ or $i = n - 2k + 1$ with $1 \leq k \leq r$. Denote by $\Im(\mathcal{D}, \mathfrak{M}, r)$ the union of all the sets $\{\mathfrak{M}(i), \mathfrak{M}(i+1)\}$ where $\mathfrak{M}(i)$ is not adjacent to $\mathfrak{M}(i+1)$ and $\{i, i+1\}$ is an r-pair. Let $\rho_{\mathcal{D},\mathfrak{M},r} : \mathcal{D} \to \mathcal{D}$ be the bijection defined by $\rho_{\mathcal{D},\mathfrak{M},r}(a) = a$ if $a \in \mathcal{D} \setminus \Im(\mathcal{D},\mathfrak{M},r)$, and by $\rho_{\mathcal{D},\mathfrak{M},r}(\mathfrak{M}(i)) = \mathfrak{M}(j)$ if $\{i,j\}$ is an r-pair and $\{\mathfrak{M}(i),\mathfrak{M}(j)\} \subset \Im(\mathcal{D},\mathfrak{M},r)$. Note that $\rho_{\mathcal{D},\mathfrak{M},r}$ is an involution.

Definition 4.6. *A marked floor diagram $\mathcal{D}$ of type $(\alpha,\beta,\gamma,\delta)$ is called r-real if $(\mathcal{D},\mathfrak{M})$ and $(\mathcal{D},\rho_{\mathcal{D},\mathfrak{M},r}\circ\mathfrak{M})$ are equivalent, and if exactly $2(\delta)_k$ edges of weight k are in $Edge^\infty(\mathcal{D})\cap\Im(\mathcal{D},\mathfrak{M},r)$ for any $k\geq 1$.*

The r-real multiplicity of a marked floor diagram of type $(\alpha,\beta,\gamma,\delta)$, denoted by $\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})$, is defined as

$$\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) = (-1)^{\frac{\sharp(Vert(\mathcal{D})\cap\Im(\mathcal{D},\mathfrak{M},r))}{2}} I^{-\delta} \prod_{e\in Edge(\mathcal{D})\setminus\mathfrak{M}(\{1,\dots,n-2r\})} w(e)$$

if $(\mathcal{D},\mathfrak{M})$ is an r-real marked floor diagram with all edges of even weight in $\Im(\mathcal{D},\mathfrak{M},r)$, and as

$$\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) = 0$$

otherwise.

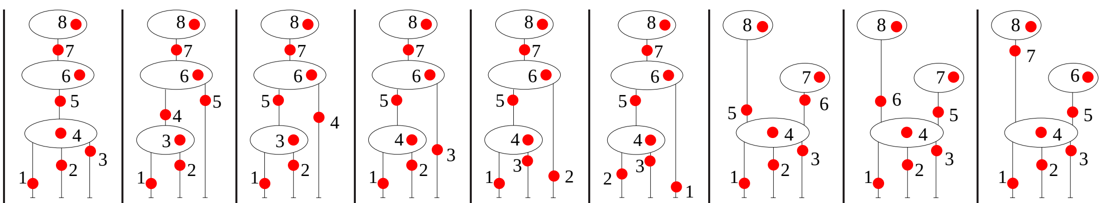
Note that as soon as $r\geq 1$, the r-real multiplicity of an r-real marked floor diagram depends not only on the underlying floor diagram but also on the marking. The following theorem is proved in [BMb].

Theorem 4.7 (Brugallé, Mikhalkin)**.** *For any $d\geq 1$ and any $r\geq 0$ such that $3d-1-2r\geq 0$, one has*

$$W_2(d,r) = \sum \mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})$$

where the sum is taken over all r-real marked floor diagrams of degree d, genus 0 and of type $(0,(d-2i)e_1,0,ie_1)$ with $0\leq i\leq \frac{d}{2}$.

Example 4.8. All marked floor diagrams of degree 3, genus 0 and type $(0,(3-2i)e_1,0,ie_1)$ are depicted in Table 1 with their multiplicities. The first floor diagram has an edge of weight 2, but we didn't mention it in the picture to avoid confusion. According to Theorems 4.4 and 4.7 one finds $N^{0,3e_1}(3) = 12$ and $W_2(3,r) = 8-2r$.



$\mu^{\mathbb{C}}$	4	1	1	1	1	1	1	1	1
$\mu_0^{\mathbb{R}}$	0	1	1	1	1	1	1	1	1
$\mu_1^{\mathbb{R}}$	0	1	1	1	1	1	0	0	1
$\mu_2^{\mathbb{R}}$	0	1	1	1	1	1	-1	-1	1
$\mu_3^{\mathbb{R}}$	0	1	0	0	1	1	-1	-1	1
$\mu_4^{\mathbb{R}}$	0	1	0	0	0	0	-1	-1	1

TABLE 1. Marked floor diagrams of degree 3 and genus 0

5. Recursive formulas

5.1. **Caporaso-Harris formula.** As a warm up for Theorem 5.2, we first remind the Caporaso-Harris formula. Note that the equivalence relation $\sim_s$ will be used in section 5.2.

Given two integer numbers $l \geq 0$ and $d \geq 0$ and two vectors α and β in $\mathbb{N}^\infty$, we denote by $\mathcal{S}(d,l,\alpha,\beta)$ the set composed by the vectors $(d_1,\ldots,d_l,\ k_1,\ldots,k_l,\ \alpha_1,\ldots,\alpha_l,\ \beta_1,\ldots,\beta_l) \in (\mathbb{N}^*)^{2l} \times (\mathbb{N}^\infty)^{2l}$ satisfying

- $\forall i, (d_i,k_i,\alpha_i,\beta_i) \leq (d_{i+1},k_{i+1},\alpha_{i+1},\beta_{i+1})$ for the lexicographic order,
- $\sum d_i = d-1$,
- $\sum \alpha_i \leq \alpha$,
- $\forall i, \beta_i \geq e_{k_i}$,
- $\sum(\beta_i - e_{k_i}) = \beta$,
- $\forall i, I\alpha_i + I\beta_i = d_i$.

To any element $s = (d_1,\ldots,d_l,k_1,\ldots,k_l,\alpha_1,\ldots,\alpha_l,\beta_1,\ldots,\beta_l)$ of $\mathcal{S}(d,l,\alpha,\beta)$, we associate the equivalence relation $\sim_s$ on the set $\{1,\ldots,l\}$ defined by

$$i \sim_s j \Leftrightarrow (d_i,k_i,\alpha_i,\beta_i) = (d_j,k_j,\alpha_j,\beta_j)$$

For each of the equivalent classes of $\sim_s$, evaluate the factorial of its cardinal, and denote by $\sigma(s)$ the product of these factorials.

Theorem 5.1 (Caporaso, Harris)**.** *The numbers $N^{\alpha,\beta}(d)$ are given by the initial value $N^{e_1,0}(1) = 1$ and the relation*

$$\begin{aligned} N^{\alpha,\beta}(d) \;=\; & \sum_{k|\beta\geq e_k} kN^{\alpha+e_k,\beta-e_k}(d) \;+ \\ & \sum_{\substack{l\geq 0 \\ s\in\mathcal{S}(d,l,\alpha,\beta)}} \left[\frac{1}{\sigma(s)} \binom{2d-2+|\beta|}{2d_1-1+|\beta_1|,\ldots,2d_l-1+|\beta_l|} \right. \\ & \left. \binom{\alpha}{\alpha_1,\ldots,\alpha_l} \prod_{i=1}^l (\beta_i)_{k_i} k_i N^{\alpha_i,\beta_i}(d_i)\right] \end{aligned}$$

5.2. **Formula for plane Welschinger invariants.** We state now the main result of this paper, a recursive formula in the spirit of Theorem 5.1 which allows one to compute the numbers $W_2(d,r)$. Let α, β, γ and δ be four vectors in $\mathbb{N}^\infty$, define $d = I\alpha + I\beta + 2I\gamma + 2I\delta$, and choose an integer number $r \geq 0$ such that $2d - 1 + |\beta + 2\delta| - 2r \geq 0$. Our recursive formula does not involve the numbers $W_2(d,r)$, but the numbers $C^{\alpha,\beta,\gamma,\delta}(d,r)$ which are defined as follows

$$C^{\alpha,\beta,\gamma,\delta}(d,r) = \sum \mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})$$

where the sum is taken over all marked floor diagrams of type $(\alpha,\beta,\gamma,\delta)$. Note that only r-real marked floor diagrams contribute to this sum. For any $d \geq 1$ and $0 \leq r \leq \frac{3d-1}{2}$, Theorem 4.7 states that

$$W_2(d,r) = \sum_{i=0}^{\frac{d}{2}} C^{0,(d-2i)e_1,0,ie_1}(d,r)$$

A vector α in $\mathbb{N}^\infty$ is said to be *odd* if $(\alpha)_{2i} = 0$ for all $i \geq 1$. It follows immediately from the definition of the multiplicity of a marked floor diagram that $C^{\alpha,\beta,\gamma,\delta}(d,r) = 0$ if α or β is not odd.

Given three integer numbers $l \geq 0$, $m \geq 0$ and $r \geq 0$ and four vectors α, β, γ and δ in $\mathbb{N}^\infty$, we denote by $\mathcal{S}_w(l,m,r,\alpha,\beta,\gamma,\delta)$ the set composed by the vectors $(d_1,\ldots,d_l,k_1,\ldots,k_l,\gamma_1,\ldots,\gamma_l,$

$\delta_1,\dots,\delta_l,d'_1,\dots,d'_m,k'_1,\dots,k'_m,r'_1,\dots,r'_m,\alpha'_1,\dots,\alpha'_m,\beta'_1,\dots,\beta'_m,\gamma'_1,\dots,\gamma'_m,\delta'_1,\dots,\delta'_m)$ in $(\mathbb{N}^*)^{2l}\times(\mathbb{N}^\infty)^{2l}\times(\mathbb{N}^*)^{2m}\times\mathbb{N}^m\times(\mathbb{N}^\infty)^{4m}$ satisfying

- $\forall i,(d'_i,k'_i,r'_i,\alpha'_i,\beta'_i,\gamma'_i,\delta'_i)\leq(d'_{i+1},k'_{i+1},r'_{i+1},,\alpha'_{i+1},\beta'_{i+1},\gamma'_{i+1},\delta'_{i+1})$ for the lexicographic order,
- $\sum\alpha'_i\leq\alpha$,
- $\forall i,k'_i$ is odd,
- $\forall i,\beta'_i\geq e_{k'_i}$,
- $\sum(\beta'_i-e_{k'_i})=\beta$,
- $\sum\gamma'_i\leq\gamma$,
- $\sum\delta'_i\leq\delta$,
- $\forall i,I\alpha'_i+I\beta'_i+2I\gamma'_i+2I\delta'_i=d'_i$.
- $(d_1,\dots,d_l,k_1,\dots,k_l,\gamma_1,\dots,\gamma_l,\ \delta_1,\dots,\delta_l)\in\mathcal{S}(\frac{d-\sum d'_i+1}{2},l,\gamma-\sum\gamma'_i,\delta-\sum\delta'_i)$,
- $\sum(2d_i-1+|\delta_i|)+\sum r'_i=r$.
- $\forall i,2d'_i-1+|\beta'_i+2\delta'_i|-r'_i\geq 0$

Given two integer numbers $l\geq 0$ and $r\geq 0$ and four vectors α, β, γ and δ in $\mathbb{N}^\infty$, we denote by $\widetilde{\mathcal{S}}_w(l,r,\alpha,\beta,\gamma,\delta)$ the set composed by the vectors $(d_1,\dots,d_l,k_1,\dots,k_l,\gamma_1,\dots,\gamma_l,\ \delta_1,\dots,\delta_l,\ d'_1,k'_1,\ r'_1,\ \gamma'_1,\delta'_1)$ in $(\mathbb{N}^*)^{2l}\times(\mathbb{N}^\infty)^{2l}\times(\mathbb{N}^*)^2\times\mathbb{N}\times(\mathbb{N}^\infty)^2$ satisfying

- $\gamma'_1\leq\gamma$,
- $0\leq\delta'_1-e_{k'_1}\leq\delta$,
- $I\alpha+I\beta+2I\gamma'_1+2I\delta'_1=d'_1$.
- $(d_1,\dots,d_l,k_1,\dots,k_l,\gamma_1,\dots,\gamma_l,\ \delta_1,\dots,\delta_l)\in\mathcal{S}(\frac{d-d'_1}{2},l,\gamma-\gamma'_1,\delta-\delta'_1+e_{k'_1})$,
- $\sum(2d_i-1+|\delta_i|)+r'_1=r$.
- $2d'_1-1+|\beta+2\delta'_1|-r'_1\geq 0$

By definition, any element s of $\mathcal{S}_w(l,m,r,\alpha,\beta,\gamma,\delta)$ (resp. $\widetilde{\mathcal{S}}_w(l,r,\alpha,\beta,\gamma,\delta)$) defines an element s' of $\mathcal{S}(\frac{d-\sum d'_i+1}{2},l,\gamma-\sum\gamma'_i,\delta-\sum\delta'_i)$ (resp. $\mathcal{S}(\frac{d-d'_1}{2},l,\gamma-\gamma'_1,\delta-\delta'_1+e_{k'_1})$). We denote by $\sigma(s)$ the integer $\sigma(s')$ defined in section 5.1.

To any element s of $\mathcal{S}_w(l,m,r,\alpha,\beta,\gamma,\delta)$, we associate the equivalence relation $\simeq_s$ on the set $\{1,\dots,m\}$ defined by

$$i\simeq_s j\Leftrightarrow(d'_i,k'_i,r'_i,\alpha'_i,\beta'_i,\gamma'_i,\delta'_i)=(d'_j,k'_j,r'_j,\alpha'_j,\beta'_j,\gamma'_j,\delta'_j)$$

For each of the equivalent classes of $\simeq_s$, evaluate the factorial of its cardinal, and denote by $\sigma'(s)$ the product of these factorials.

To any element s of $\mathcal{S}_w(l,m,r,\alpha,\beta,\gamma,\delta)$, we associate the set $\mathcal{E}(s)$ of all j in $\{1,\dots,m\}$ such that $\beta'_j\geq e_{k'_j}$ and $2d'_j-1+|\beta'_j+2\delta'_j|-2r'_j=1$. Given an element $j\in\{1,\dots,m\}$, we denote by $\simeq_s^j$ the restriction to $\{1,\dots,m\}\setminus\{j\}$ of the equivalence relation $\simeq_s$. For each of the equivalent classes of $\simeq_s^j$, evaluate the factorial of its cardinal, and denote by $\sigma'(s,j)$ the product of these factorials.

Let us introduce one more notation. Given an element s of $\mathcal{S}_w(l,m,r,\alpha,\beta,\gamma,\delta)$ or $\widetilde{\mathcal{S}}_w(l,r,\alpha,\beta,\gamma,\delta)$, we put

$$\Theta(s) = \binom{r}{2d_1-1+|\delta_1|,\dots,2d_l-1+|\delta_l|,r'_1,\dots,r'_m}\binom{\alpha}{\alpha'_1,\dots,\alpha'_m}\binom{\gamma}{\gamma_1,\dots,\gamma_l,\gamma'_1,\dots,\gamma'_m}$$
$$\prod_{i=1}^l(-1)^{d_i}(\delta_i)_{k_i}k_i2^{2d_i-2+|\delta_i+\gamma_i|}N^{\gamma_i,\delta_i}(d_i)$$

where if s is in $\widetilde{\mathcal{S}}_w(l,r,\alpha,\beta,\gamma,\delta)$, then there are no α'_i elements and we set the value of the corresponding multinomial coefficient equal to 1.

Theorem 5.2. *The numbers $C^{\alpha,\beta,\gamma,\delta}(d,r)$, with α and β odd, are given by the initial values $C^{e_1,0,0,0}(1,0) = C^{0,e_1,0,0}(1,1) = 1$ and the relations*

(1) if $2d-1+|\beta+2\delta|-2r>0$

$$\begin{aligned} C^{\alpha,\beta,\gamma,\delta}(d,r) &= \sum_{k\ odd|\beta\geq e_k} C^{\alpha+e_k,\beta-e_k,\gamma,\delta}(d,r) \ + \\ & \sum_{\substack{l,m\geq 0\\ s\in\mathcal{S}_w(l,m,r,\alpha,\beta,\gamma,\delta)}} \Bigg[\frac{\Theta(s)}{\sigma(s)\sigma'(s)}\prod_{i=1}^m (\beta'_i)_{k'_i} C^{\alpha'_i,\beta'_i,\gamma'_i,\delta'_i}(d'_i,r'_i) \\ & \binom{2d-2+|\beta+2\delta|-2r}{2d'_1-1+|\beta'_1+2\delta'_1|-2r'_1,\dots,2d'_m-1+|\beta'_m+2\delta'_m|-2r'_m}\Bigg] \end{aligned}$$

(2) if $2d-1+|\beta+2\delta|-2r=0$

$$\begin{aligned} C^{\alpha,\beta,\gamma,\delta}(d,r) &= \sum_{k|\delta\geq e_k} kC^{\alpha,\beta,\gamma+e_k,\delta-e_k}(d,r-1) \ + \\ & \sum_{\substack{l,m\geq 0\\ K\ odd|\beta\geq e_K\\ s\in\mathcal{S}_w(l,m,r-1,\alpha,\beta-e_K,\gamma,\delta)}} \frac{K\Theta(s)}{\sigma(s)\sigma'(s)}\prod_{i=1}^m (\beta'_i)_{k'_i} C^{\alpha'_i,\beta'_i,\gamma'_i,\delta'_i}(d'_i,r'_i) \\ + & \sum_{\substack{l,m\geq 0\\ s\in\mathcal{S}_w(l,m,r-1,\alpha,\beta,\gamma,\delta)\\ j\in\mathcal{E}(s)}} \frac{\Theta(s)k'_jC^{\alpha'_j+e_{k'_j},\beta'_j-e_{k'_j},\gamma'_j,\delta'_j}(d'_j,r'_j)}{\sigma(s)\widetilde{\sigma}'(s,j)} \prod_{i=1,i\neq j}^m (\beta'_i)_{k'_i} C^{\alpha'_i,\beta'_i,\gamma'_i,\delta'_i}(d'_i,r'_i) \\ - & \sum_{\substack{l\geq 0\\ s\in\widetilde{\mathcal{S}}_w(l,r-1,\alpha,\beta,\gamma,\delta)}} \frac{\Theta(s)}{\sigma(s)} 2^{|\gamma-\gamma'_1-\sum_{i=1}^l\gamma_i|+1}(\delta'_1)_{k'_1}k'_1C^{\alpha,\beta,\gamma'_1,\delta'_1}(d'_1,r'_1) \end{aligned}$$

Remark 5.3. If one plugs $r=0$ in Theorem 5.2, then one finds the Itenberg-Kharlamov-Shustin formula from [IKS09].

6. Proof of Theorem 5.2

The proof of Theorem 5.2 is divided into two parts, depending on wether $2d-1+|\beta+2\delta|-2r>0$ or not. In this whole section α and β are two odd vectors in $\mathbb{N}^\infty$.

6.1. **The case $2d-1+|\beta+2\delta|-2r>0$.** By definition, one has

$$C^{\alpha,\beta,\gamma,\delta}(d,r) = \sum_{(\mathcal{D},\mathfrak{M})\in A}\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) + \sum_{(\mathcal{D},\mathfrak{M})\in B}\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})$$

where A (resp. B) consists of all r-real marked floor diagrams of degree d and type $(\alpha,\beta,\gamma,\delta)$ such that $\mathfrak{M}(|\alpha+2\gamma|+1)\in \text{Edge}^\infty(\mathcal{D})$ (resp. $\mathfrak{M}(|\alpha+2\gamma|+1)\in\text{Vert}(\mathcal{D})$).

6.1.1. *Marked floor diagrams in A.* Let us de denote by A_k the set of all r-real marked floor diagrams of degree d and type $(\alpha+e_k,\beta-e_k,\gamma,\delta)$. There exists a bijection Φ from the set A to the union of all the sets A_k for k such that $\beta\geq e_k$. If $(\mathcal{D},\mathfrak{M})$ is a marked floor diagram in A and k is the weight of the edge $\mathfrak{M}(|\alpha+2\gamma|+1)$, we define $\Phi(\mathcal{D},\mathfrak{M}) = (\mathcal{D},\mathfrak{M}')$ where

- $\mathfrak{M}'(i) = \mathfrak{M}(i)$ if $i\leq\sum_{j=1}^k\alpha_j$ or $i\geq|\alpha+2\gamma|+2$,
- $\mathfrak{M}'(\sum_{j=1}^k\alpha_j+1)$ is the source adjacent to $\mathfrak{M}(|\alpha+2\gamma|+1)$,

- $\mathfrak{M}'(i) = \mathfrak{M}(i-1)$ if $\sum_{j=1}^{k} \alpha_j + 2 \leq i \leq |\alpha + 2\gamma| + 1$.

By definition of the multiplicity, one has $\mu_r^{\mathbb{R}}(\mathcal{D}, \mathfrak{M}) = \mu_r^{\mathbb{R}}(\mathcal{D}, \mathfrak{M}')$, and then

$$\sum_{(\mathcal{D},\mathfrak{M})\in A} \mu_r^{\mathbb{R}}(\mathcal{D}, \mathfrak{M}) = \sum_{k|\beta\geq e_k} \sum_{(\mathcal{D},\mathfrak{M}')\in A_k} \mu_r^{\mathbb{R}}(\mathcal{D}, \mathfrak{M}')$$

Note that k is odd since β is odd, hence one has

$$\sum_{(\mathcal{D},\mathfrak{M})\in A} \mu_r^{\mathbb{R}}(\mathcal{D}, \mathfrak{M}) = \sum_{k\ odd|\beta\geq e_k} C^{\alpha+e_k,\beta-e_k,\gamma,\delta}(d,r)$$

6.1.2. *Marked floor diagrams in B.* Let us first explain backward the idea to obtain the second sum in the right-hand side of the formula in Theorem 5.2, equation (1). If $(\mathcal{D}, \mathfrak{M})$ is a marked floor diagram in B, by "cutting" the vertex $\mathfrak{M}(|\alpha + 2\gamma| + 1)$ from $\mathcal{D}$, one obtains several marked floor diagrams of genus 0 and of lower degrees. Since $(\mathcal{D}, \mathfrak{M})$ is r-real, the new marked floor diagrams contained in $\Im(\mathcal{D}, \mathfrak{M}, r)$ are naturally coupled in pairs by the involution $\rho_{\mathcal{D},\mathfrak{M},r}$ (see definition 4.6). Moreover, none of the edges in $\text{Edge}^\infty(\mathcal{D})$ adjacent to the vertex $\mathfrak{M}(|\alpha + 2\gamma| + 1)$ is in the image of $\mathfrak{M}$.

Let l and m be two nonnegative integer numbers such that the set $\mathcal{S}_w(l, m, r, \alpha, \beta, \gamma, \delta)$ is not empty. For s in $\mathcal{S}_w(l, m, r, \alpha, \beta, \gamma, \delta)$, denote by $\mathfrak{P}(s)$ the set of all $2l + m$-tuples of marked floor diagrams $(\mathcal{D}_1, \mathfrak{M}_1), \ldots, (\mathcal{D}_{2l}, \mathfrak{M}_{2l}), (\mathcal{D}'_1, \mathfrak{M}'_1), \ldots, (\mathcal{D}'_m, \mathfrak{M}'_m)$ such that

- $(\mathcal{D}_{2i-1}, \mathfrak{M}_{2i-1})$ and $(\mathcal{D}_{2i}, \mathfrak{M}_{2i})$ are two equivalent marked floor diagrams of degree d_i and type $(\gamma_i, \delta_i, 0, 0)$,
- $(\mathcal{D}'_i, \mathfrak{M}'_i)$ is r'_i-real of degree d'_i and type $(\alpha'_i, \beta'_i, \gamma'_i, \delta'_i)$.

Let ϕ_i be a homeomorphism of the oriented graph identifying $(\mathcal{D}_{2i-1}, \mathfrak{M}_{2i-1})$ and $(\mathcal{D}_{2i}, \mathfrak{M}_{2i})$.

Starting from an element of $\mathfrak{P}(s)$, we construct several elements of B in the following way

(1) For all i in $\{1, \ldots, l\}$ choose an element a_i of $\text{Edge}^\infty(\mathcal{D}_{2i-1})$ which is in the image of $\mathfrak{M}_{2i-1}$ and of weight k_i. Since $\delta_i \geq e_{k_i}$, it is always possible to choose such an a_i.
(2) For all i in $\{1, \ldots, m\}$ choose an element a'_i of $\text{Edges}^\infty(\mathcal{D}'_i)$ which is in the image of $\mathfrak{M}'_i$ but not in $\Im(\mathcal{D}'_i, \mathfrak{M}'_i, r'_i)$, and of weight k'_i. Since $\beta'_i \geq e_{k'_i}$, it is always possible to choose such an a'_i.
(3) Construct a new oriented tree $\widetilde{\mathcal{D}}$ out of $(\mathcal{D}_1, \mathfrak{M}_1), \ldots, (\mathcal{D}_{2l}, \mathfrak{M}_{2l}), (\mathcal{D}'_1, \mathfrak{M}'_1), \ldots, (\mathcal{D}'_m, \mathfrak{M}'_m)$ by identifying all the sources adjacent to the edges a_i, $\phi_i(a_i)$, and a'_j. Denote this vertex by v.
(4) By adding sources and edges adjacent to the vertex v, complete $\widetilde{\mathcal{D}}$ into a (unique) floor diagram $\mathcal{D}$ of degree d, genus 0, with $(\alpha)_j + (\beta)_j + 2(\gamma)_j + 2(\delta)_j$ edges in $\text{Edges}^\infty(\mathcal{D})$ of weight j for all $j \geq 1$. Denote by $v_1, \ldots, v_t$ the sources added.
(5) Define $\alpha'_{m+1} = \alpha - \sum_{i=1}^{m} \alpha'_i$ and $\gamma'_{m+1} = \gamma - \sum_{i=1}^{l} \gamma_i - \sum_{i=1}^{m} \gamma'_i$.
(6) For all $j \geq 1$, choose a partition $(I_i^j)_{1\leq i\leq m+1}$ of the set $\{1, \ldots, (\alpha)_j\}$ such that $\sharp I_i^j = (\alpha'_i)_j$ for all i.
(7) For all $j \geq 1$, choose a partition $(\widehat{I}_i^j)_{1\leq i\leq l} \cup (\widetilde{I}_i^j)_{1\leq i\leq m+1}$ of the set $\{1, \ldots, (\gamma)_j\}$ such that $\sharp\widehat{I}_i^j = (\gamma_i)_j$ and $\sharp\widetilde{I}_i^j = (\gamma'_i)_j$ for all i.
(8) Choose a partition $(J_i)_{1\leq i\leq m}$ of the set $\{1, \ldots, 2d - 2 + |\beta + 2\delta| - 2r\}$ such that $\sharp J_i = 2d'_i - 1 + |\beta'_i + 2\delta'_i| - 2r'_i$ for all i.
(9) Choose a partition $(\widehat{J}_i)_{1\leq i\leq l} \cup (\widetilde{J}_i)_{1\leq i\leq m}$ of the set $\{1, \ldots, r\}$ such that $\sharp\widehat{J}_i = 2d_i - 1 + |\delta_i|$ and $\sharp\widetilde{J}_i = r'_i$ for all i.
(10) For all i in $\{1, \ldots, l\}$, choose a vector ε_i in $\{0,1\}^{2d_i-1+|\gamma_i+\delta_i|}$.
(11) Choose a marking $\mathfrak{M}$ of $\mathcal{D}$ of type $(\alpha, \beta, \gamma, \delta)$ such that

(a) $\mathfrak{M}(|\alpha| + 2|\gamma| + 1) = v$,

(b) for all $j \geq 1$ and all k in I^j_{m+1}, $\mathfrak{M}(\sum_{t=1}^{j-1}(\alpha)_t + k)$ is a source v_q (see step (4)) of $\mathcal{D}$ of divergence $-j$ (note that different choices of v_q produce equivalent marked floor diagrams),

(c) for all $j \geq 1$ and all k in $\widetilde{I}^j_{m+1}$, $\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - 1\right)$ and $\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k\right)$ are two sources v_q and $v_{q'}$ of $\mathcal{D}$ of divergence $-j$ (again, different choices of v_q and $v_{q'}$ produce equivalent marked floor diagrams),

(d) for all $j \geq 1$ and all i in $\{1, \ldots, m\}$, if k is the h-th element (for the natural ordering on I^j_i) of I^i_j, then

$$\mathfrak{M}\left(\sum_{t=1}^{j-1}(\alpha)_t + k\right) = \mathfrak{M}'_j\left(\sum_{t=1}^{j-1}(\alpha'_j)_t + h\right)$$

(e) for all $j \geq 1$ and all i in $\{1, \ldots, l\}$, if k is the h-th element of $\widehat{I}^j_i$, then

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - 1 + (\varepsilon_i)_{\sum_{t=1}^{j-1}(\gamma_i)_t + h}\right) = \mathfrak{M}_{2i-1}\left(\sum_{t=1}^{j-1}(\gamma_i)_t + h\right)$$

and

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - (\varepsilon_i)_{\sum_{t=1}^{j-1}(\gamma_i)_t + h}\right) = \phi_i \circ \mathfrak{M}_{2i-1}\left(\sum_{t=1}^{j-1}(\gamma_i)_t + h\right)$$

(f) for all $j \geq 1$ and all i in $\{1, \ldots, m\}$, if k is the h-th element of $\widetilde{I}^j_i$, then

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - 1\right) = \mathfrak{M}'_i\left(|\alpha'_i| + 2\sum_{t=1}^{j-1}(\gamma'_i)_t + 2h - 1\right)$$

and

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k\right) = \mathfrak{M}'_i\left(|\alpha'_i| + 2\sum_{t=1}^{j-1}(\gamma'_i)_t + 2h\right)$$

(g) for all i in $\{1, \ldots, m\}$, if k is the h-th element of J_i, then

$$\mathfrak{M}\left(|\alpha| + 2|\gamma| + k + 1\right) = \mathfrak{M}'_i\left(|\alpha'_i| + 2|\gamma'_i| + h\right)$$

(h) for all i in $\{1, \ldots, l\}$, if k is the h-th element of $\widehat{J}_i$, then (recall that $n = 2d - 1 + |\alpha + \beta + 2\gamma + 2\delta|$)

$$\mathfrak{M}\left(n - 2k + 1 + (\varepsilon_i)_{|\gamma_i| + h}\right) = \mathfrak{M}_{2i-1}\left(2d_i - 1 + |\delta_i| - h + 1\right)$$

and

$$\mathfrak{M}\left(n - 2k + 2 - (\varepsilon_i)_{|\gamma_i| + h}\right) = \phi_i \circ \mathfrak{M}_{2i-1}\left(2d_i - 1 + |\delta_i| - h + 1\right)$$

(i) for all i in $\{1, \ldots, m\}$, if k is the h-th element of $\widetilde{J}_i$, then

$$\mathfrak{M}\left(n - 2k + 1\right) = \mathfrak{M}'_i\left(2d'_i - 1 + |\beta'_i + 2\delta'_i| - 2h + 1\right)$$

and

$$\mathfrak{M}\left(n - 2k + 2\right) = \mathfrak{M}'_i\left(2d'_i - 1 + |\beta'_i + 2\delta'_i| - 2h + 2\right)$$

In this way, we construct all marked floor diagrams in B. Moreover, two distinct elements of $\mathcal{S}_w(l,m,r,\alpha,\beta,\gamma,\delta)$ will produce non equivalent marked floor diagrams. If a marked floor diagram $(\mathcal{D},\mathfrak{M})$ of type $(\alpha,\beta,\gamma,\delta)$ is constructed out of an element s of $\mathcal{S}_w(l,m,r,\alpha,\beta,\gamma,\delta)$, then $(\mathcal{D},\mathfrak{M})$ is obtained exactly $2^l\sigma(s)\sigma'(s)$ times. Indeed, since $(\mathcal{D}_{2i-1},\mathfrak{M}_{2i-1})$ and $(\mathcal{D}_{2i},\mathfrak{M}_{2i})$ are equivalent, the two vectors ε_i and $(1,\dots,1)-\varepsilon_i$ produce 2 equivalent marked floor diagrams. The factor $\sigma(s)\sigma'(s)$ is clear.

By construction, one has

$$\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) = \prod_{i=1}^{l}(-1)^{d_i}k_i\mu^{\mathbb{C}}(\mathcal{D}_{2i},\mathfrak{M}_{2i})\prod_{i=1}^{m}\mu^{\mathbb{R}}_{r'_i}(\mathcal{D}'_i,\mathfrak{M}'_i)$$

Now, the second sum in the right-hand side of the formula in Theorem 5.2, equation (1), follows from all possible choices in our construction, and from Theorem 4.4.

6.2. **The case** $2d-1+|\beta+2\delta|-2r=0$**.** In this case, if $(\mathcal{D},\mathfrak{M})$ is an r-real marked floor diagram of degree d and type $(\alpha,\beta,\gamma,\delta)$, then $\{|\alpha+2\gamma|+1,|\alpha+2\gamma|+2\}$ is an r-pair of $(\mathcal{D},\mathfrak{M})$. The four terms in the right-hand side of the formula in Theorem 5.2, equation (2), come from consideration of four subcases. Let A', B', C' and D' be the the sets of all r-real marked floor diagrams of degree d and type $(\alpha,\beta,\gamma,\delta)$ satisfying respectively

- both $\mathfrak{M}(|\alpha+2\gamma|+1)$ and $\mathfrak{M}(|\alpha+2\gamma|+2)$ are in $\mathrm{Edge}^\infty(\Gamma)$,
- $\mathfrak{M}(|\alpha+2\gamma|+1)$ is in $\mathrm{Edge}^\infty(\Gamma)$ and $\mathfrak{M}(|\alpha+2\gamma|+2)$ is in $\mathrm{Vert}(\Gamma)$,
- $\mathfrak{M}(|\alpha+2\gamma|+1)$ is in $\mathrm{Vert}(\Gamma)$ and $\mathfrak{M}(|\alpha+2\gamma|+2)$ is in $\mathrm{Edge}(\Gamma)$,
- both $\mathfrak{M}(|\alpha+2\gamma|+1)$ and $\mathfrak{M}(|\alpha+2\gamma|+2)$ are in $\mathrm{Vert}(\Gamma)$.

6.2.1. *Marked floor diagrams in* A'. There exists a bijection Φ' from the set A' to the union of all $(r-1)$-real marked floor diagrams of degree d and type $(\alpha,\beta,\gamma+e_k,\delta-e_k)$ for k such that $\delta\geq e_k$. If $(\mathcal{D},\mathfrak{M})$ is a marked floor diagram in A' and k is the weight of the edge $\mathfrak{M}(|\alpha+2\gamma|+1)$, we define $\Phi'(\mathcal{D},\mathfrak{M})=(\mathcal{D},\mathfrak{M}')$ where

- $\mathfrak{M}'(i)=\mathfrak{M}(i)$ if $i\leq|\alpha|+2\sum_{j=1}^k\gamma_j$ or $i\geq|\alpha+2\gamma|+3$,
- $\mathfrak{M}'(|\alpha|+2\sum_{j=1}^k\gamma_j+1)$ is the source adjacent to $\mathfrak{M}(|\alpha+2\gamma|+1)$,
- $\mathfrak{M}'(|\alpha|+2\sum_{j=1}^k\gamma_j+2)$ is the source adjacent to $\mathfrak{M}(|\alpha+2\gamma|+2)$,
- $\mathfrak{M}'(i)=\mathfrak{M}(i-2)$ if $|\alpha|+2\sum_{j=1}^k\gamma_j+3\leq i\leq|\alpha+2\gamma|+2$.

Note that since $(\mathcal{D},\mathfrak{M})$ is r-real, the weight of the edges $\mathfrak{M}(|\alpha+2\gamma|+1)$ and $\mathfrak{M}(|\alpha+2\gamma|+2)$ is the same. One has $\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})=k\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}')$, hence the first sum of the right-hand side of the formula in Theorem 5.2, equation (2), is given by $\sum_{(\mathcal{D},\mathfrak{M})\in A'}\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})$.

6.2.2. *Marked floor diagrams in* B'. Choose $K\geq 1$ such that $\beta\geq e_K$. Let l and m be two non-negative integers such that the set $S_w(l,m,d,k,r-1,\alpha,\beta-e_K,\gamma,\delta)$ is not empty, and define the set $\mathfrak{P}(s)$ as in section 6.1.2 for any s in $S_w(l,m,d,k,r-1,\alpha,\beta-e_K,\gamma,\delta)$. Then, starting from an element of $\mathfrak{P}(s)$ we construct several elements of B' as in the step (1) - (11) of section 6.1.2, except for the following modifications

(8B') Define $J_i=\emptyset$ for all i in $\{1,\dots,m\}$.

(9B') Choose a partition $(\widehat{J}_i)_{1\leq i\leq l}\cup(\widetilde{J}_i)_{1\leq i\leq m}$ of the set $\{1,\dots,r-1\}$ such that $\sharp\widehat{J}_i=2d_i-1+|\delta_i|$ and $\sharp\widetilde{J}_i=r'_i$ for all i.

(11B') (a) $\mathfrak{M}(|\alpha|+2|\gamma|+2)=v$, and $\mathfrak{M}(|\alpha|+2|\gamma|+1)$ is an edge in $\mathrm{Edge}^\infty(\mathcal{D})$ of weight K adjacent to v

By construction, one has

$$\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) = K\prod_{i=1}^{l}(-1)^{d_i}k_i\mu^{\mathbb{C}}(\mathcal{D}_{2i},\mathfrak{M}_{2i})\prod_{i=1}^{m}\mu_{r'_i}^{\mathbb{R}}(\mathcal{D}'_i,\mathfrak{M}'_i)$$

Now, the second sum in the right-hand side of the formula in Theorem 5.2, equation (2), follows from all possible choices in our construction.

6.2.3. *Marked floor diagrams in C'.* Let l and m be two nonnegative integers such that the set $S_w(l,m,d,k,r-1,\alpha,\beta,\gamma,\delta)$ is not empty, and define the set $\mathfrak{P}(s)$ as in section 6.1.2 for any s in $S_w(l,m,d,k,r-1,\alpha,\beta,\gamma,\delta)$. Then, starting from an element of $\mathfrak{P}(s)$ we construct several elements of C' as in the step (1) - (11) of section 6.1.2, except for the following modifications

(0C') Choose j in $\{1,\dots,m\}$ such that $\beta_j \geq k'_j$, the weight of $\mathfrak{M}'_j(|\alpha'_j|+2|\gamma'_j|+1)$ is k'_j, and $2d'_j-1+|\beta'_j+2\gamma'_j|-2r'_j=1$.

(2C') For all $1\leq i\leq m$, $i\neq j$, choose an element a'_i of $\text{Edges}^\infty(\mathcal{D}'_i)$ which is in the image of $\mathfrak{M}'_i$ but not in $\Im(\mathcal{D}'_i,\mathfrak{M}'_i,r'_i)$, and of weight k'_i.

(8C') Define $J_i=\emptyset$ for all i in $\{1,\dots,m\}$.

(9C') Choose a partition $(\widehat{J}_i)_{1\leq i\leq l}\cup(\widetilde{J}_i)_{1\leq i\leq m}$ of the set $\{1,\dots,r-1\}$ such that $\sharp\widehat{J}_i=2d_i-1+|\delta_i|$ and $\sharp\widetilde{J}_i=r'_i$ for all i.

(11C') (a) $\mathfrak{M}(|\alpha|+2|\gamma|+1)=v$, and $\mathfrak{M}(|\alpha|+2|\gamma|+2)=\mathfrak{M}'_j(|\alpha'_j|+2|\gamma'_j|+1)$.

By construction, one has

$$\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) = k'_j\prod_{i=1}^{l}(-1)^{d_i}k_i\mu^{\mathbb{C}}(\mathcal{D}_{2i},\mathfrak{M}_{2i})\prod_{i=1}^{m}\mu_{r'_i}^{\mathbb{R}}(\mathcal{D}'_i,\mathfrak{M}'_i)$$

Now, the third sum in the right-hand side of the formula in Theorem 5.2, equation (2), follows from all possible choices in our construction.

6.2.4. *Marked floor diagrams in D'.* In this case, by "cutting" the vertices $\mathfrak{M}(|\alpha+2\gamma|+1)$ and $\mathfrak{M}(|\alpha+2\gamma|+2)$ from $\mathcal{D}$, one obtains several marked floor diagrams of genus 0 and of lower degrees. Since $\mathcal{D}$ is a tree and is r-real, exactly one of these marked floor diagrams is adjacent to both cut vertices, and all the other are naturally coupled in pairs by the map $\rho_{\mathcal{D},\mathfrak{M},r}$ (see definition 4.6). Moreover, any edge in $\text{Edge}^\infty(\mathcal{D})$ adjacent to the vertex $\mathfrak{M}(|\alpha+2\gamma|+1)$ is not in the image of $\mathfrak{M}$ and is naturally coupled to an edge adjacent to the vertex $\mathfrak{M}(|\alpha+2\gamma|+2)$ since the diagram is r-real. In particular, both edges have the same weight.

Let l be a nonnegative integer such that the set $\widetilde{S}_w(l,r-1,\alpha,\beta,\gamma,\delta)$ is not empty. For s in this set, define the set $\mathfrak{P}(s)$ as in section 6.1.2 with $m=1$.

Starting from an element of $\mathfrak{P}(s)$, we construct several elements of D' in the following way

1. For all $0\leq i\leq l$ choose an element a_i of $\text{Edge}^\infty(\mathcal{D}_{2i-1})$ which is in the image of $\mathfrak{M}_{2i-1}$ and of weight k_i.
2. Choose an edge a'_1 of $\mathcal{D}'_1$ in $\text{Edge}^\infty(\mathcal{D}'_1)\cap\Im(\mathcal{D}'_1,m'_1,r'_1)$ of weight k'_1.
3. Construct a new oriented tree $\widetilde{\mathcal{D}}$ out of $(\mathcal{D}_1,\mathfrak{M}_1),\dots,(\mathcal{D}_{2l},\mathfrak{M}_{2l}),(\mathcal{D}'_1,\mathfrak{M}'_1)$ by identifying
   (a) all the sources adjacent to the edges a_i and a'_1
   (b) all the sources adjacent to the edges $\phi_i(a_i)$ and $\rho_{\mathcal{D}'_1,m'_1,r'_1}(a'_1)$.
   Denote by v and v' the 2 vertices added.
4. Construct a degree d and genus 0 floor diagram $\mathcal{D}$ out of $\widetilde{\mathcal{D}}$ by adding sources $v_1,\dots,v_t,v'_1$, $\dots$, v'_t and edges (v_1,v), $\dots$, (v_t,v), (v'_1,v'), $\dots$, (v'_t,v'), such that $\mathcal{D}$ has $(\alpha)_j+(\beta)_j+2(\gamma)_j+$

$2(\delta)_j$ edges in $\mathrm{Edges}^\infty(\mathcal{D})$ of weight j for all $j \geq 1$, and such that there are as many edges (v_i, v) of weight j as edges (v'_i, v') of weight j for all $j \geq 1$.

(5) Define $\gamma'_2 = \gamma - \sum_{i=1}^{l} \gamma_i - \gamma'_1$.

(6) For all $j \geq 1$, choose a partition $(\widehat{I}_i^j)_{1\leq i\leq l} \cup (\widetilde{I}_1^j) \cup (\widetilde{I}_2^j)$ of the set $\{1, \ldots, (\gamma)_j\}$ such that $\sharp \widehat{I}_i^j = (\gamma_i)_j$ and $\sharp \widetilde{I}_i^j = (\gamma'_i)_j$.

(7) Choose a partition $(\widehat{J}_i)_{1\leq i\leq l} \cup (\widetilde{J}_1)$ of the set $\{1, \ldots, r-1\}$ such that $\sharp \widehat{J}_i = 2d_i - 1 + |\delta_i|$ and $\sharp \widetilde{J}_1 = r'_1$.

(8) Choose a vector ε in $\{0,1\}^{|\gamma'_2|}$.

(9) For all i in $\{1, \ldots, l\}$, choose a vector ε_i in $\{0,1\}^{2d_i - 1 + |\gamma_i + \delta_i|}$.

(10) Choose a marking $\mathfrak{M}$ of $\mathcal{D}$ of type $(\alpha, \beta, \gamma, \delta)$ such that

(a) $\mathfrak{M}(|\alpha| + 2|\gamma| + 1) = v$ and $\mathfrak{M}(|\alpha| + 2|\gamma| + 2) = v'$,

(b) for all $j \geq 1$ and if k is the h-th element of $\widetilde{I}_2^j$, then $\mathfrak{M}\Big(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - 1 + (\varepsilon)_{\sum_{t=1}^{j-1}(\gamma'_i)_t + h}\Big)$ and $\mathfrak{M}\Big(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k \ - (\varepsilon)_{\sum_{t=1}^{j-1}(\gamma'_i)_t + h}\Big)$ are respectively a source v_q and $v'_{q'}$ of divergence $-j$,

(c) If $1 \leq i \leq |\alpha|$, $\mathfrak{M}(i) = \mathfrak{M}'_1(i)$,

(d) for all $j \geq 1$ and all i in $\{1, \ldots, l\}$, if k is the h-th element of $\widehat{I}_i^j$, then

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - 1 + (\varepsilon_i)_{\sum_{t=1}^{j-1}(\gamma_i)_t + h}\right) = \mathfrak{M}_{2i-1}\left(\sum_{t=1}^{j-1}(\gamma_i)_t + h\right)$$

and

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - (\varepsilon_i)_{\sum_{t=1}^{j-1}(\gamma_i)_t + h}\right) = \phi_i \circ \mathfrak{M}_{2i-1}\left(\sum_{t=1}^{j-1}(\gamma_i)_t + h\right)$$

(e) for all $j \geq 1$, if k is the h-th element of $\widetilde{I}_1^j$, then

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - 1\right) = \mathfrak{M}'_1\left(|\alpha'_1| + 2\sum_{t=1}^{j-1}(\gamma'_1)_t + 2h - 1\right)$$

and

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k\right) = \mathfrak{M}'_1\left(|\alpha'_1| + 2\sum_{t=1}^{j-1}(\gamma'_1)_t + 2h\right)$$

(f) for all i in $\{1, \ldots, l\}$, if k is the h-th element of $\widehat{J}_i$, then

$$\mathfrak{M}\left(n - 2k + 1 + (\varepsilon_i)_{|\gamma_i| + h}\right) = \mathfrak{M}_{2i-1}\left(2d_i - 1 + |\delta_i| - h + 1\right)$$

and

$$\mathfrak{M}\left(n - 2k + 2 - (\varepsilon_i)_{|\gamma_i| + h}\right) = \phi_i \circ \mathfrak{M}_{2i-1}\left(2d_i - 1 + |\delta_i| - h + 1\right)$$

(g) if if k is the h-th element of $\widetilde{J}_1$,

$$\mathfrak{M}\left(n - 2k + 1\right) = \mathfrak{M}'_1\left(2d'_1 - 1 + |\beta'_1 + 2\delta_i| - 2h + 1\right)$$

and

$$\mathfrak{M}\left(n - 2k + 2\right) = \mathfrak{M}'_1\left(2d'_1 - 1 + |\beta'_1 + 2\delta_i| - 2h + 2\right)$$

In this way, we construct all marked floor diagrams in D'. Moreover, any element of D' is obtained exactly $\sigma(s)\sigma'(s)$ times for some s in $\widetilde{S}_w(l, r-1, \alpha, \beta, \gamma, \delta)$.

By construction, one has

$$\mu_r^{\mathbb{R}}(\mathcal{D}, \mathfrak{M}) = -k_1' \mu_{r_1'}^{\mathbb{R}}(\mathcal{D}_1', \mathfrak{M}_1') \prod_{i=1}^{l} (-1)^{d_i} k_i \mu^{\mathbb{C}}(\mathcal{D}_{2i}, \mathfrak{M}_{2i})$$

Now, the fourth sum in the right-hand side of the formula in Theorem 5.2, equation (2), follows from all possible choices in our construction.

7. Further remarks

7.1. **More recursive formulas.** Using floor diagrams, one can enumerate complex curves in other complex varieties than $\mathbb{C}P^2$, for example in Hirzebruch surfaces, or in $\mathbb{C}P^n$ blown up in a small number of points. Floor diagrams allow one to compute Welschinger invariants as soon as they are defined and a theorem similar to Theorem 4.7 holds (see [BMb] and [BM07]). Hence, one can easily write formulas in the style of Theorem 5.2 for other varieties than $\mathbb{C}P^2$, using the same technic as here. As an example, we give below a formula to compute Weslchinger invariants of $\mathbb{R}P^3$ for rational curves passing through a configuration of real points. This invariants are defined in [Wel05b], and were computed for the first time in [BMa] and [BM07] using floor diagrams.

Let us denote by $W_3(d)$ the Welschinger invariant of degree d of $\mathbb{R}P^3$ for configurations of $2d$ real points. Given an integer number $l \geq 0$ and two odd vectors α and β in $\mathbb{N}^\infty$, we denote by $\mathcal{S}_3(l, \alpha, \beta)$ the set composed by the vectors $(d_1, \ldots, d_l,\ k_1, \ldots, k_l,\ \alpha_1, \ldots, \alpha_l,\ \beta_1, \ldots, \beta_l) \in (\mathbb{N}^*)^{2l} \times (\mathbb{N}^\infty)^{2l}$ satisfying

- $\forall i, (d_i, k_i, \alpha_i, \beta_i) \leq (d_{i+1}, k_{i+1}, \alpha_{i+1}, \beta_{i+1})$ for the lexicographic order,
- $\sum d_i < I\alpha + I\beta$,
- $\sum \alpha_i \leq \alpha$,
- $\forall i,\ k_i$ is odd,
- $\forall i, \beta_i \geq e_{k_i}$,
- $\sum(\beta_i - e_{k_i}) = \beta$,
- $\forall i, I\alpha_i + I\beta_i = d_i$,
- $l + |\alpha - \sum \alpha_i| = 3d - 3\sum d_i - 2$.

Given an integer number $d \geq 1$ and two odd vectors α and βin $\mathbb{N}^\infty$ satisfying $I\alpha + I\beta = d$, we define the numbers $W_3^{\alpha,\beta}(d)$ by the initial value $W_3^{e_1,0}(1) = 1$ and the relation

$$\begin{aligned} W_3^{\alpha,\beta}(d) \quad = \quad & \textstyle\sum_{k\ odd|\beta \geq e_k} W_3^{\alpha+e_k, \beta-e_k}(d) \ + \\ & \sum_{\substack{l \geq 0 \\ s \in \mathcal{S}_3(l,\alpha,\beta)}} \left[\frac{1}{\sigma(s)} W_2\left(d - \textstyle\sum d_i, 0\right) \begin{pmatrix} \frac{3d - |\alpha| + |\beta|}{2} - 1 \\ \frac{3d_1 - |\alpha_1| + |\beta_1|}{2}, \ldots, \frac{3d_l - |\alpha_l| + |\beta_l|}{2} \end{pmatrix} \right. \\ & \left. \begin{pmatrix} \alpha \\ \alpha_1, \ldots, \alpha_l \end{pmatrix} \textstyle\prod_{i=1}^{l} (\beta_i)_{k_i} W_3^{\alpha_i, \beta_i}(d_i) \right] \end{aligned}$$

where the integer number $\sigma(s)$ is defined as in section 5.1.

Using [BM07, Theorem 2] and the technic of this paper, one can easily prove the following result.

Theorem 7.1. *For any $d \geq 1$, one has $W_3(d) = (-1)^{\frac{(d-1)(d-2)}{2}} W_3^{0,de_1}(d)$.*

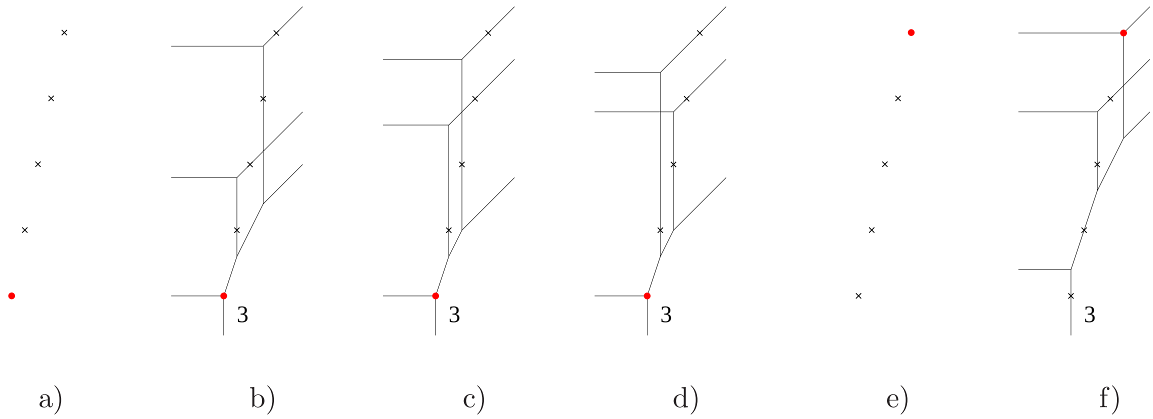


FIGURE 2. Tropical Welschinger multiplicity does not provide tropical relative invariants

7.2. **Tropical relative Welschinger invariants.** As mentioned in section 3.2, one does not get any relative Welschinger invariant just by counting with Welschinger sign the real rational curves of a given degree satisfying incidence and tangency conditions. Surprisingly enough, Itenberg, Kharlamov and Shustin proved in [IKS09] that the situation is totally different in tropical geometry: counting tropical curves with the tropical Welschinger multiplicity provides tropical relative invariants in any genus.

Such a result does not hold when one considers situations related to configurations of points with pairs of complex conjugated points. A counter-example is given in Figure 2 where we look at rational tropical cubics in $\mathbb{R}^2$ with an unbounded edge of weight 3 and passing through 5 points x'_1, x'_2, x'_3, x'_4 and x''_1 (for brevity, we use the same notation as in [Shu06, Theorem 3.1] and we refer to this article for more details). In Figures 2a and e, crosses represent the points x'_i and the disk represents the point x''_1. There exist exactly 3 tropical cubics of genus 0 with an unbounded edge of weight 3 passing through the configuration of Figure 2a with a non zero real multiplicity (see Figure 2b, c and d), and all of them are of multiplicity 3. There exists only 1 tropical cubic of genus 0 with an unbounded edge of weight 3 passing through the configuration of Figure 2e with a non zero real multiplicity (see Figure 2f), and it is of multiplicity 1.

7.3. **Some computations.** We list below the first values of $N^{0,de_1}(d)$, $W_2(d,r)$ and $W_3(d)$. These computations have been made using a Maple program, available at ???.

d	1	2	3	4	5	6	7	8	9
$N^{0,de_1}(d)$	1	1	12	620	87304	26312976	14616808192	13525751027392	19385778269260800

r	0	1	2	3	4
$W_2(3,r)$	8	6	4	2	0

r	0	1	2	3	4	5
$W_2(4,r)$	240	144	80	40	16	0

r	0	1	2	3	4	5	6	7
$W_2(5,r)$	18264	9096	4272	1872	744	248	64	64

r	0	1	2	3	4	5	6	7	8
$W_2(6,r)$	2845440	1209600	490368	188544	67968	22400	6400	1536	1024

r	0	1	2	3	4	5	6	7
$W_2(7,r)$	792731520	293758272	104600448	35670576	11579712	3538080	995904	248976

r	8	9	10
$W_2(7,r)$	54272	11776	−14336

r	0	1	2	3	4	5
$W_2(8,r)$	359935488000	118173265920	37486448640	11463469056	3367084032	944056320

r	6	7	8	9	10	11
$W_2(8,r)$	249999360	61424640	13643776	2705408	499712	−280576

r	0	1	2	3	4
$W_2(9,r)$	248962406889600	73359212457600	20972001869568	5807486276352	1553952238848

r	5	6	7	8	9	10
$W_2(9,r)$	400246421760	98632018560	23031485568	5021757312	1003137408	181785600

r	11	12	13
$W_2(9,r)$	30391296	3932160	17326080

In particular, the values $W_2(9,12)$ and $W_2(9,13)$ disprove the monotonicity conjecture of the function $r \mapsto W_2(d,r)$ by Itenberg, Kharlamov and Shustin (see [IKS04, Conjecture 6]). Note that the positivity conjecture of $W_2(d,r)$ in [IKS04, Conjecture 6] has already been disproved by Welschinger in [Wel].

d	1	3	5	7	9	11	13
$W_3(d)$	1	−1	45	−14589	17756793	−58445425017	426876362998821

As $W_3(2k) = 0$ for any $k \geq 1$, we listed only the first values of $W_3(2k+1)$.

7.4. **Congruences.** Mikhalkin observed that the numbers $W_2(d,0)$ and $N_2(d)$ are equal modulo 4. More generally, it seems that the numbers $W_2(d,r)$ and $N_2(d)$ are equal modulo $2^{f(d)}$, where $f : \mathbb{N} \to \mathbb{N}$ is some function which goes to infinity as d goes to infinity. Using Kontsevich formula (see [KM94] or [DFI95]), one can prove immediatly the following proposition, where $[x]$ denotes the integer part of the real number x.

Proposition 7.2. *For any $d \geq 1$, the number $N_2(d)$ is divisible by $2^{\left[\frac{d-1}{2}\right]}$.*

Up to our knowledge, no such result is known in full generality for Welschinger invariants $W_2(d,r)$, although computations suggest that an analog result holds in this case. If $W_2(d,r)$ and $N_2(d)$ are truly equal modulo big powers of 2, it would be very interesting to find a geometrical reason. Note that Welschinger used symplectic field theory in [Wel] to prove that a big power of 2 divides $W_2(d,r)$ when r is close to $\frac{3d-1}{2}$.

It is proved in [BMa] (see also [BM07]) that the number $W_3(d)$ is also equal modulo 4 to the corresponding number of complex curves. However, unlike in the case of enumerative invariants of $\mathbb{C}P^2$, a stronger congruence does not seem to hold.

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