

# SYMMETRIC PLANE CURVES OF DEGREE 7 : PSEUDOHOLOMORPHIC AND ALGEBRAIC CLASSIFICATIONS

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## Abstract

This paper is motivated by the real symplectic isotopy problem : does there exist a nonsingular real pseudoholomorphic curve not isotopic in the projective plane to any real algebraic curve of the same degree? Here, we focus our study on symmetric real curves on the projective plane. We give a classification of real schemes (resp. complex schemes) realizable by symmetric real curves of degree 7 with respect to the type of the curve (resp.  $M$ -symmetric real curves of degree 7). In particular, we exhibit two real schemes which are realizable by real symmetric dividing pseudoholomorphic curves of degree 7 on the projective plane but not by algebraic ones.

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# 1 Introduction and statements of results

## 1.1 Brief History

The origin of the topology of real algebraic curves can be traced back to the paper of A. Harnack [Har76] which was published in 1876. D. Hilbert formulated the following question in the 16-th problem of his famous list (see [Hil01]): up to ambient isotopy of  $\mathbb{R}P^2$ , how can be arranged the connected components of the real point set of a nonsingular real algebraic curve of a given degree  $m$  on  $\mathbb{R}P^2$ ?

In 1900, the answer was known only for  $m \leq 5$  and nowadays the problem is also solved for  $m = 6$  (D. A. Gudkov 1969, [Gud69]) and  $m = 7$  (O. Ya. Viro 1979).

**Theorem 1.1 (Viro, [Vir84a], [Vir84b])** *Any nonsingular real algebraic curve of degree 7 on  $\mathbb{R}P^2$  has one of the following real schemes :*

- $\langle J \amalg \alpha \amalg 1 \langle \beta \rangle \rangle$  with  $\alpha + \beta \leq 14$ ,  $0 \leq \alpha \leq 13$ ,  $1 \leq \beta \leq 13$ ,
- $\langle J \amalg \alpha \rangle$  with  $0 \leq \alpha \leq 15$ ,
- $\langle J \amalg 1 \langle 1 \langle 1 \rangle \rangle \rangle$ .

*Moreover, any of these 121 real schemes is realizable by nonsingular real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .*

**Remark.** The notations used to encode real schemes are the usual ones introduced in [Vir84b]. For example,  $\langle J \amalg \alpha \amalg 1 \langle \beta \rangle \rangle$  means a non-contractible (in  $\mathbb{R}P^2$ ) component, an oval with  $\beta$  ovals in its interior and  $\alpha$  ovals in its exterior. All the ovals  $\alpha$  and  $\beta$  lie outside each other. An oval is a contractible (in  $\mathbb{R}P^2$ ) component of the real curve.

In the study of the topology of real algebraic varieties, one can distinguish two directions : *constructions* and *prohibitions*. Historically, the first prohibition result, the Harnack theorem, asserts that the real part of a real algebraic curve of genus  $g$  cannot have more than  $g + 1$  connected components (curves with the maximal number of connected components are called *M-curves*). In [Arn71], V. I. Arnold paved the way to the use of powerful topological methods in the study of this problem. Almost all known prohibitions can be obtained via a topological study of the double covering of  $\mathbb{C}P^2$  branched along the complex point set of a curve of even degree or looking at the braid defined by the intersection of a curve and a certain sphere  $S^3$  in  $\mathbb{C}P^2$  (the interested reader can see the surveys [DK00], [Vir84b] and [Wil78]). In particular, almost all known restrictions on the topology of real algebraic curves are also valid for a category of more flexible objects : *pseudoholomorphic curves*. These objects, introduced by M. Gromov to study symplectic 4-manifolds in [Gro85], share a lot of properties with the algebraic curves (for example, the Harnack theorem is still valid in the real pseudoholomorphic case) and are much easier to deal with. Indeed, there are methods to construct pseudoholomorphic curves which are not necessarily algebraic : S. Yu. Orevkov (see [Ore99]) proved that it is sufficient to show the quasipositivity of some braid, and I. Itenberg and E. Shustin (see [IS02]) proved that any T-construction gives a pseudoholomorphic curve even if the convexity condition on the triangulation is not fulfilled. Up to now, the classification up to isotopy of real pseudoholomorphic *M-curves* of degree 8 on  $\mathbb{R}P^2$  is achieved by Orevkov [Ore02a]. However, it remains 6 real schemes for which it is unknown whether they are realizable by algebraic *M-curves* of degree 8 on  $\mathbb{R}P^2$ .

Then a natural question arises : does there exist a real scheme on  $\mathbb{R}P^2$  which would be realizable pseudoholomorphically but not algebraically? This problem is the real counterpart of the symplectic isotopy problem. Strictly speaking (i.e., dealing with nonsingular curves on the projective plane), this problem is still open. However, dealing with singular curves or curves in ruled surfaces up to fiberwise isotopy, the answer is *yes* : S. Fiedler Le Touzé and S. Orevkov (see [Ore02a], [FLTO02]) exhibit mutual arrangements of two nonsingular curves on  $\mathbb{R}P^2$  which are realizable pseudoholomorphically but not algebraically (examples with many irreducible components are easy to construct, see the “pseudoholomorphic Pappus theorem” in [FLTO02]), and Orevkov and Shustin (see [OS02] and [OS03]) exhibit nonsingular  $\mathcal{L}$ -schemes on the second rational geometrically ruled surface  $\Sigma_2$  which are realizable pseudoholomorphically but not algebraically. One can note that in our definition of pseudoholomorphic curves on ruled surfaces, (which is the same than Orevkov and Shustin’s one), we consider only almost complex structures on  $\Sigma_n$  for which the exceptional divisor (if any) is pseudoholomorphic (see section 2). Forgetting this condition, J-Y. Welschinger constructed in [Wel02] examples of real pseudoholomorphic curves on  $\Sigma_n$  for  $n \geq 2$  which are not isotopic to any real algebraic curve realizing the same homology class.

However in the case of nonsingular curves on  $\mathbb{R}P^2$ , for each known algebraic classification, the pseudoholomorphic classification is the same (and even the proofs for both classifications are alike!). Thus, Theorem 1.1 is still true replacing ”algebraic” by ”pseudoholomorphic”. The real symplectic isotopy problem turns out to be difficult. So, one can tackle a simpler question, looking for example at complex curves which admit more symmetries than the action of  $\mathbb{Z}/2\mathbb{Z}$  given by the complex conjugation. The first natural action is an additional holomorphic action of  $\mathbb{Z}/2\mathbb{Z}$ , which can be given by a symmetry of the projective plane (see section 2.4). Such a real plane curve, invariant under a symmetry, is called a *symmetric curve*. The systematic study of symmetric curves was started by T. Fiedler ([Fie]) and continued by S. Trille ([Tri01], [Tri03]). The *rigid isotopy classes* (two nonsingular curves of degree  $m$  on  $\mathbb{R}P^2$  are said to be rigidly isotopic if they belong to the same connected component of the complement of the discriminant hypersurface in the space of curves of degree  $m$ ) of nonsingular sextics in  $\mathbb{R}P^2$  which contain a symmetric curve can be obtained from [Ite95]. Recently (see [II01]), using auxiliary conics, I. Itenberg and V. Itenberg found an elementary proof of this classification. Once again, algebraic and pseudoholomorphic classifications coincide. On the other hand, in [OS03], Orevkov and Shustin showed that there exists a real scheme which is realizable by nonsingular symmetric pseudoholomorphic  $M$ -curves of degree 8, but which is not realizable by real symmetric algebraic curves of degree 8.

Hence, it is natural to wonder about the degree 7 and this is the subject of this paper. It turns out that the classification of real schemes which are realizable by nonsingular symmetric curves of degree 7 on  $\mathbb{R}P^2$  are the same in both algebraic and pseudoholomorphic cases, as well as the classification of *complex schemes* which are realizable by nonsingular symmetric  $M$ -curves of degree 7 on  $\mathbb{R}P^2$  (Theorem 1.4 and Corollary 1.6). However, if we look at real schemes which are realizable by *nonsingular dividing symmetric curves* of degree 7 on  $\mathbb{R}P^2$ , the answers are different. In Theorems 1.7 and 1.8, we exhibit two real schemes which are realizable by real symmetric dividing pseudoholomorphic curves of degree 7 on  $\mathbb{R}P^2$  but not by algebraic ones.

We state our classification results in the next subsection. In section 2, we give some definitions and properties of the objects used in this paper (*rational geometrically ruled surfaces, braid associated to an  $\mathcal{L}$ -scheme*). The algebraic prohibitions are obtained by means of *real trigonal graphs* and *combs* associated to an  $\mathcal{L}$ -scheme. We present these objects and their link with real algebraic trigonal curves in section 3. We give there an efficient algorithm to deal with combs. In section 4, we give results related to the pseudoholomorphic category. Algebraic results are given in section 5.

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## 1.2 Classification results

First we state two simple lemmas. Surprisingly, we did not succeed to find these statements in the literature. The proof of these prohibitions is a straightforward application of the Bezout theorem, the Fiedler orientation alternating rule (see [Vir84b]) and the Rokhlin-Mischachev orientation formula (see [Rok74] or section 2). All the constructions are performed in [Sou01] and [FLT97]. A real algebraic (or pseudoholomorphic) curve is said to be a *dividing* curve if the set of real points of the curve disconnects the set of complex points.

**Lemma 1.2** *Any nonsingular dividing real pseudoholomorphic curve of degree 7 on  $\mathbb{R}P^2$  has one of the following real schemes :*

- $\langle J \amalg \alpha \amalg 1 \langle \beta \rangle \rangle$  with  $\alpha + \beta \leq 14$ ,  $\alpha + \beta = 0 \pmod{2}$ ,  $0 \leq \alpha \leq 13$ ,  $1 \leq \beta \leq 13$ , if  $\alpha = 0$  then  $\beta \neq 2, 6, 8$  and if  $\alpha = 1$  then  $\beta \geq 5$ ,
- $\langle J \amalg \alpha \rangle$  with  $7 \leq \alpha \leq 15$ ,  $\alpha = 1 \pmod{2}$ ,
- $\langle J \amalg 1 \amalg 1 \langle 1 \langle 1 \rangle \rangle \rangle$ .

Moreover, any of these real schemes is realizable by nonsingular dividing real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .

**Lemma 1.3** *Any nonsingular non-dividing real pseudoholomorphic curve of degree 7 on  $\mathbb{R}P^2$  has one of the following real schemes :*

- $\langle J \amalg \alpha \amalg 1 \langle \beta \rangle \rangle$  with  $\alpha + \beta \leq 13$ ,  $0 \leq \alpha \leq 12$ ,  $1 \leq \beta \leq 13$ ,
- $\langle J \amalg \alpha \rangle$  with  $0 \leq \alpha \leq 14$ .

Moreover, any of these real schemes is realizable by nonsingular non-dividing real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .

We now introduce *symmetric curves*. Denote by  $s$  the holomorphic involution of  $\mathbb{C}P^2$  given by  $[x : y : z] \mapsto [x : -y : z]$ . A real curve on  $\mathbb{R}P^2$  is called *symmetric* if  $s(A) = A$ .

Here we state the five main classifications of this article.

**Theorem 1.4** *The following real schemes are not realizable by nonsingular symmetric real pseudoholomorphic curves of degree 7 on  $\mathbb{R}P^2$  :*

- $\langle J \amalg (14 - \alpha) \amalg 1 \langle \alpha \rangle \rangle$  with  $\alpha = 6, 7, 8, 9$ ,
- $\langle J \amalg (13 - \alpha) \amalg 1 \langle \alpha \rangle \rangle$  with  $\alpha = 6, 7, 9$ .

Moreover, any other real scheme realizable by nonsingular real algebraic curves of degree 7 on  $\mathbb{R}P^2$  is realizable by nonsingular symmetric real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .

*Proof.* The pseudoholomorphic prohibitions are proved in Propositions 4.6 and 4.7. All the other curves are constructed algebraically in Propositions 5.7, 5.12, 5.13, and Corollary 5.11.  $\square$

If  $C$  is a symmetric curve of degree 7, the quotient curve  $C/s$  is a curve of bidegree  $(3, 1)$  on  $\Sigma_2$  which has a special position with respect to the base  $\{y = 0\}$  (see section 2.4). In the case of  $M$ -curves of degree 7, we give a classification of the possible mutual arrangements of those two curves.

As explained in section 2.1, the real part of  $\Sigma_2$  is a torus. In Figure 1, the rectangles with parallel edges identified according to the arrows represent  $\mathbb{R}\Sigma_2$ . The two horizontal edges represent the real

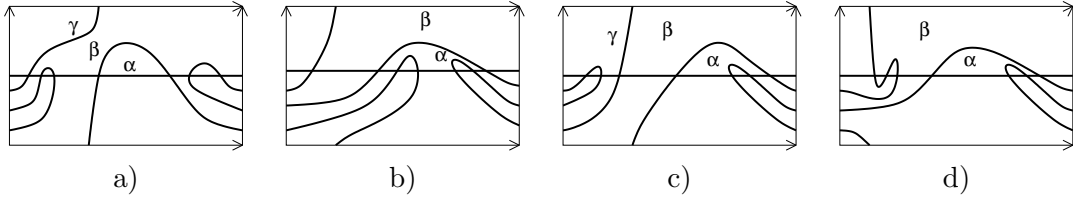


Figure 1:

part of the exceptional divisor  $E$ , and the two vertical edges represent the real part of a fiber. An  $\mathcal{L}$ -scheme is a collection of embedded circles in  $\mathbb{R}\Sigma_n$  up to isotopy which respects the pencil of lines in  $\mathbb{R}\Sigma_n$  (see section 2.1). In the  $\mathcal{L}$ -schemes depicted in Figure 1, the numbers  $\alpha, \beta$  and  $\gamma$  represent as many ovals lying all outside each other.

**Theorem 1.5** *Let  $C$  be a nonsingular symmetric pseudoholomorphic  $M$ -curve of degree 7 on  $\mathbb{R}P^2$ . Then the  $\mathcal{L}$ -scheme realized by the union of its quotient curve  $C/s$  and the base  $\{y = 0\}$  in  $\Sigma_2$  is one of those depicted in*

- Figure 1a) with  $(\alpha, \beta, \gamma) = (1, 5, 0), (5, 1, 0), (0, 1, 5)$  and  $(0, 5, 1)$ ,
- Figures 1b) and d) with  $(\alpha, \beta) = (1, 5)$  and  $(5, 1)$ ,
- Figure 1c) with  $(\alpha, \beta, \gamma) = (2, 4, 0), (6, 0, 0), (0, 0, 6)$  and  $(0, 4, 2)$ .

Moreover, all these arrangements are realizable by the quotient curves of nonsingular symmetric algebraic  $M$ -curves of degree 7 on  $\mathbb{R}P^2$ .

*Proof.* Let  $C$  be a symmetric  $M$ -curve of degree 7 on  $\mathbb{R}P^2$ . According to Theorem 2.12, Proposition 2.11, the Bezout theorem, Lemmas 4.1 and 2.5, the only possibilities for the  $\mathcal{L}$ -schemes realized by the union of the quotient curve of  $C$  and the base  $\{y = 0\}$  are depicted in Figures 15a) and c) with  $\alpha + \beta + \gamma = 6$  and in Figures 15b) and d) with  $\alpha + \beta = 6$ . Now the prohibitions follow from Lemmas 4.2, 4.3, 4.4 and 4.5. All the constructions are performed in Proposition 5.7.  $\square$

**Remark.** Theorem 1.5 has an immediate consequence : if a symmetric  $M$ -curve of degree 7 has a nest (see section 2.2) and at least 2 inner ovals, then the  $\mathcal{L}$ -scheme realized by the union of its quotient curve and the base  $\{y = 0\}$  in  $\Sigma_2$  is uniquely determined by the real scheme of the symmetric curve. If a symmetric  $M$ -curve has a nest and only one inner oval, or has no nest, then there are two possibilities for the  $\mathcal{L}$ -scheme realized by the union of its quotient curve and the base  $\{y = 0\}$ .

This latter theorem allows us to classify the complex schemes realized by symmetric  $M$ -curves of degree 7. The notations used to encode complex schemes are the usual ones proposed in [Vir84b].

**Corollary 1.6** *A complex scheme is realizable by nonsingular symmetric real algebraic (or pseudoholomorphic)  $M$ -curves of degree 7 on  $\mathbb{R}P^2$  if and only if it is contained in the following list :*

$$\begin{array}{lll}
 \langle J \amalg 9_+ \amalg 6_- \rangle_I & \langle J \amalg 4_+ \amalg 6_- \amalg 1_- \langle 3_+ \amalg 1_- \rangle \rangle_I & \langle J \amalg 2_- \amalg 1_- \langle 7_+ \amalg 5_- \rangle \rangle_I \\
 \langle J \amalg 7_+ \amalg 6_- \amalg 1_- \langle 1_+ \rangle \rangle_I & \langle J \amalg 5_+ \amalg 4_- \amalg 1_- \langle 3_+ \amalg 2_- \rangle \rangle_I & \langle J \amalg 1_+ \amalg 1_- \langle 7_+ \amalg 6_- \rangle \rangle_I \\
 \langle J \amalg 5_+ \amalg 7_- \amalg 1_- \langle 2_+ \rangle \rangle_I & \langle J \amalg 1_+ \amalg 3_- \amalg 1_- \langle 6_+ \amalg 4_- \rangle \rangle_I & \\
 \langle J \amalg 6_+ \amalg 5_- \amalg 1_+ \langle 1_+ \amalg 2_- \rangle \rangle_I & \langle J \amalg 2_+ \amalg 1_- \amalg 1_+ \langle 5_+ \amalg 6_- \rangle \rangle_I & 
 \end{array}$$

*Proof.* If  $C$  is a nonsingular dividing symmetric curve of degree 7 in  $\mathbb{R}P^2$ , the  $\mathcal{L}$ -scheme realized by the union of its quotient curve and the base  $\{y = 0\}$  determines uniquely the complex orientations of the initial symmetric curve (see section 2.4). Now the corollary follows from Theorem 1.5.  $\square$

Looking at dividing symmetric curves of degree 7, the pseudoholomorphic and the algebraic classifications differ.

**Theorem 1.7 (Pseudoholomorphic classification)** *The following real schemes are not realizable by nonsingular dividing symmetric real pseudoholomorphic curves of degree 7 on  $\mathbb{R}P^2$  :*

$$\langle J \amalg 2 \amalg 1 \langle 10 \rangle \rangle, \langle J \amalg 6 \amalg 1 \langle 6 \rangle \rangle \text{ and } \langle J \amalg 4 \amalg 1 \langle 4 \rangle \rangle.$$

Moreover, any other real scheme mentioned in Lemma 1.2 and not forbidden by Theorem 1.4 is realizable by nonsingular dividing symmetric real pseudoholomorphic curves of degree 7 on  $\mathbb{R}P^2$ ; any real scheme mentioned in Lemma 1.3 which is not forbidden by Theorem 1.4 is realizable by non-dividing nonsingular symmetric real pseudoholomorphic curves of degree 7 on  $\mathbb{R}P^2$ .

*Proof.* All the pseudoholomorphic prohibitions are proved in Propositions 4.13 and 4.15. All the constructions are done in Propositions 5.8, 5.17, 5.18, 5.19 and in Corollary 5.10, and 4.19.  $\square$

**Theorem 1.8 (Algebraic classification)** *The real schemes*

$$\langle J \amalg 8 \amalg 1 \langle 4 \rangle \rangle \text{ and } \langle J \amalg 4 \amalg 1 \langle 8 \rangle \rangle$$

*are not realizable by a nonsingular dividing symmetric real algebraic curve of degree 7 on  $\mathbb{R}P^2$ . Any other real scheme which is realizable by nonsingular dividing symmetric real pseudoholomorphic curves of degree 7 on  $\mathbb{R}P^2$  is realizable by nonsingular dividing symmetric real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .*

*Any real scheme which is realizable by non-dividing nonsingular symmetric real pseudoholomorphic curves of degree 7 on  $\mathbb{R}P^2$  is realizable by non-dividing nonsingular symmetric real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .*

*Proof.* The two algebraic prohibitions are proved in Propositions 4.14, 5.3 and 5.6. All the constructions are done in Propositions 5.8, 5.17, 5.18, 5.19, and in Corollary 5.10.  $\square$

## 2 Preliminaries

### 2.1 Rational geometrically ruled surfaces

Let us define the  $n^{\text{th}}$  rational geometrically ruled surface, denoted by  $\Sigma_n$ , the surface obtained by taking four copies of  $\mathbb{C}^2$  with coordinates  $(x, y)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$  and  $(x_4, y_4)$ , and by gluing them along  $(\mathbb{C}^*)^2$  with the identifications  $(x_2, y_2) = (1/x, y/x^n)$ ,  $(x_3, y_3) = (x, 1/y)$  and  $(x_4, y_4) = (1/x, x^n/y)$ . Define on  $\Sigma_n$  the algebraic curve  $E$  (resp.  $B$  and  $F$ ) by the equation  $\{y_3 = 0\}$  (resp.  $\{y = 0\}$  and  $\{x = 0\}$ ). The coordinate system  $(x, y)$  is called *standard*. The projection  $\pi : (x, y) \mapsto x$  on  $\Sigma_n$  defines a  $\mathbb{C}P^1$ -bundle over  $\mathbb{C}P^1$ . The intersection numbers of  $B$  and  $F$  are  $B \circ B = n$ ,  $F \circ F = 0$  and  $B \circ F = 1$ . The surface  $\Sigma_n$  has a natural real structure induced by the complex conjugation on  $\mathbb{C}^2$ , and the real part of  $\Sigma_n$  is a torus if  $n$  is even and a Klein bottle if  $n$  is odd. The restriction of  $\pi$  on  $\mathbb{R}\Sigma_n$  defines a pencil of lines denoted by  $\mathcal{L}$ .

The group  $H_2(\Sigma_n; \mathbb{Z})$  is isomorphic to  $\mathbb{Z} \times \mathbb{Z}$  and is generated by the classes of  $B$  and  $F$ . Moreover, one has  $E = B - nF$ . An algebraic or pseudoholomorphic curve on  $\Sigma_n$  is said to be of *bidegree*  $(k, l)$  if it realizes the homology class  $kB + lF$  in  $H_2(\Sigma_n; \mathbb{Z})$ .

**Definition 2.1** *A curve of bidegree  $(1, 0)$  is called a base.  
A curve of bidegree  $(3, 0)$  is called a trigonal curve.*

In the rational geometrically ruled surfaces, we study real curves up to *isotopy with respect to  $\mathcal{L}$* . Two curves are said to be isotopic with respect to the fibration  $\mathcal{L}$  if there exists an isotopy of  $\mathbb{R}\Sigma_n$  which brings the first curve to the second one, and which maps each line of  $\mathcal{L}$  to another line of  $\mathcal{L}$ . The work of Gromov (see [Gro85]) shows that the following proposition, trivial in the algebraic case, it is also true in the pseudoholomorphic case.

**Proposition 2.2** *There exists a unique base of  $\Sigma_2$  which passes through 3 generic points.*

An arrangement of circles  $A$ , which may be nodal, in  $\mathbb{R}\Sigma_n$  up to isotopy of  $\mathbb{R}\Sigma_n \setminus E$  which respects the pencil of lines  $\mathcal{L}$  is called an  $\mathcal{L}$ -*scheme*.

In this paper, we represent  $\mathbb{R}\Sigma_n$  as a rectangle whose parallel edges are identified according to the arrows. The two horizontal edges represent the real part of the exceptional divisor  $E$ , and the two vertical edges represent the real part of a fiber. In the  $\mathcal{L}$ -schemes depicted in this paper, the numbers  $\alpha, \beta$  and  $\gamma$  represent as many ovals lying all outside each other.

## 2.2 Real curves

A *real pseudoholomorphic curve*  $C$  on  $\mathbb{C}P^2$  or  $\Sigma_n$  is an immersed Riemann surface which is a  $J$ -holomorphic curve in some tame almost complex structure  $J$  such that the exceptional section (in  $\Sigma_n$  with  $n \geq 1$ ) is  $J$ -holomorphic (see [Gro85]),  $conj(C) = C$ , and  $conj_* \circ J_p = -J_p \circ conj_*$ , where  $conj$  is the complex conjugation and  $p$  is any point of  $C$ . If the curve  $C$  is immersed in  $\mathbb{C}P^2$  (resp.  $\Sigma_n$ ) and realizes the homology class  $d[CP^1]$  in  $H_2(\mathbb{C}P^2; \mathbb{Z})$  (resp.  $kB + lF$  in  $H_2(\Sigma_n; \mathbb{Z})$ ),  $C$  is said to be of degree  $d$  (resp. of bidegree  $(k, l)$ ). For example, a real algebraic curve of degree  $d$  on  $\mathbb{C}P^2$  is a real pseudoholomorphic curve of degree  $d$  for the standard complex structure. All intersections of two  $J$ -holomorphic curves are positive, so the Bezout theorem is still true for two  $J$ -holomorphic curves.

Gromov pointed out that given 2 (resp. 5) given generic points in  $\mathbb{C}P^2$ , there exists a unique  $J$ -line (resp.  $J$ -conic) passing through these points. As soon as the degree is greater than 3, such a statement is no more true. This is a direction to find some difference between algebraic and pseudoholomorphic curves (see [FLTO02] where the authors use pencils of cubics).

From now on, we state results about "curves" if they are valid in both cases, algebraic and pseudoholomorphic. A nonsingular real curve of genus  $g$  is called an  $(M - i)$ -*curve* if its real part has  $g + 1 - i$  connected components. If  $i = 0$ , we simply speak about an  $M$ -*curve* or *maximal curve*. A connected component of the real part of a nonsingular real curve on a real surface is called an *oval* if it is contractible in the surface, and is called a *pseudo-line* otherwise. In  $\mathbb{R}P^2$  the complement of an oval is formed by two connected components, one of which is homeomorphic to a disk (called the *interior* of the oval) and the other to a Möbius strip (called the *exterior* of the oval). Two ovals in  $\mathbb{R}P^2$  are said to constitute an *injective pair* if one of them is enclosed by the other. A set of ovals such that any two ovals of this set form an injective pair is called a nest. For a nonsingular real curve of degree 7 with a nest of depth 2, the oval which contains some other ovals is called the *non-empty oval* of the curve. The ovals lying inside this oval are called the *inner ovals* while those lying outside are called the *outer ovals*.

A nonsingular real curve  $C$  is a 2-dimensional manifold and  $C \setminus \mathbb{R}C$  is either connected or it has two connected components. In the former case, we say that  $\mathbb{R}C$  is a *non dividing curve*, or of type *II*, and in the latter case, we say that  $\mathbb{R}C$  is a *dividing curve*, or of type *I*.

Let  $C$  be a dividing curve. The two halves of  $C \setminus \mathbb{R}C$  induce two opposite orientations on  $\mathbb{R}C$  which are called *complex orientations* of the curve. Fix such a complex orientation of  $\mathbb{R}C$ . An injective pair of ovals in  $\mathbb{R}P^2$  is called *positive* if the orientations of the two ovals are induced by one of the orientations of the annulus in  $\mathbb{R}P^2$  bounded by the two ovals, and *negative* otherwise. Now suppose that  $C$  is of odd degree. If the integral homology classes realized by the odd component of the curve and an oval in the Möbius strip defined by the exterior of this oval have the same sign, we say that this oval is *negative*, and *positive* otherwise. Denote by  $\Pi_+$  (resp.  $\Pi_-$ ) the number of positive (resp. negative) injective pairs of ovals and by  $\Lambda_+$  (resp.  $\Lambda_-$ ) the number of positive (resp. negative) ovals.

**Proposition 2.3 (Rokhlin-Mischachev's orientation formula, [Rok74], [Mis75])** *If  $C$  is a dividing nonsingular curve of degree  $2k + 1$  on  $\mathbb{R}P^2$  with  $l$  ovals, then*

$$\Lambda_+ - \Lambda_- + 2(\Pi_+ - \Pi_-) = l - k(k + 1)$$

In this paper, we will use the the Fiedler orientations alternating rule (see [Vir84b], [Fie83]) to determine complex orientations of dividing curves in  $\Sigma_n$ .

The following fact about curves on toric surfaces is well known (see, for example, [Ful93]).

**Proposition 2.4** *Let  $C$  be a nonsingular real curve with Newton polygon  $\Delta$  on a toric surface. Then the genus of  $C$  is equal to the number of integer points in the interior of  $\Delta$ .*

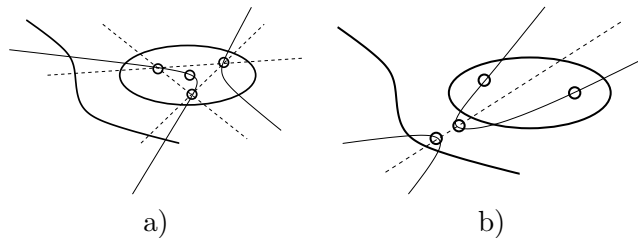


Figure 2:

Given a curve of odd degree on  $\mathbb{R}P^2$ , one can speak about convexity : the segment defined by two points  $a$  and  $b$  is the connected component of the line  $(a, b) \setminus \{a, b\}$  which has an even number of intersection points with the odd component of the curve.

**Lemma 2.5** *Let  $C$  be a real curve of degree 7 with at least 6 ovals and a nest.*

*Pick a point in each inner oval. Then these points are the vertices of a convex polygon in  $\mathbb{R}P^2$ .*

*Moreover, if a line  $L$  passes through two outer ovals  $O_1$  and  $O_2$  and separates the inner ovals, then  $O_1$  and  $O_2$  are not on the same connected component of  $L \setminus (\text{Int}(O) \cup \mathcal{J})$  where  $O$  is the non-empty oval.*

*Proof.* Suppose there exist four empty ovals contradicting the lemma as depicted in Figure 2a) and b). Then the conic passing through these ovals and another one intersects the curve in at least 16 points which contradicts the Bezout theorem.  $\square$

### 2.3 Braid theoretical methods

Here we recall the method exposed by Orevkov in [Ore99]. Our setting is a little bit different from [Ore99] since we consider curves which intersect the exceptional divisor. However, all the results stated in [Ore99] remain valid for such curves. The proofs of [Ore99] should be adapted using the



local model  $z = \text{constant}$  for the lines of the pencil, and  $z = \pm \frac{1}{w}$  (resp.  $z = \pm \frac{1}{w^2}$ ) for the curve at each point of the curve which is not (resp. which is) a tangency point with the pencil and which lie on the exceptional divisor.

The *group of braids with  $m$ -strings* is defined as

$$B_m = \langle \sigma_1, \dots, \sigma_{m-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \rangle.$$

We call the *exponent sum* of a braid  $b = \prod_{j=1}^n \sigma_{i_j}^{k_j}$  the integer  $e(b) = \sum_{j=1}^n k_j$ . A braid  $b$  is called *quasipositive* if it is a product of conjugated elements of  $\sigma_1$  in  $B_m$ .

From now on, we fix a base  $B$  of  $\Sigma_n$ , a fiber  $l_\infty$  of  $\mathcal{L}$  and an  $\mathcal{L}$ -scheme  $A$  on  $\Sigma_n$ . Suppose moreover that all the intersection points of  $A$  and  $E$  are nonsingular points of  $A$ , and that  $A$  intersects each fiber in  $m$  or  $m - 2$  real points (counted with multiplicities) and  $l_\infty$  in  $m$  distinct real points. Choose a standard coordinate system on  $\Sigma_n$  such that  $l_\infty$  has equation  $\{x_2 = 0\}$  (see section 2.1) and a trivialization of the  $\mathbb{C}P^1$ -bundle over  $B \setminus (B \cap l_\infty)$ .

Now, we describe how to encode such an  $\mathcal{L}$ -scheme  $A$ . Examine the real part of the fibration from  $x = -\infty$  to  $x = +\infty$ . Each time the pencil of real lines is not transversal to  $A$  or meets an intersection point of  $A$  with  $E$ , do the following :

- if the pencil of lines has a tangency point  $p$  with  $A$  which is not on  $E$ , write  $\supset_k$  if  $A$  intersects a fiber in  $m$  real points before  $p$ , and  $\subset_k$  otherwise,
- if the pencil of lines meets a double point  $p$  of  $A$ , write  $\subset_k \supset_k$  if the tangents are non-real and  $\times_k$  otherwise,
- if the pencil of lines has no tangency point with  $A$  but meets an intersection point  $p$  of  $A$  and  $E$ , write  $/$  if the branch of  $A$  passing through  $p$  lies in the region  $\{y > 0\}$  before  $p$ , and  $\backslash$  otherwise.
- if the pencil of lines has a tangency point  $p$  with  $A$  which is on  $E$ , write  $\backslash \supset_{m-1}$  if  $A$  intersects a fiber in  $m$  real points before  $p$ , and  $\subset_{m-1} /$  otherwise,

In the first two cases  $k$  is the number of real intersection points of the fiber and  $A$  strictly below the point  $p$ , incremented by 1. Thus, we have now an encoding  $s_1 \dots s_r$  of the  $\mathcal{L}$ -scheme  $A$ . In order to obtain a braid from this encoding, perform the following substitutions :

- replace each  $\times_j$  which appears between  $\subset_s$  and  $\supset_t$  by  $\sigma_j^{-1}$ ,
- replace each  $\backslash$  which appears between  $\subset_s$  and  $\supset_t$  by  $\sigma_1 \sigma_2 \dots \sigma_{m-1}$ ,
- replace each  $/$  which appears between  $\subset_s$  and  $\supset_t$  by  $\sigma_{m-1} \sigma_{m-2} \dots \sigma_1$ ,
- replace each subword  $\supset_s \times_{i_{1,1}} \dots \times_{i_{r_1,1}} ?_1 \times_{i_{1,2}} \dots \times_{i_{r_2,2}} ?_2 \dots \times_{i_{1,p}} \dots \times_{i_{r_p,p}} \subset_t$  with  $?_i = \backslash$  or  $/$  by  $\sigma_s^{-1} \tau_{s,m-1} \sigma_{i_{1,1}}^{-1} \dots \sigma_{i_{r_1,1}}^{-1} v_1 \sigma_{i_{1,2}}^{-1} \dots \sigma_{i_{r_2,2}}^{-1} v_2 \dots v_p \sigma_{i_{1,p}}^{-1} \dots \sigma_{i_{r_p,p}}^{-1} \tau_{m-1,t}$ , where

$$v_l = \begin{cases} \sigma_1 \sigma_2 \dots \sigma_{m-3} \sigma_{m-2}^2 & \text{if } ?_l = \backslash, \\ \sigma_{m-2}^{-1} \sigma_{m-1}^2 \sigma_{m-2} \dots \sigma_1 & \text{if } ?_l = /, \end{cases}$$

$$\tau_{s,t} = \begin{cases} (\sigma_{s+1}^{-1} \sigma_s) (\sigma_{s+2}^{-1} \sigma_{s+1}) \dots (\sigma_t^{-1} \sigma_{t-1}) & \text{if } t > s, \\ (\sigma_{s-1}^{-1} \sigma_s) (\sigma_{s-2}^{-1} \sigma_{s-1}) \dots (\sigma_t^{-1} \sigma_{t+1}) & \text{if } t < s, \\ 1 & \text{if } t = s. \end{cases}$$

Then we obtain a braid  $b_{\mathbb{R}}$ . We define the braid associated to the  $\mathcal{L}$ -scheme  $A$ , denoted  $b_A$ , as the braid  $b_{\mathbb{R}}\Delta_m^n$ , where  $\Delta_m$  is the Garside element of  $B_m$  and which is given by

$$\Delta_m = (\sigma_1 \dots \sigma_{m-1})(\sigma_1 \dots \sigma_{m-2}) \dots (\sigma_1 \sigma_2) \sigma_1.$$

**Example.** The encoding and the braid corresponding to the real  $\mathcal{L}$ -scheme on  $\Sigma_2$  depicted in Figure 3 are (we have abbreviated the pattern  $\subset_k \supset_k$  by  $o_k$ ) :

$$\supset_3 o_3^2 \times_1 o_2^2 \times_1^4 / \subset_3 \times_2^2 \supset_3 \subset_3 \quad \text{and} \quad \sigma_3^{-3} \sigma_1^{-1} \sigma_2^{-1} \sigma_3 \sigma_2^{-2} \sigma_3^{-1} \sigma_2 \sigma_1^{-4} \sigma_2^{-1} \sigma_3^2 \sigma_2 \sigma_1^{-2} \sigma_3^{-1} \Delta_4^2$$

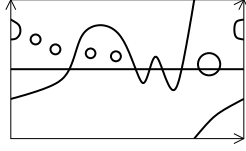


Figure 3:

Orevkov has proved the following results, where  $r$  is the number of (real) intersection points of  $A$  with  $E$ .

**Theorem 2.6 (Orevkov, [Ore99])** *The  $\mathcal{L}$ -scheme  $A$  is realizable by a pseudoholomorphic curve of bidegree  $(m, r)$  if and only if the braid  $b_A$  is quasipositive.*

**Proposition 2.7 (Orevkov, [Ore99])** *Let  $A$  be an  $\mathcal{L}$ -scheme, and  $A'$  be the  $\mathcal{L}$ -scheme obtained from  $A$  by one of the following elementary operations :*

$$\begin{aligned} \times_j \supset_{j\pm 1} &\rightarrow \times_{j\pm 1} \supset_j \quad , \quad \subset_{j\pm 1} \times_j \rightarrow \subset_j \times_{j\pm 1} \quad , \quad \times_j u_k \rightarrow u_k \times_j \quad , \\ \setminus \supset_{m-1} \leftrightarrow / \supset_1 \quad , \quad \subset_{m-1} / \leftrightarrow \subset_1 \setminus \quad , \quad \setminus u_k \leftrightarrow u_k \setminus \quad , \quad / u_k \leftrightarrow u_k / \\ o_k \subset_k \supset_{k-1} &\longleftrightarrow \subset_k \times_{k-1} \supset_k \longleftrightarrow \subset_{k-1} \supset_k o_k \quad , \\ \subset_j \supset_{j\pm 1} &\rightarrow \emptyset \quad , \quad \subset_j \supset_k \rightarrow \supset_k \subset_j \quad , \quad o_j \rightarrow \emptyset \quad , \end{aligned}$$

where  $|k - j| > 1$  and  $u$  stands for  $\times$ ,  $\subset$ , or  $\supset$ .

Then if  $A$  is realizable by a nonsingular real pseudoholomorphic curve of bidegree  $(m, r)$  in  $\Sigma_n$ , so is  $A'$ .

Moreover, if  $A$  is realizable by a dividing pseudoholomorphic curve of bidegree  $(m, r)$  and if  $A'$  is obtained from  $A$  by one of the previous elementary operations but the last one, so is  $A'$ .

In this paper, we use the following tests to show that a given braid is not quasipositive. Propositions 2.8 and 2.9 are a corollary of the Murasugi-Tristram inequality (see [Ore99]) and Proposition 2.10 is a corollary of the generalized Fox-Milnor theorem (see [Ore02a]).

**Proposition 2.8 (Orevkov, [Ore99])** *If a braid  $b$  in  $B_m$  is quasipositive and  $e(b) < m - 1$ , then the Alexander polynomial of  $b$  is identically zero.*

**Proposition 2.9 (Orevkov, [Ore99])** *If a braid  $b$  in  $B_m$  is quasipositive and  $e(b) = m - 1$ , then all the roots of the Alexander polynomial of  $b$  situated on the unit circle are of order at least two.*

**Proposition 2.10 (Orevkov, [Ore02a])** *If a braid  $b$  in  $B_m$  is quasipositive and  $e(b) = m - 1$ , then  $\det(b)$  is a square in  $\mathbb{Z}$ .*

One can refer to [Lic97] or [Ore99] for the definitions of the Alexander polynomial and the determinant of a braid.

## 2.4 General facts about symmetric curves on the real plane

In this article, we will use two results by Fiedler and one result by Trille. As these results have never been published before, we give here an outline of their proof.

We denote by  $B_0$  the line  $\{y = 0\}$ .

**Proposition 2.11 (Fiedler, [Fie])** *If  $C$  is a dividing symmetric curve of degree  $d$  on  $\mathbb{R}P^2$ , then*

$$\text{Card}(\mathbb{R}C \cap B_0) = d \text{ or } \text{Card}(\mathbb{R}C \cap B_0) = 0$$

*Proof.* The involution  $s$  acts locally on  $C$  at all its fixed points like a rotation. Suppose that  $p$  is a point of  $\mathbb{R}C \cap B_0$ . Then,  $p$  is a fixed point of  $s$  and  $s$  exchanges the two halves of  $C \setminus \mathbb{R}C$  locally at  $p$ . Hence,  $s$  exchanges globally the two halves of  $C \setminus \mathbb{R}C$ . So  $s$  cannot have any non real fixed point and the proposition is proved.  $\square$

Thus, in the case of a dividing symmetric curve of odd degree, all the common points of the curve and  $B_0$  are real.

The involutions  $s$  and  $\text{conj}$  commute, so  $s \circ \text{conj}$  is an anti-holomorphic involution on  $\mathbb{C}P^2$ . The real part of this real structure is a real projective plane  $\widetilde{\mathbb{R}P^2} = \{[x_0 : ix_1 : x_2] \in \mathbb{C}P^2 \mid (x_0, x_1, x_2) \in \mathbb{R}^3 \setminus \{0\}\}$ , and it is clear that  $\widetilde{\mathbb{R}P^2} \cap \mathbb{R}P^2 = \mathbb{R}B_0 \cup \{[0 : 1 : 0]\}$ . A symmetric curve  $C$  is real for the structures defined by  $\text{conj}$  and  $s \circ \text{conj}$ . Denote  $\widetilde{\mathbb{R}C}$  its real part for  $s \circ \text{conj}$ . This is called the *mirror curve* of  $\mathbb{R}C$ .

For a maximal symmetric curve, the real scheme realized by the mirror curve is uniquely determined.

**Theorem 2.12 (Fiedler, [Fie])** *The mirror curve of a maximal symmetric curve of degree  $2k + 1$  is a nest of depth  $k$  with a pseudo-line.*

*Proof.* Let  $C$  be a maximal symmetric curve of degree  $2k + 1$  and put  $C \setminus \mathbb{R}C = C^+ \cup C^-$ . According to Proposition 2.11, all the fixed points of  $s$  are real, and  $s \circ \text{conj}$  defines an involution of  $C^+$ . Glue some disks to  $C^+$  along  $\mathbb{R}C$  in order to obtain a sphere  $S$  and extend  $s \circ \text{conj}$  to the whole  $S$ . Then, one sees that the map  $s \circ \text{conj}$  is isotopic to a reflexion on  $S$ . Now, the lift of the fixed point set of  $s \circ \text{conj}$  on  $C^+$  to  $C$  shows that  $\widetilde{\mathbb{R}C}$  contains at least  $k$  ovals and a pseudo-line and that these  $k + 1$  components divide  $C$ . So,  $\widetilde{\mathbb{R}C}$  cannot have other components, and according to the Rokhlin-Mischachev orientation formula, it is a nest of depth  $k$  with a pseudo-line.  $\square$

Denote by  $\mathcal{L}_p$  the pencil of lines through the point  $[0 : 1 : 0]$  in  $\mathbb{C}P^2$ . If  $C$  is a real symmetric curve of degree  $2k + 1$  on  $\mathbb{R}P^2$ , the curve  $X = C/s$  is a real curve of bidegree  $(k, 1)$  on  $\Sigma_2$  and is called the *quotient curve* of  $C$ . The  $\mathcal{L}$ -scheme realized by  $\mathbb{R}X$  is obtained by gluing the  $\mathcal{L}_p$ -schemes realized by  $\mathbb{R}C/s$  and  $\widetilde{\mathbb{R}C}/s$  along  $B_0$ . Since there is no ambiguity, we will denote by  $B_0$  the (symmetric) line  $\{y = 0\}$  in  $\mathbb{C}P^2$  as well as its quotient curve in  $\Sigma_2$ .

Conversely, a symmetric curve of degree  $2k + 1$  is naturally associated to any arrangement of a curve  $X$  of bidegree  $(k, 1)$  and a base on  $\Sigma_2$ .

**Proposition 2.13** *Let  $C$  be a nonsingular real pseudoholomorphic symmetric curve on  $\mathbb{R}P^2$ . Then  $C$  is smoothly and equivariantly isotopic to a nonsingular real pseudoholomorphic symmetric curve  $C'$  on  $\mathbb{R}P^2$  such that all the tangency points of the invariant components of  $\widetilde{\mathbb{R}C}'$  with the pencil of lines  $\mathcal{L}_p$  lie on  $B_0$ .*

*Proof.* Suppose that  $\widetilde{\mathbb{R}C}'$  has tangency points with a line of  $\mathcal{L}_p$  not on  $B_0$ . Then push all the corresponding tangent points of the quotient curve above  $B_0$  applying the first elementary operation

of Proposition 2.7 to the curve  $\widetilde{\mathbb{R}C'} \cup B_0$ . The resulting symmetric curve satisfies the conditions of the proposition.  $\square$

**Example.** The symmetric curves of degree 4 depicted in Figures 4a) and d) are equivariantly isotopic in  $\mathbb{C}P^2$ . The dashed curve represent their mirror curves. The corresponding quotient curves are depicted in Figures 4b) and c).

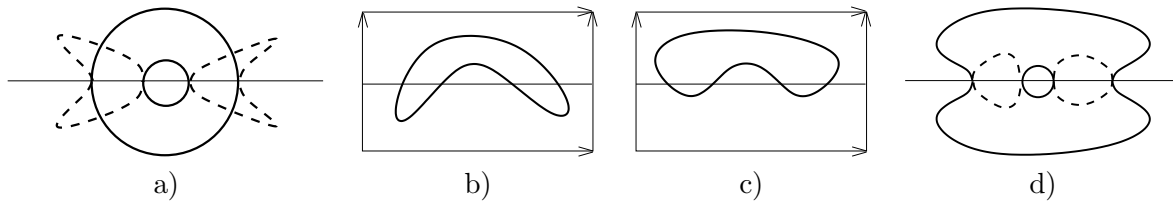


Figure 4:

The curves  $\mathbb{R}C$  and  $\mathbb{R}C'$  have the same complex orientations. Using the invariant components of  $\widetilde{\mathbb{R}C'}$ , one can apply the Fiedler orientations alternating rule (see [Vir84b]) to  $\mathbb{R}C'$ . The following proposition is an application of this observation.

**Proposition 2.14** *If  $C$  is a dividing symmetric curve of odd degree on  $\mathbb{R}P^2$ , then an oval of  $\mathbb{R}C$  and an oval of  $\widetilde{\mathbb{R}C}$  cannot intersect in more than 1 point.*

*Proof.* Suppose that a curve  $C$  contradicts the lemma and denote by  $O$  (resp.  $\tilde{O}$ ) an invariant oval of  $\mathbb{R}C$  (resp.  $\widetilde{\mathbb{R}C}$ ) such that  $O$  and  $\tilde{O}$  have two points in common. By Proposition 2.13, one can suppose that all the tangency points of the invariant components of  $\widetilde{\mathbb{R}C}$  with the pencil of lines  $\mathcal{L}_p$  lie on  $B_0$ .

The homology class realized by  $(O \cup \tilde{O})/s$  in  $H_1(\mathbb{R}\Sigma_2, \mathbb{Z})$  is equal to  $a\mathbb{R}B_0$ . We have  $a = 0$  because otherwise, the quotient curve  $X$  would be singular. Indeed,  $\mathbb{R}X$  is equal to  $b\mathbb{R}B_0 + \mathbb{R}F$  in this group,  $\mathbb{R}B_0 \circ \mathbb{R}B_0 = \mathbb{R}F \circ \mathbb{R}F = 0$  and  $\mathbb{R}B_0 \circ \mathbb{R}F = 1$ . Hence,  $(O \cup \tilde{O})/s$  is an oval of  $\mathbb{R}X$  and  $O$  and  $\tilde{O}$  are arranged in  $\mathbb{C}P^2$  as shown in Figure 5. Then, transport the orientation of  $O$  at  $p_1$  to  $p_2$  using the Fiedler orientations alternating rule. The two orientations are not consistent, so there is a contradiction.  $\square$

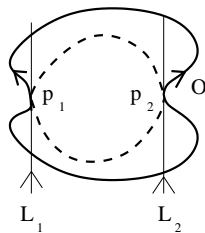


Figure 5:

In general, there is no link between the type of the symmetric curve and the type of its quotient curve. However, if both the symmetric curve and its mirror curve are of type I, there is no ambiguity.

**Proposition 2.15 (Trille, [Tri03])** *If a symmetric curve on  $\mathbb{R}P^2$  and its mirror curve are of type I, so is the quotient curve.*

*Proof.* The set  $C \setminus (\mathbb{R}C \cup \widetilde{\mathbb{R}C})$  has four connected components, so its quotient by  $s$  has two complex conjugated connected components.  $\square$

We finish this section by a simple observation. Denote by  $\pi$  the projection  $\Sigma_2 \rightarrow B_0$  and by  $\pi_p$  the projection  $\mathbb{C}P^2 \setminus \{p\} \rightarrow B_0$ . Then the set  $\pi_p^{-1}(\mathbb{R}B_0) \cap C$  can be deduced from the set  $\pi^{-1}(\mathbb{R}B_0) \cap X$ . Hence some information on the complex orientations of  $C$  (if any) can be deduced from  $\pi^{-1}(\mathbb{R}B_0) \cap X$ . This observation will be very useful in this paper since if  $C$  is of degree 7, the curve  $X$  is of bidegree  $(3, 1)$ . In particular, the set  $\pi^{-1}(\mathbb{R}B_0) \cap X$  can be extracted only from the  $\mathcal{L}$ -scheme realized by  $X$ .

### 3 Real trigonal graphs on $\mathbb{C}P^1$ and real trigonal algebraic curves

In [Ore03], Orevkov reformulates the existence of trigonal real algebraic curves realizing a given  $\mathcal{L}$ -scheme on  $\Sigma_n$  in terms of some colored graphs on  $\mathbb{C}P^1$ . Guided by this article, we give in this section an efficient algorithm to check whether an  $\mathcal{L}$ -scheme is realizable by a trigonal real algebraic curve on  $\Sigma_n$ .

Using these so-called *real trigonal graphs on  $\mathbb{C}P^1$* , Orevkov ([Ore02b]) obtained a classification of trigonal real algebraic curves on  $\Sigma_n$  up to isotopy which respects the pencil of lines, in terms of gluing of cubics.

Note that real trigonal graphs on  $\mathbb{C}P^1$  are a particular case of *real rational graphs* defined in [Bru06] and used to deal with *root schemes*.

#### 3.1 Root scheme associated to a trigonal $\mathcal{L}$ -scheme

**Definition 3.1** *A root scheme is a  $k$ -uplet  $((l_1, m_1), \dots, (l_k, m_k)) \in (\{p, q, r\} \times \mathbb{N})^k$  with  $k$  a natural number (here,  $p, q$  and  $r$  are symbols and do not stand for natural numbers).*

*A root scheme  $((l_1, m_1), \dots, (l_k, m_k))$  is realizable by polynomials of degree  $n$  if there exist two real polynomials in one variable of degree  $n$ , with no common roots,  $P(x)$  and  $Q(x)$  such that if  $x_1 < x_2 < \dots < x_k$  are the real roots of  $P, Q$  and  $P + Q$ , then  $l_i = p$  (resp.,  $q, r$ ) if  $x_i$  is a root of  $P$  (resp.,  $Q, P + Q$ ) and  $m_i$  is the multiplicity of  $x_i$ .*

*The polynomials  $P, Q$  and  $P + Q$  are said to realize the root scheme  $((l_1, m_1), \dots, (l_k, m_k))$ .*

In what follows,  $n$  is a positive integer and  $C(x, y) = y^3 + b_2(x)y + b_3(x)$  is a real polynomial, where  $b_j(x)$  is a real polynomial of degree  $jn$  in  $x$ . By a suitable change of coordinates in  $\Sigma_n$ , any real algebraic trigonal curve on  $\Sigma_n$  can be put into this form.

Denote by  $D = -4b_2^3 - 27b_3^2$  the discriminant of  $C$  with respect to the variable  $y$ . The knowledge of the root scheme realized by  $-4b_2^3, 27b_3^2$  and  $D$  allows one to recover the  $\mathcal{L}$ -scheme realized by  $C$ , up to the transformation  $y \mapsto -y$ . Indeed, the position of  $C$  with respect to the pencil of lines is given by the sign of the double root of  $C(x_0, Y)$  at each root  $x_0$  of  $D$ , which is the sign of  $b_3(x_0)$ .

Consider a trigonal  $\mathcal{L}$ -scheme  $A$  on  $\Sigma_n$  such that  $A$  intersects some fiber  $l_\infty$  in 3 distinct real points. Consider also the encoding  $s_1 \dots s_r$  of  $A$  defined in section 2.3, using the symbols  $\subset, \supset$  and  $\times$ . In this encoding, replace all the occurrences  $\times_k$  by  $\supset_k \subset_k$ . This encoding is denoted by  $r_1 \dots r_q$ . Define a root scheme  $RS_A = (S_1, \dots, S_q)$  where the  $S_i$ 's are sequences of pairs  $(l_1, m_1), \dots, (l_{k_i}, m_{k_i})$ , as follows :

$$\bullet S_1 = \begin{cases} (r, 1) & \text{if } n \text{ is even and } r_1 = \supset_k \text{ and } r_q = \subset_k, \\ & \text{or } n \text{ is odd and } r_1 = \supset_k \text{ and } r_q = \subset_{k \pm 1}, \\ (q, 2), (r, 1) & \text{otherwise,} \end{cases}$$

- for  $i > 1$ ,
 
$$S_i = \begin{cases} (r, 1) & \text{if } r_i = \subset_k \text{ and } r_{i-1} = \supset_k, \\ & \text{or } r_i = \supset_k \text{ and } r_{i-1} = \subset_k, \\ (p, 3), (q, 2), (p, 3), (r, 1) & \text{if } r_i = \supset_k \text{ and } r_{i-1} = \subset_{k\pm 1}, \\ (q, 2), (r, 1) & \text{if } r_i = \subset_k \text{ and } r_{i-1} = \supset_{k\pm 1}. \end{cases}$$

**Definition 3.2** The root scheme  $RS_A$  is called the root scheme associated to  $A$ .

This root scheme encode the mutual cyclic order in  $\mathbb{R}P^1$  realized by the roots of the polynomials  $b_2^3$  (encoded by the letter  $p$ ),  $b_3^2$  (encoded by the letter  $q$ ) and  $D$  (encoded by the letter  $r$ ).

**Example.** The encoding of the  $\mathcal{L}$ -scheme depicted in Figure 13a) is  $\supset_2 \subset_1 \supset_2$   $o_2$   $\subset_2$  and the associated root scheme is

$$\left( (q, 2), (r, 1), (p, 3), (q, 2), (p, 3), (r, 1), (q, 2), (r, 1), (r, 1), (r, 1), (r, 1) \right).$$

The realizability of a root scheme associated to a trigonal  $\mathcal{L}$ -scheme on  $\Sigma_n$  can be studied via what we call real trigonal graphs.

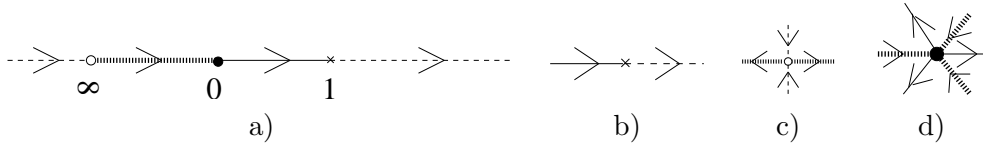


Figure 6:

**Definition 3.3** Let  $\Gamma$  be a graph on  $\mathbb{C}P^1$  invariant under the action of the complex conjugation and  $\pi : \Gamma \rightarrow \mathbb{R}P^1$  a continuous map. Then the coloring and orientation of  $\mathbb{R}P^1$  shown in Figure 6a) defines a coloring and an orientation of  $\Gamma$  via  $\pi$ .

The graph  $\Gamma$  equipped with this coloring and this orientation is called a real trigonal graph of degree  $n$  if

- any vertex of  $\Gamma$  has an even valence,
- for any connected component  $D$  of  $\mathbb{C}P^1 \setminus \Gamma$ , then  $\pi|_{\partial D}$  is a covering of  $\mathbb{R}P^1$ ,
- $\Gamma$  has exactly  $6n$  vertices of the kind depicted in Figure 6b),  $3n$  vertices of the kind depicted on 6c) and  $2n$  vertices of the kind depicted on 6d), and no other non-real multiple points,
- The set  $\pi^{-1}([\infty; 0])$  is connected.

Since all the graphs on  $\mathbb{C}P^1$  considered in this article are invariant under the complex conjugation, we draw only one half of them.

Now, suppose that the trigonal curve  $C$  on  $\Sigma_n$  realizes the  $\mathcal{L}$ -scheme  $A$  and that  $-4b_2^3$ ,  $27b_3^2$  and  $D$  realize the root scheme  $RS_A$ . Color and orient  $\mathbb{R}P^1$  as depicted in Figure 6a). Consider the rational function  $f = \frac{-4b_2^3}{27b_3^2}$  defined on  $\mathbb{C}P^1$  and let  $\Gamma$  be  $f^{-1}(\mathbb{R}P^1) \subset \mathbb{C}P^1$  with the coloring and the orientation induced by those chosen on  $\mathbb{R}P^1$ . Let  $\tilde{\Gamma}$  be the colored and oriented graph on  $\mathbb{C}P^1$  obtained out of  $\Gamma$  by smoothing equivariantly its non-real double points as depicted in Figure 7a), and by performing operations depicted in Figures 7b), c) and d) in order to minimize the number of its real double points. The colored and oriented graph on  $\mathbb{R}P^1$  obtained as the intersection of  $\tilde{\Gamma}$  and  $\mathbb{R}P^1$  can clearly be extracted from  $RS_A$ .

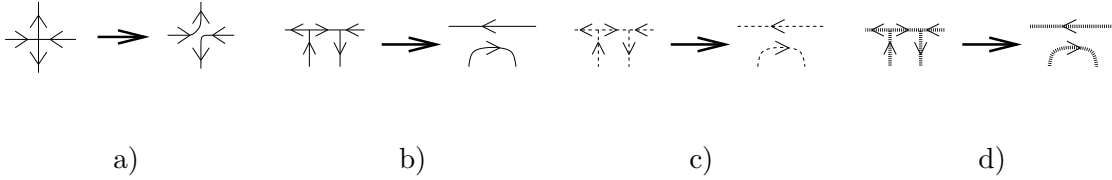


Figure 7:

**Definition 3.4** *The colored and oriented graph on  $\mathbb{R}P^1$  constructed above is called the real graph associated to  $A$ .*

The real graph associated to  $A$  is obtained from  $A$  as depicted in Figure 8 (we omitted the arrows).

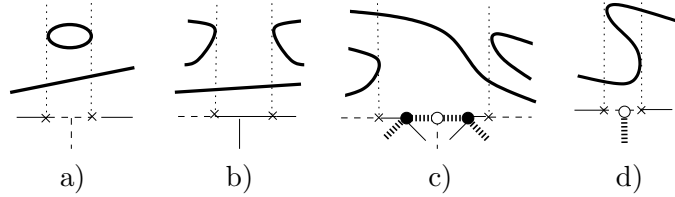


Figure 8:

**Example.** In Figure 9a) we have depicted a trigonal  $\mathcal{L}$ -scheme on  $\Sigma_1$  and its real graph.

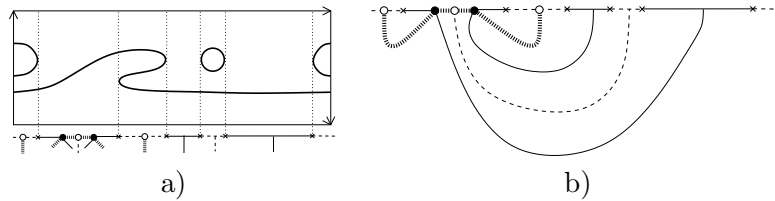


Figure 9:

The importance of the real graph associated to an  $\mathcal{L}$ -scheme is given by the following theorem.

**Theorem 3.5** *Let  $A$  be a trigonal  $\mathcal{L}$ -scheme on  $\Sigma_n$  and  $G$  its real graph. Then  $A$  is realizable by a nonsingular trigonal real algebraic curve on  $\Sigma_n$  if and only if there exists a trigonal graph  $\Gamma$  of degree  $n$  such that  $\Gamma \cap \mathbb{R}P^1 = G$ .*

*Proof.* If there exists such a trigonal graph of degree  $n$ , the existence of the desired real algebraic curve is proved in [Ore03] (see also [NSV02]).

Suppose there exists a real algebraic curve  $C$  realizing  $A$ . Define  $f$  and  $\tilde{\Gamma}$  as above. We will perform some operations on one of the halves of  $\mathbb{C}P^1 \setminus \mathbb{R}P^1$ . The final picture will be obtained by gluing the obtained graph with its image under the complex conjugation.

If  $f^{-1}([\infty; 0])$  is not connected, choose  $p$  in  $f^{-1}(0)$  and  $q$  in  $f^{-1}(\infty)$  belonging to different connected components of  $f^{-1}([\infty; 0])$ . If  $p$  and  $q$  belong to the same connected components of  $\tilde{\Gamma}$ , choose  $p$  and  $q$  such that they are connected in  $\tilde{\Gamma}$  by an arc of  $f^{-1}(]0; \infty[)$  and perform on  $\tilde{\Gamma}$  the operation depicted in Figure 10a). Otherwise, choose  $p$  and  $q$  lying on the boundary of one connected components of

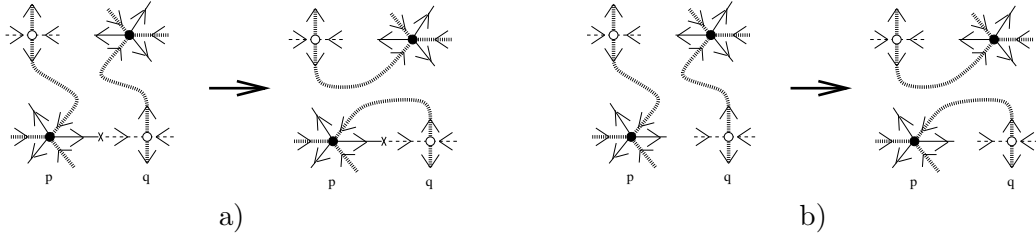


Figure 10:

$\mathbb{C}P^1 \setminus \tilde{\Gamma}$  and perform on  $\tilde{\Gamma}$  the operation depicted in Figure 10b). Perform these operations until  $f^{-1}([\infty; 0])$  is connected.  $\square$

**Remark.** The connectedness of  $\pi^{-1}([\infty; 0])$  is not necessary to the existence of the algebraic curve. We will use this condition only in the next section.

**Example.** In Figure 9b), we exhibit a real trigonal graph which ensures the existence of a nonsingular real trigonal curve in  $\Sigma_1$  realizing the  $\mathcal{L}$ -scheme depicted in Figure 9a).

According to Proposition 2.7, if an  $\mathcal{L}$ -scheme  $A$  is realizable by a pseudoholomorphic curve, so is any  $\mathcal{L}$ -scheme obtained from  $A$  by some  $\subset_j \supset_{j\pm 1} \rightarrow \emptyset$  operations. Orevkov showed ([Ore02b]) that this is not the case for algebraic curves. Nevertheless some operations are allowed on real algebraic trigonal curves.

**Proposition 3.6** *Let  $A$  be an  $\mathcal{L}$ -scheme, and  $A'$  be the  $\mathcal{L}$ -scheme obtained from  $A$  by one of the following elementary operations :*

$$\supset_j \subset_{j\pm 1} \supset_j \rightarrow \supset_j \quad , \quad \subset_j \supset_{j\pm 1} \subset_j \rightarrow \subset_j .$$

*Then if  $A$  is realizable by a nonsingular real algebraic trigonal curve in  $\Sigma_n$ , so is  $A'$ .*

*Proof.* Suppose that  $A$  is realizable by a nonsingular real algebraic trigonal curve in  $\Sigma_n$ . Let  $\Gamma$  be a trigonal graph of degree  $n$  as in Theorem 3.5. The first operation corresponds to the operation depicted in Figure 11 performed on  $\Gamma$ . The second operation is symmetric to the first one.

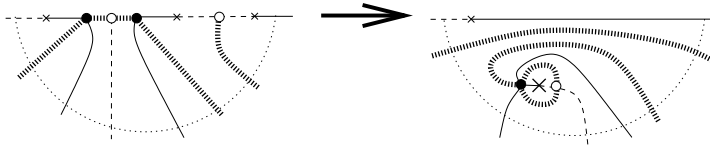


Figure 11:

Proposition 3.6 follows now from Theorem 3.5.  $\square$

### 3.2 Comb theoretical method

In this section, we reformulate in an algorithmic way the following problem : given a trigonal  $\mathcal{L}$ -scheme  $A$ , does there exist a real trigonal graph  $\Gamma$  of degree  $n$  such that  $\Gamma \cap \mathbb{R}P^1$  is the real graph of  $A$ ?

Denote by  $\mathfrak{M}$  the semigroup generated by the elements  $g_1, \dots, g_6$  in  $\mathbb{R}^2$  depicted in Figure 12. The multiplication of two elements  $\mathfrak{m}_1$  and  $\mathfrak{m}_2$  in  $\mathfrak{M}$  is the attachment of the right endpoint of  $\mathfrak{m}_1$  to the left endpoint of  $\mathfrak{m}_2$ .



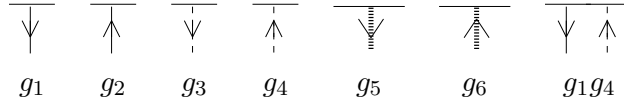


Figure 12:

**Definition 3.7** *The elements of  $\mathfrak{M}$  are called combs.*

For example, the comb  $g_1g_4$  is depicted in Figure 12. The unit element of  $\mathfrak{M}$  is denoted by 1.

**Definition 3.8** *A weighted comb is a quadruplet  $(\mathbf{m}, \alpha, \beta, \gamma)$  in  $\mathfrak{M} \times \mathbb{Z}^3$ .*

Consider a trigonal  $\mathcal{L}$ -scheme  $A$  on  $\Sigma_n$  which intersects some fiber  $l_\infty$  in 3 distinct real points. Consider also the encoding  $r_1 \dots r_q$  of  $A$  defined in section 3.1, using the symbols  $\subset, \supset$ . Define the weighted combs  $(\mathbf{m}_i, \alpha_i, \beta_i, \gamma_i)$  as follows :

- $(\mathbf{m}_0, \alpha_0, \beta_0, \gamma_0) = (1, 6n, 3n, 2n)$
  - $(\mathbf{m}_1, \alpha_1, \beta_1, \gamma_1) = \begin{cases} (g_3, 6n - 1, 3n, 2n) & \text{if } n \text{ is even and } r_1 = \supset_k \text{ and } r_q = \subset_k, \\ & \text{or } n \text{ is odd and } r_1 = \supset_k \text{ and } r_q = \subset_{k\pm 1}, \\ (g_5, 6n - 1, 3n - 1, 2n) & \text{otherwise,} \end{cases}$
  - for  $i > 1$ ,
- $$(\mathbf{m}_i, \alpha_i, \beta_i, \gamma_i) = \begin{cases} (\mathbf{m}_{i-1}g_2, \alpha_{i-1} - 1, \beta_{i-1}, \gamma_{i-1}) & \text{if } r_i = \subset_k \text{ and } r_{i-1} = \supset_k, \\ (\mathbf{m}_{i-1}g_3, \alpha_{i-1} - 1, \beta_{i-1}, \gamma_{i-1}) & \text{if } r_i = \supset_k \text{ and } r_{i-1} = \subset_k, \\ (\mathbf{m}_{i-1}g_5, \alpha_{i-1} - 1, \beta_{i-1} - 1, \gamma_{i-1}) & \text{if } r_i = \supset_k \text{ and } r_{i-1} = \subset_{k\pm 1}, \\ (\mathbf{m}_{i-1}g_6g_1g_4g_1g_6, \alpha_{i-1} - 1, \beta_{i-1} - 1, \gamma_{i-1} - 2) & \text{if } r_i = \subset_k \text{ and } r_{i-1} = \supset_{k\pm 1}. \end{cases}$$

**Definition 3.9** *The weighted comb  $(\mathbf{m}_q, \frac{\alpha_q}{2}, \frac{\beta_q}{2}, \frac{\gamma_q}{2})$  is said to be associated to the  $\mathcal{L}$ -scheme  $A$ .*

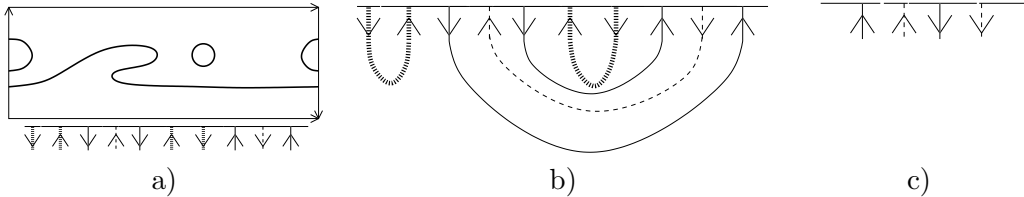


Figure 13:

**Definition 3.10** *Let  $\mathbf{m}$  be a comb. A closure of  $\mathbf{m}$  is a subset of  $\mathbb{R}^2$  obtained by joining each generator  $g_1$  (resp.,  $g_3, g_5$ ) in  $\mathbf{m}$  to a generator  $g_2$  (resp.,  $g_4, g_6$ ) in  $\mathbf{m}$  by a path on  $\mathbb{R}^2$  in such a way that these paths do not intersect.*

*If there exists a closure of  $\mathbf{m}$ , one says that  $\mathbf{m}$  is closed.*

**Example.** The weighted comb associated to the  $\mathcal{L}$ -scheme on  $\Sigma_1$  depicted in Figure 13a) is  $(g_5g_6g_1g_4g_1g_6g_5g_2g_3g_2, 0, 0, 0)$ . A closure of this comb is shown in Figure 13b) (compare with Figure 9). The comb depicted in Figure 13c) is not closed.

**Definition 3.11** *A chain of weighted combs is a sequence  $(w_i)_{1 \leq i \leq k}$  of weighted combs, with  $w_i = (\mathbf{m}_i, \alpha_i, \beta_i, \gamma_i)$ , such that :*

$w_k = (\mathbf{m}, 0, 0, 0)$ , where  $\mathbf{m}$  is a closed comb,

$\forall i \in \{1 \dots k-1\}$ , the weighted comb  $w_{i+1}$  is obtained from  $w_i$  by one of the following operations :

- (1) if  $\gamma_i > 0$  :  $g_2 \rightarrow (g_6 g_1)^2 g_6, \alpha_{i+1} = \alpha_i, \beta_{i+1} = \beta_i, \gamma_{i+1} = \gamma_i - 1$   
or  $g_5 \rightarrow (g_3 g_6)^2 g_3, \alpha_{i+1} = \alpha_i - 3, \beta_{i+1} = \beta_i, \gamma_{i+1} = \gamma_i - 1$   
(2) else if  $\alpha_i > 0$  :  $g_1 \rightarrow g_3, \alpha_{i+1} = \alpha_i - 1, \beta_{i+1} = \beta_i, \gamma_{i+1} = 0$   
(3) else :  $g_5 \rightarrow g_4 g_5 g_4, \alpha_{i+1} = 0, \beta_{i+1} = \beta_i - 1, \gamma_{i+1} = 0$

where  $g_j \rightarrow a$  means “replace one  $g_j$  in  $\mathbf{m}_i$  by  $a$ ”. One says that the chain  $(w_i)_{1 \leq i \leq k}$  starts at  $w_1$ .

**Definition 3.12** Let  $w$  be a weighted comb. The multiplicity of  $w$ , denoted by  $\mu(w)$ , is defined as the number of chains of weighted combs which start at  $w$ .

**Theorem 3.13** Let  $A$  be a trigonal  $\mathcal{L}$ -scheme on  $\Sigma_n$ , and  $w$  its associated weighted comb. Then  $A$  is realizable by nonsingular real algebraic trigonal curves on  $\Sigma_n$  if and only if  $\mu(w) > 0$  or  $w = (1, 6n, 3n, 2n)$ .

*Proof.* Let  $w = (\mathbf{m}, \alpha, \beta, \gamma)$ , and  $G$  be the real graph associated to the  $\mathcal{L}$ -scheme  $A$ . If  $\mathbf{m} = 1$ , it is well known that  $A$  is realizable by a real algebraic trigonal curve on  $\Sigma_n$ . Otherwise, a chain of weighted combs starting at  $w$  is a reformulation of the statement :

”choose a half  $D$  of  $\mathbb{C}P^1 \setminus \mathbb{R}P^1$ ; then there exists a finite sequence  $(G_i)_{0 \leq i \leq k}$  of subsets of  $\mathbb{C}P^1$  such that:

- $G_0 = G$ ,
- for  $i$  in  $\{1, \dots, \gamma\}$ , the subset  $G_i$  is obtained from  $G_{i-1}$  by one of the gluings in  $D$  depicted in Figures 14a) and b); denote by  $c$  the number of times that  $G_i$  is obtained from  $G_{i-1}$  by the gluing depicted in Figure 14b) for  $i$  in  $\{1, \dots, \gamma\}$ ,
- for  $i$  in  $\{\gamma+1, \dots, \alpha+\gamma-3c\}$ , the subset  $G_i$  is obtained from  $G_{i-1}$  by the gluing in  $D$  depicted in Figure 14c),
- for  $i$  in  $\{\alpha+\gamma-3c+1, \dots, k-2\}$ , the subset  $G_i$  is obtained from  $G_{i-1}$  by the gluing in  $D$  depicted in Figure 14d),
- $G_{k-1}$  has no boundary and contains  $G_{k-2}$ ,
- $k = \alpha + \beta + \gamma - 3c + 2$ ,
- $G_k$  is the gluing of  $G_{k-1}$  and its image under the complex conjugation,
- $G_k$  is a trigonal graph such that  $G_k \cap \mathbb{R}P^1 = G$ .

So, according to Theorem 3.5, there exists a chain of weighted combs starting at  $w$  if and only if  $A$  is realizable by a real algebraic trigonal curve on  $\Sigma_n$ .  $\square$

Theorem 3.13 provides an algorithm to check whether an  $\mathcal{L}$ -scheme is realizable by a real algebraic trigonal curve on  $\Sigma_n$ . In order to reduce computations, one can use the following observations.

**Lemma 3.14** Let  $\mathbf{m}$  be a closed comb, and  $\overline{\mathbf{m}}$  one of its closures. Suppose that  $\mathbf{m} = \mathbf{m}_1 g_{i_1} \mathbf{m}_2 g_{i_2} \mathbf{m}_3$ , and that  $g_{i_1}$  and  $g_{i_2}$  are joined in  $\overline{\mathbf{m}}$ . Then the combs  $\mathbf{m}_1 \mathbf{m}_3$  and  $\mathbf{m}_2$  contain the same number of generators  $g_1$  (resp.,  $g_3, g_5$ ) and  $g_2$  (resp.,  $g_4, g_6$ ).

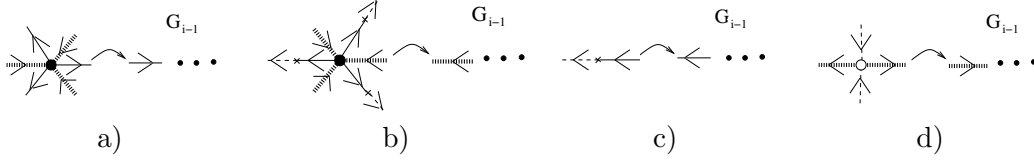


Figure 14:

*Proof.* Straightforward. □

**Lemma 3.15** *Let  $(\mathbf{m}, 0, \beta, 0)$  be an element of a chain of weighted combs. Then it is possible to join each  $g_1$  in  $\mathbf{m}$  to a  $g_2$  in  $\mathbf{m}$  by pairwise non-intersecting paths on  $\mathbb{R}^2$  such that if  $\mathbf{m} = \mathbf{m}_1 g_{i_1} \mathbf{m}_2 g_{i_2} \mathbf{m}_3$  with  $g_{i_1}$  and  $g_{i_2}$  joined, then the combs  $\mathbf{m}_1 \mathbf{m}_3$  and  $\mathbf{m}_2$  contain the same number of generators  $g_1$  and  $g_2$ .*

*Proof.* Straightforward. □

**Lemma 3.16** *Let  $(\mathbf{m}, \alpha, \beta, 0)$  be an element of a chain of weighted combs, where  $\mathbf{m} = \prod_{j=1}^k g_{i_j}$ . Define the equivalence relation  $\sim$  on  $\{j \mid i_j = 1 \text{ or } 2\}$  as follows :*

$$r \sim s \text{ if the cardinal of } \{j \mid r < j < s \text{ and } i_j = 1, 2, 3 \text{ or } 4\} \text{ is odd.}$$

Denote by  $E_1^{\mathbf{m}}$  and  $E_2^{\mathbf{m}}$  the two equivalence classes of  $\sim$ . Then

$$|\text{Card}(E_1^{\mathbf{m}}) - \text{Card}(E_2^{\mathbf{m}})| \leq \alpha.$$

*Proof.* Choose a chain of weighted combs  $(w_i)_{1 \leq i \leq k}$  which contains  $(\mathbf{m}, \alpha, \beta, 0)$ . Let  $(\tilde{\mathbf{m}}, 0, \beta, 0)$  be an element of this chain. Then there exists  $l \in \{1 \dots \alpha\}$  such that

$$\text{Card}(E_1^{\tilde{\mathbf{m}}}) = \text{Card}(E_1^{\mathbf{m}}) - l \text{ and } \text{Card}(E_2^{\tilde{\mathbf{m}}}) = \text{Card}(E_2^{\mathbf{m}}) - \alpha + l.$$

It is obvious that in a closure of  $\tilde{\mathbf{m}}$ , an element of  $E_1^{\tilde{\mathbf{m}}}$  has to be joined to an element of  $E_2^{\tilde{\mathbf{m}}}$ , hence the cardinal of these two sets are equal. □

The algorithm given by Theorem 3.13 improved by Lemmas 3.14, 3.15, and 3.16, is quite efficient. It will allow us in section 5 to prohibit algebraically two  $\mathcal{L}$ -schemes pseudoholomorphically realizable.

## 4 Pseudoholomorphic statements

### 4.1 Prohibitions for curves of bidegree $(3, 1)$ on $\Sigma_2$

We obtain all our results on symmetric curves of degree 7 via the study of the possible quotient curves. Hence, the first idea to prove that a real scheme is not realizable by a symmetric curve is to show that no quotient curve is admissible.

**Lemma 4.1** *Let  $X$  be a real curve of bidegree  $(3, 1)$  on  $\Sigma_2$ , and consider the encoding of the  $\mathcal{L}$ -scheme realized by  $X$ , as defined in section 2.3 using the symbols  $\subset$ ,  $\supset$ ,  $\times$  and  $o$ . Suppose that this encoding contains a subsequence of the form  $o_1^{k_1} o_2^{k_2} o_1^{k_3} o_2^{k_4} \dots o_1^{k_{r-1}} o_2^{k_r} /$ . Then  $k_i = 0$  for  $i \geq 4$  and  $k_1 k_3 = 0$ .*

Moreover, if  $k_1 + k_2 + k_3 \geq 3$ , then  $k_3 = 0$ .

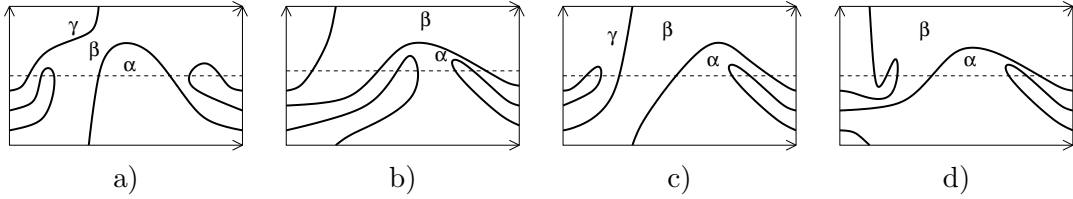


Figure 15:

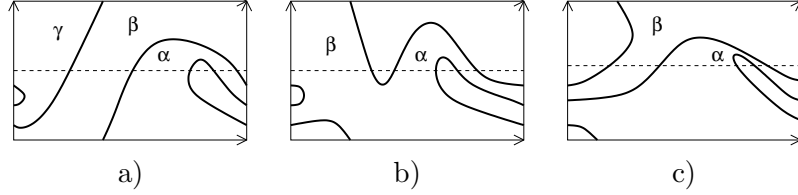


Figure 16:

*Proof.* Suppose there exist 3 ovals contradicting the lemma. Then the base passing through these ovals intersects the curve in at least 9 points which contradicts the Bezout theorem.  $\square$

In Figures 15 and 16, the dashed line represents the base  $\{y = 0\}$ .

**Lemma 4.2** *If there exists a pseudoholomorphic curve of bidegree  $(3, 1)$  on  $\Sigma_2$  realizing the  $\mathcal{L}$ -scheme depicted in Figure 15a) with  $\alpha + \beta + \gamma = 6$ , then  $(\alpha, \beta) = (0, 1), (0, 5), (1, 5)$  or  $(5, 1)$ .*

*Proof.* The corresponding braid is

$$b_{\alpha, \beta, \gamma}^1 = \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_2^{-\gamma} \sigma_1^{-1} \sigma_2^2 \sigma_1 \sigma_2^{-\beta} \sigma_1^{-1} \sigma_2 \sigma_1^{-\alpha} \sigma_2^{-1} \sigma_1 \Delta_3^2.$$

According to the Bezout theorem,  $\alpha$  and  $\gamma$  cannot be simultaneously non null. Moreover, it is clear by symmetry of the  $\mathcal{L}$ -scheme that  $b_{\alpha, \beta, \gamma}^1$  is quasipositive if and only if  $b_{\gamma, \beta, \alpha}^1$  is quasipositive. Some calculations give the following Alexander polynomials

$$\begin{aligned} p_{0,6,0}^1 &= p_{6,0,0}^1 = (t-1)(t^4 - t^3 + t^2 - t + 1), & p_{2,4,0}^1 &= p_{4,2,0}^1 = (t-1)(t^2 - t + 1), \\ p_{3,3,0}^1 &= (t-1)^3, & p_{1,5,0}^1 &= p_{5,1,0}^1 = 0. \end{aligned}$$

We have  $e(b_{\alpha, \beta, \gamma}^1) = 7 - (\alpha + \beta + \gamma)$  so according to Proposition 2.8, the braid is not quasipositive as soon as its Alexander polynomial is not identically zero, and the lemma is proved.  $\square$

**Lemma 4.3** *If there exists a pseudoholomorphic curve of bidegree  $(3, 1)$  on  $\Sigma_2$  realizing one of the  $\mathcal{L}$ -schemes depicted in Figures 15b) and d) or in Figure 16c) with  $\alpha + \beta = 6$ , then  $\alpha = 1$  or  $5$ .*

*Proof.* First, note that the three  $\mathcal{L}$ -schemes give rise to the same braid which is

$$b_{\alpha, \beta}^2 = \sigma_1 \sigma_2^{-\beta} \sigma_1^{-1} \sigma_2 \sigma_1^{-\alpha} \Delta_3^2.$$

Some calculations show that  $p_{\alpha, \beta}^2 = p_{\alpha, \beta, 0}^1$  as soon as  $\alpha + \beta = 6$ . We have  $e(b_{\alpha, \beta}^2) = 7 - (\alpha + \beta)$  so according to Proposition 2.8, the braid is not quasipositive as soon as its Alexander polynomial is not identically zero, and the lemma is proved.  $\square$

**Lemma 4.4** *If there exists a pseudoholomorphic curve of bidegree  $(3, 1)$  on  $\Sigma_2$  realizing one of the  $\mathcal{L}$ -schemes depicted in Figure 15c) or in Figure 16a) with  $\alpha + \beta + \gamma = 6$ , then  $(\alpha, \beta) = (0, 4), (0, 0), (2, 4)$  or  $(6, 0)$ .*

*Proof.* First, note the two  $\mathcal{L}$ -schemes are the same. The corresponding braid is

$$b_{\alpha, \beta, \gamma}^3 = \sigma_2^{-(1+\gamma)} \sigma_1^{-1} \sigma_2^2 \sigma_1 \sigma_2^{-\beta} \sigma_1^{-1} \sigma_2 \sigma_1^{-\alpha} \Delta_3^2.$$

According to the Bezout theorem,  $\alpha$  and  $\gamma$  cannot be simultaneously non null. Moreover, it is clear by symmetry of the  $\mathcal{L}$ -scheme that  $b_{\alpha, \beta, \gamma}^3$  is quasipositive if and only if  $b_{\gamma, \beta, \alpha}^3$  is quasipositive. Some calculations give the following Alexander polynomials

$$\begin{aligned} p_{0,6,0}^3 &= (t^2 + t + 1)(t^2 - t + 1)(t - 1)^3, & p_{1,5,0}^3 &= (t - 1)(t^4 - t^3 + t^2 - t + 1), \\ p_{3,3,0}^3 &= p_{5,1,0}^3 = (t - 1)(t^2 - t + 1), & p_{4,2,0}^3 &= (t - 1)^3, \\ p_{2,4,0}^3 &= p_{6,0,0}^3 = 0. \end{aligned}$$

We have  $e(b_{\alpha, \beta, \gamma}^3) = 7 - (\alpha + \beta + \gamma)$  so according to Proposition 2.8, the braid is not quasipositive as soon as its Alexander polynomial is not identically zero, and the lemma is proved.  $\square$

**Lemma 4.5** *There does not exist a pseudoholomorphic curve of bidegree  $(3, 1)$  on  $\Sigma_2$  realizing the  $\mathcal{L}$ -scheme depicted in Figure 16b) with  $\alpha + \beta = 6$ ,*

*Proof.* The corresponding braid and its Alexander polynomial are

$$b^4 = \sigma_2^{-7} \sigma_1 \sigma_2 \Delta_3^2 \text{ and } p^4 = (t - 1)(t^4 - t^3 + t^2 - t + 1).$$

We have  $e(b^4) = 1$ , so according to Proposition 2.8, this braid is not quasipositive.  $\square$

**Proposition 4.6** *The following real schemes cannot be realized by symmetric pseudoholomorphic curves of degree 7 on  $\mathbb{R}P^2$ :*

$$\langle J \amalg \beta \amalg 1 \langle \alpha \rangle \rangle \text{ with } \alpha = 6, 7, 8, 9 \text{ and } \beta = 14 - \alpha, \alpha = 7, 9 \text{ and } \beta = 13 - \alpha.$$

*Proof.* According to Proposition 2.12, 2.11, the Bezout theorem, Lemmas 4.1 and 2.5, the only possibilities for the  $\mathcal{L}$ -schemes of the quotient curves are depicted in Figure 15a), c) and d) with  $(\alpha, \beta) = (3, 3)$  or  $(4, 2)$  and in Figure 15b) with  $(\alpha, \beta) = (3, 3)$  or  $(2, 4)$  for the four  $M$ -curves, and in Figure 16 with  $(\alpha, \beta) = (3, 3)$ , for the two  $(M - 1)$ -curves. Now the proposition follows from Lemmas 4.2, 4.3, 4.4 and 4.5.  $\square$

## 4.2 Prohibitions for reducible curves of bidegree $(4, 1)$ on $\Sigma_2$

Here, we have to study more carefully the  $\mathcal{L}$ -schemes realized by the hypothetical quotient curves. Indeed, some of them are realized by curves of bidegree  $(3, 1)$  in  $\Sigma_2$ . To prohibit the symmetric curves, we have to take into account the mutual position of the quotient curves and some base of  $\Sigma_2$ .

**Proposition 4.7** *The real scheme  $\langle J \amalg 7 \amalg 1 \langle 6 \rangle \rangle$  is not realizable by a nonsingular symmetric pseudoholomorphic curve of degree 7 on  $\mathbb{R}P^2$ .*

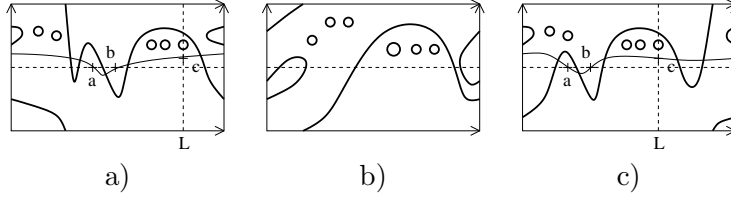


Figure 17:

*Proof.* The possible  $\mathcal{L}$ -schemes realized by the corresponding quotient curves with respect to the base  $\{y = 0\}$  are depicted in Figure 17.

The  $\mathcal{L}$ -scheme in Figures 17b) cannot be realized pseudoholomorphically. The braid corresponding to this  $\mathcal{L}$ -scheme and its Alexander polynomial are :

$$b^5 = \sigma_1^{-4} \sigma_2^2 \sigma_1^{-3} \sigma_2^{-1} \sigma_1 \Delta_3^2 \quad \text{and} \quad p^5 = (t - 1)^3$$

The Alexander polynomial is not null although  $e(b^5) = 1$ , so according to Proposition 2.8, the braid is not quasipositive.

Consider the base  $H$  passing through the points  $a$ ,  $b$  and  $c$  in Figures 17a) and c), where  $c$  is a point of the fiber  $L$ . If the point  $c$  varies on  $L$  from 0 to  $\infty$  in  $\{y \geq 0\}$ , then, because of the choice of  $L$ , for some  $c$ , the base  $H$  passes through an oval. Let  $H$  be a base which passes through the first oval we meet as  $c$  varies from 0 to  $\infty$ . The only possible mutual arrangements for  $H$  and the quotient curves which do not contradict the Bezout theorem are shown in Figure 18. The corresponding braids are :

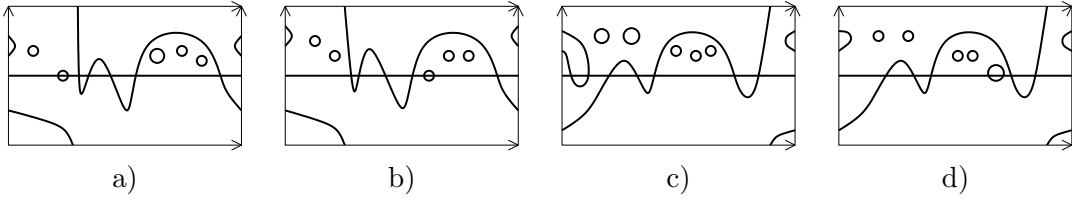


Figure 18:

$$\begin{aligned} b^6 &= \sigma_3^{-2} \sigma_2^{-2} \sigma_3^{-1} \sigma_1 \sigma_2^2 \sigma_1^{-4} \sigma_2^{-1} \sigma_3 \sigma_2^{-3} \sigma_3^{-1} \sigma_2 \sigma_1^{-1} \Delta_4^2, \\ b^7 &= \sigma_3^{-3} \sigma_1 \sigma_2^2 \sigma_1^{-4} \sigma_2^{-1} \sigma_3 \sigma_1^{-2} \sigma_2^{-3} \sigma_3^{-1} \sigma_2 \sigma_1^{-1} \Delta_4^2, \\ b^8 &= \sigma_2^{-2} \sigma_3^{-3} \sigma_1^{-3} \sigma_2^{-1} \sigma_3 \sigma_2^{-3} \sigma_3^{-1} \sigma_2 \sigma_1^{-2} \sigma_2^{-1} \sigma_3^2 \sigma_2 \sigma_1 \Delta_4^2, \\ b^9 &= \sigma_3^{-3} \sigma_1^{-3} \sigma_2^{-1} \sigma_3 \sigma_2^{-2} \sigma_1^{-2} \sigma_2^{-1} \sigma_3^{-1} \sigma_2 \sigma_1^{-2} \sigma_2^{-1} \sigma_3^2 \sigma_2 \sigma_1 \Delta_4^2. \end{aligned}$$

The computation of the corresponding Alexander polynomials gives :

$$\begin{aligned} p^6 &= (t^2 - t + 1)(t^6 - 3t^5 + 6t^4 - 5t^3 + 6t^2 - 3t + 1)(-1 + t)^3, & p^8 &= (-1 + t)^7, \\ p^7 &= (2t^4 - 2t^3 + 3t^2 - 2t + 2)(t^2 - t + 1)^2(-1 + t)^3, & p^9 &= (t^2 - t + 1)(-1 + t)^3. \end{aligned}$$

In each case,  $e(b) = 2$ , so according to Proposition 2.8, none of these braids is quasipositive.  $\square$

### 4.3 Restrictions for dividing symmetric curves

In this section, we use extensively the observation made at the end of section 2.4 : useful information on the complex orientations of a symmetric curve can be extracted from the topology of its quotient curve.

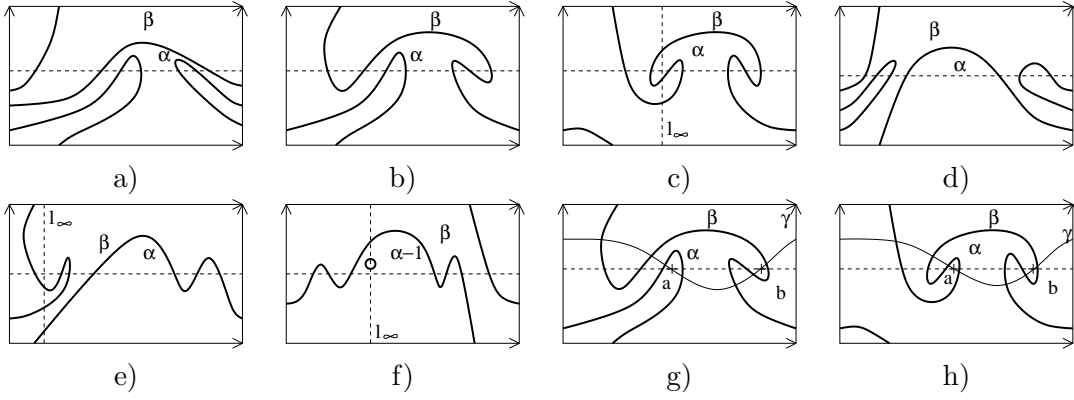


Figure 19:

**Lemma 4.8** *There does not exist a symmetric dividing pseudoholomorphic curve of degree 7 on  $\mathbb{R}P^2$  with a quotient curve realizing the  $\mathcal{L}$ -scheme depicted in Figures 19a) and d) with  $\alpha + \beta$  odd.*

*Proof.* Such a quotient curve is of type II because it is an  $(M - 2i - 1)$ -curve. The mirror curve of the initial symmetric curve is a nest of depth 3 with an odd component, and so is of type I. Thus, according to Proposition 2.15, the initial symmetric curve cannot be of type I.  $\square$

**Lemma 4.9** *There does not exist a symmetric dividing pseudoholomorphic curve of degree 7 on  $\mathbb{R}P^2$  with a quotient curve realizing the  $\mathcal{L}$ -scheme depicted in Figure 19b) with  $\alpha + \beta$  odd.*

*Proof.* According to the Fiedler orientation alternating rule on symmetric curves corresponding to these quotient curves, the symmetric curves cannot be of type I if  $\alpha + \beta$  is odd, as the two invariant empty ovals have opposite orientations.  $\square$

**Lemma 4.10** *There does not exist a symmetric dividing pseudoholomorphic curve of degree 7 on  $\mathbb{R}P^2$  with a quotient curve realizing the  $\mathcal{L}$ -scheme depicted in Figure 19c) with  $(\alpha, \beta) = (4, 1), (3, 2), (2, 3),$  and  $(1, 2)$ .*

*Proof.* According to the Fiedler orientation alternating rule on symmetric curves corresponding to these quotient curves, the three invariant ovals are positive and we have  $\Lambda_+ - \Lambda_- = 1, \Pi_+ - \Pi_- = 0$  if  $\alpha$  is odd, and  $\Pi_+ - \Pi_- = -2$  if  $\alpha$  is even. Thus, the Rokhlin-Mischachev orientation formula is fulfilled only for  $(\alpha, \beta) = (3, 2)$ . Choose  $l_\infty$  as depicted in Figure 19c). The braid corresponding to the quotient curve and its Alexander polynomial are :

$$\sigma_1^{-4} \sigma_2^{-1} \sigma_1 \sigma_2^{-2} \sigma_1 \Delta_3^2 \quad \text{and} \quad (-1 + t)^3.$$

Since  $e(b) = 1$ , according to Proposition 2.8, the braid is not quasipositive.  $\square$

**Lemma 4.11** *There does not exist a symmetric dividing pseudoholomorphic curve of degree 7 on  $\mathbb{R}P^2$  with a quotient curve realizing the  $\mathcal{L}$ -scheme depicted in Figure 19e) with  $(\alpha, \beta) = (5, 0), (4, 1), (3, 2), (2, 3),$  and  $(2, 1)$ .*

*Proof.* According to the Fiedler orientation alternating rule on symmetric curves corresponding to these quotient curves, the two invariant empty ovals have opposite orientations, the non-empty oval is negative, and we have  $\Lambda_+ - \Lambda_- = 1, \Pi_+ - \Pi_- = 0$  if  $\alpha$  is even, and  $\Pi_+ - \Pi_- = -2$  if  $\alpha$  is

odd. Hence, the Rokhlin-Mischachev orientation formula is fulfilled only for  $(\alpha, \beta) = (4, 1)$  or  $(2, 3)$ . Choose  $l_\infty$  as depicted in Figure 19e). Then the braids corresponding to the union of the quotient curves and of the base  $\{y = 0\}$  are :

$$b_{\alpha, \beta}^{10} = \sigma_3^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_3^{-(1+\beta)} \sigma_2^{-1} \sigma_3 \sigma_2^{-\alpha} \sigma_3^{-1} \sigma_2 \sigma_1^{-3} \sigma_3 \sigma_2 \sigma_1 \sigma_3^{-1} \Delta_4^2.$$

The computation of the corresponding Alexander polynomials gives

$$p_{4,1}^{10} = (t^2 + 1)(t^2 - t + 1)(-1 + t)^3, \quad p_{2,3}^{10} = (t^4 - 2t^3 + 4t^2 - 2t + 1)(-1 + t)^3$$

In each case we have  $e(b) = 2$ , so according to Proposition 2.8, both braids are not quasipositive.  $\square$

**Lemma 4.12** *There does not exist a symmetric dividing pseudoholomorphic curve of degree 7 on  $\mathbb{R}P^2$  with a quotient curve realizing the  $\mathcal{L}$ -scheme depicted in Figure 19f) with  $(\alpha, \beta) = (5, 0)$ ,  $(3, 2)$ , and  $(2, 3)$ .*

*Proof.* Choose  $l_\infty$  as depicted in Figure 19f). Then the braids corresponding to the union of the quotient curves and of the base  $\{y = 0\}$  are :

$$b_{\alpha, \beta}^{11} = \sigma_2^{-\alpha} \sigma_3^{-1} \sigma_2 \sigma_3^{-\beta} \sigma_1^{-3} \sigma_1 \sigma_2^2 \sigma_1^{-4} \sigma_2^{-1} \sigma_3 \Delta_4^2.$$

The computation of the determinant gives 976 for  $b_{5,0}^{11}$  and 592 for  $b_{3,2}^{11}$  which are not squares in  $\mathbb{Z}$  although  $e(b) = 3$ . So according to Proposition 2.9, these two braids are not quasipositive.

The computation of the Alexander polynomial of  $b_{2,3}^{11}$  gives

$$(t^2 + 1)(t^6 - 5t^5 + 12t^4 - 14t^3 + 12t^2 - 5t + 1)(-1 + t)^2.$$

The number  $i$  is a simple root of this polynomial and  $e(b) = 3$ . Thus, according to Proposition 2.10, this braid is not quasipositive.  $\square$

**Proposition 4.13** *The real schemes  $\langle J \amalg 2 \amalg 1 \langle 10 \rangle \rangle$  and  $\langle J \amalg 6 \amalg 1 \langle 6 \rangle \rangle$  are not realizable by nonsingular symmetric dividing pseudoholomorphic curves of degree 7 on  $\mathbb{R}P^2$ .*

*Proof.* According to the Bezout theorem, Proposition 2.14, Lemma 4.1 and Lemma 2.5, the only possibilities for the  $\mathcal{L}$ -scheme of the quotient curve of such a dividing symmetric curve of degree 7 on  $\mathbb{R}P^2$  are depicted in Figures 19a), g) and h) with  $(\alpha, \beta + \gamma) = (4, 1)$  and  $(2, 3)$  and in Figures 19d), e) and f) with  $(\alpha, \beta) = (5, 0)$ , and  $(3, 2)$ . If a curve of bidegree  $(3, 1)$  realizes one of the two  $\mathcal{L}$ -schemes depicted in Figures 19g) and h) then  $\gamma = 0$ . Otherwise, the base passing through the points  $a$  and  $b$  and through an oval  $\gamma$  intersects the quotient curve in more than 7 points, which contradicts the Bezout theorem.

The remaining quotient curves have been prohibited in Lemmas 4.8, 4.9, 4.10, 4.11, and 4.12  $\square$

**Proposition 4.14** *If a nonsingular symmetric dividing pseudoholomorphic curve of degree 7 on  $\mathbb{R}P^2$  realizes the real scheme  $\langle J \amalg 8 \amalg 1 \langle 4 \rangle \rangle$ , then the  $\mathcal{L}$ -scheme of its quotient curve is as depicted in Figure 19c) with  $(\alpha, \beta) = (1, 4)$ .*

*If a nonsingular symmetric dividing pseudoholomorphic curve of degree 7 on  $\mathbb{R}P^2$  realizes the real scheme  $\langle J \amalg 2\beta + 2 \amalg 1 \langle 2\alpha \rangle \rangle$  with  $(\alpha, \beta) = (4, 1)$  or  $(2, 1)$ , then the  $\mathcal{L}$ -scheme of its quotient curve is as depicted in Figure 19f).*

*Proof.* The proof is the same as for the previous proposition.  $\square$



**Proposition 4.15** *The real scheme  $\langle \text{III4III1}\langle 4 \rangle \rangle$  is not realizable by nonsingular symmetric dividing pseudoholomorphic curves of degree 7 on  $\mathbb{R}P^2$ .*

*Proof.* Suppose that there exists a pseudoholomorphic curve  $C$  which contradicts Proposition 4.15. Then, according to Proposition 4.14, its quotient curve is as depicted in Figure 19f) with  $(\alpha, \beta) = (2, 1)$ . Using the Fiedler orientation alternating rule, and denoting by  $\epsilon$  the sign of the two non-invariant outer ovals of  $C$ , we have  $\Lambda_+ - \Lambda_- = -1 + 2\epsilon$  and  $\Pi_+ - \Pi_- = 0$ . Thus, the Rokhlin-Mischachev orientation formula is fulfilled only if  $\epsilon = -1$ . Hence, one of the two complex orientations of the curve is as depicted in Figure 20a). Using once again the Fiedler orientation alternating rule, we see that the pencil of lines through the point  $p$  induces a cyclic order on the 6 non-invariant ovals of  $C$  as depicted in Figure 20a). So the ovals 4 and 1 are not on the same connected component of  $\mathbb{R}P^2 \setminus (L_1 \cup L_2)$ . A symmetric conic passing through all the non-invariant ovals (in bold line in Figure 20a)) intersects  $C$  in at least 18 points what contradicts the Bezout Theorem.  $\square$

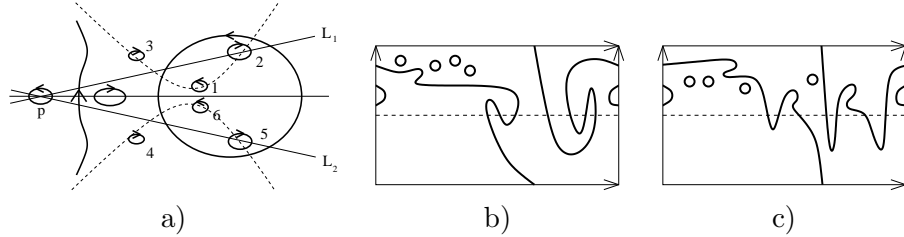


Figure 20:

#### 4.4 Constructions

**Proposition 4.16** *There exist nonsingular real pseudoholomorphic curves of bidegree  $(3, 1)$  on  $\Sigma_2$  such that the  $\mathcal{L}$ -scheme realized by the union of this curve and a base is as shown in Figures 20b) and c). In particular, all the real tangency points of the curve with the pencil  $\mathcal{L}$  are above the base  $\{y = 0\}$ .*

*Proof.* The braids associated to these  $\mathcal{L}$ -schemes are

$$\begin{aligned} b^{12} &= \sigma_2^{-1} \sigma_3^{-1} \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_3^{-3} \sigma_2^{-1} \sigma_3 \sigma_1^{-1} \sigma_2^{-2} \sigma_3^{-1} \sigma_1 \sigma_2^2 \sigma_1^{-1} \sigma_2^{-2} \sigma_1^{-1} \sigma_2^{-1} \Delta_4^2, \\ b^{13} &= \sigma_2^{-3} \sigma_3^{-1} \sigma_2^{-1} \sigma_3 \sigma_2^{-1} \sigma_3^{-1} \sigma_2 \sigma_1^{-2} \sigma_2^{-1} \sigma_1^{-1} \sigma_3^{-1} \sigma_1 \sigma_2^2 \sigma_1^{-2} \sigma_2^{-1} \sigma_1^{-2} \sigma_2^{-1} \sigma_3 \Delta_4^2. \end{aligned}$$

Using the Garside normal form (see [Gar69] or [Jac90]), we see that these braids are trivial, so quasipositive.  $\square$

Denote by  $C$  (resp.  $C'$ ) the strict transform by the blow up of  $\mathbb{C}P^2$  at  $[0 : 1 : 0]$  of a symmetric nonsingular pseudoholomorphic curve of degree 7 on  $\mathbb{R}P^2$  corresponding to the quotient curve depicted in Figure 20b) (resp. c)). If we denote by  $p$  the intersection point of  $C$  with the exceptional section  $E$  and  $F_p$  the fiber of  $\Sigma_1$  through  $p$ , the curve  $C$  has a tangency point of order 2 with  $F_p$  at  $p$ .

Let us introduce some notations. For  $\epsilon > 0$ , denote by  $\gamma_{1,\epsilon}$  and  $\gamma_{2,\epsilon}$  the following paths

$$\gamma_{1,\epsilon} : \begin{array}{ccc} [-1, 1] & \rightarrow & \mathbb{C} \\ t & \mapsto & \frac{1}{\epsilon}t + i\epsilon \end{array} \quad \text{and} \quad \gamma_{2,\epsilon} : \begin{array}{ccc} [0, 1] & \rightarrow & \mathbb{C} \\ t & \mapsto & \frac{1}{\epsilon}e^{i\pi t} + i\epsilon \end{array} .$$

Let  $\gamma_\epsilon$  be the union of the images of  $\gamma_{1,\epsilon}$  and  $\gamma_{2,\epsilon}$  (see Figure 22). Denote also by  $\pi$  the projection  $\Sigma_1 \rightarrow \mathbb{C}P^1$  on the base  $\{y = 0\}$ ,  $S_C = \pi^{-1}(\mathbb{R}P^1)$ ,  $D_\epsilon$  the compact region of  $\mathbb{C}$  bounded by  $\gamma_\epsilon$ ,  $b_\epsilon = \pi^{-1}(\gamma_\epsilon) \cap C$ ,  $\hat{b}_\epsilon$  the closure of the braid  $b_\epsilon$ , and  $N_\epsilon = \pi^{-1}(D_\epsilon) \cap C$ .

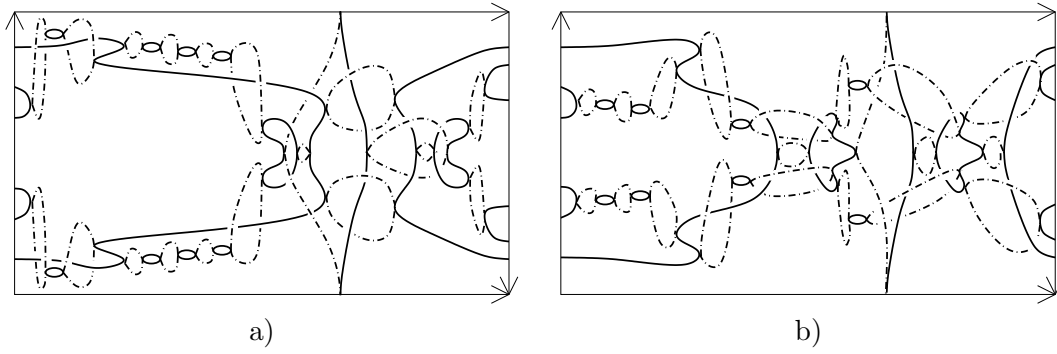


Figure 21:

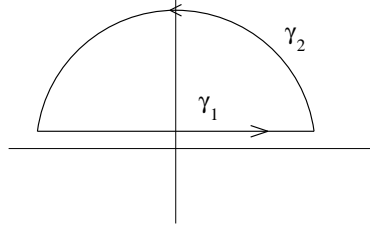


Figure 22:

As  $C$  is a real curve,  $S_C$  is formed by a real part which is  $\mathbb{R}C$ , and by a non-real part. This latter space has several connected components which are globally invariant by the complex conjugation. One can deduce  $S_C$  (resp.  $S_{C'}$ ) from the quotient curve of  $C$ . The curve  $S_C$  (resp.  $S_{C'}$ ) is depicted in Figure 21a) (resp. c)), where the bold lines are used to draw  $\mathbb{R}C$  and the dashed lines are used to draw  $S_C \setminus \mathbb{R}C$ .

**Proposition 4.17** *The real pseudoholomorphic curve  $C$  constructed above is a dividing curve.*

*Proof.* By the Riemann-Hurwitz formula, we have

$$\mu(N_\epsilon) = g(N_\epsilon) + \frac{\mu(\hat{b}_\epsilon) + 6 - e(b_\epsilon)}{2},$$

where  $\mu(N_\epsilon)$  is the number of connected components of  $N_\epsilon$  and  $g(N_\epsilon)$  the sum of the genus of the connected components of  $N_\epsilon$ . We have  $\mu(\hat{b}_\epsilon) = 6$ ,  $e(b_\epsilon) = 0$  and  $\mu(N_\epsilon) \leq 6$ , so  $N_\epsilon$  is composed by 6 disks. Denote these disks  $D_{1,\epsilon}, \dots, D_{6,\epsilon}$  and their boundaries  $\partial D_{i,\epsilon} = L_{i,\epsilon}$ . Define also  $\overline{D_{i,\epsilon}} = \text{conj}(D_{i,\epsilon})$ . As  $\epsilon \rightarrow 0$ , these 12 disks glue together along  $S_C$  as depicted in Figure 23a), and  $C$  is the result of this gluing. Moreover,  $C \setminus \mathbb{R}C$  is the result of the gluing of these 12 disks along  $S_C \setminus \mathbb{R}C$ . Hence, to find the type of  $C$ , we just have to study how the 12 disks glue along  $S_C \setminus \mathbb{R}C$ .

Denote by  $D_{i,\epsilon} || D_{j,\epsilon}$  the relation “ $D_{i,\epsilon}$  glues with  $D_{j,\epsilon}$  along a connected component of  $S_C \setminus \mathbb{R}C$  as  $\epsilon \rightarrow 0$ ”. Using the fact that each connected component of  $S_C \setminus \mathbb{R}C$  is globally invariant by the complex conjugation, we have (see Figure 23a)) :

$$\begin{aligned} D_{1,\epsilon} || \overline{D_{2,\epsilon}}, \quad D_{1,\epsilon} || \overline{D_{6,\epsilon}}, \quad D_{1,\epsilon} || \overline{D_{4,\epsilon}}, \quad D_{2,\epsilon} || \overline{D_{3,\epsilon}}, \quad D_{2,\epsilon} || \overline{D_{5,\epsilon}}, \\ D_{3,\epsilon} || \overline{D_{6,\epsilon}}, \quad D_{3,\epsilon} || \overline{D_{4,\epsilon}}, \quad D_{4,\epsilon} || \overline{D_{5,\epsilon}}, \quad D_{5,\epsilon} || \overline{D_{6,\epsilon}}, \quad D_{i,\epsilon} || \overline{D_{j,\epsilon}} \implies D_{j,\epsilon} || \overline{D_{i,\epsilon}}. \end{aligned}$$

The curve  $C$  is a dividing curve if and only if there exist two equivalence classes for  $||$ . Here the equivalence classes are  $\{D_{1,\epsilon}, D_{3,\epsilon}, D_{5,\epsilon}, \overline{D_{2,\epsilon}}, \overline{D_{4,\epsilon}}, \overline{D_{6,\epsilon}}\}$  and  $\{D_{2,\epsilon}, D_{4,\epsilon}, D_{6,\epsilon}, \overline{D_{1,\epsilon}}, \overline{D_{3,\epsilon}}, \overline{D_{5,\epsilon}}\}$ . Hence,  $C$  is a dividing curve.  $\square$

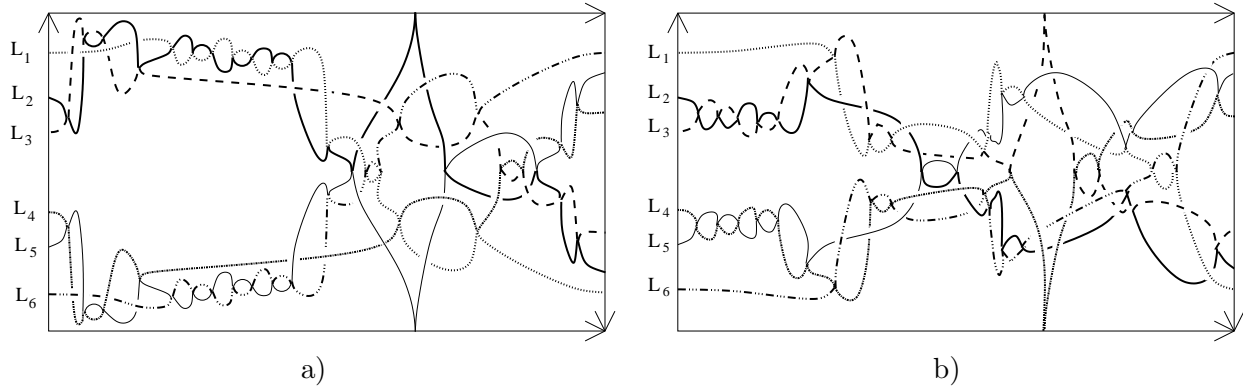


Figure 23:

**Proposition 4.18** *The real pseudoholomorphic curve  $C'$  constructed above is a dividing curve.*

*Proof.* We keep the same notations than in Proposition 4.17. As in this proposition, the closure of the braid  $b_\epsilon$  has 6 components, the surface  $N_\epsilon$  is composed by 6 disks and the two equivalence classes for the relation  $\parallel$  are  $\{D_{1,\epsilon}, D_{2,\epsilon}, D_{4,\epsilon}, \overline{D_{3,\epsilon}}, \overline{D_{5,\epsilon}}, \overline{D_{6,\epsilon}}\}$  and  $\{D_{3,\epsilon}, D_{5,\epsilon}, D_{6,\epsilon}, \overline{D_{1,\epsilon}}, \overline{D_{2,\epsilon}}, \overline{D_{4,\epsilon}}\}$  (see Figures 21b) and 23b)). Hence,  $C'$  is a dividing curve.  $\square$

**Corollary 4.19** *The complex schemes  $\langle J\Pi 4_+ \Pi 4_- \Pi 1_+ \langle 2_+ \Pi 2_- \rangle \rangle_I$  and  $\langle J\Pi 2_+ \Pi 2_- \Pi 1_+ \langle 4_+ \Pi 4_- \rangle \rangle_I$  are realizable by nonsingular symmetric real pseudoholomorphic curves of degree 7 on  $\mathbb{R}P^2$ .*

## 5 Algebraic statements

### 5.1 Prohibitions

We prove in this section the algebraic prohibitions stated in Theorem 1.8. The main tools are the real trigonal graphs (see section 3) and the cubic resolvent of an algebraic curve of bidegree  $(4, 0)$  in  $\Sigma_n$  (see [OS03]).

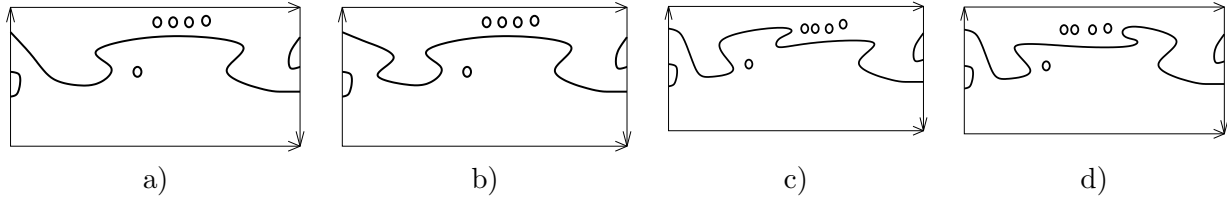


Figure 24:

**Lemma 5.1** *The  $\mathcal{L}$ -schemes depicted in Figures 24b), c) and d) are not realizable by trigonal nonsingular real pseudoholomorphic curves on  $\Sigma_3$ .*

*Proof.* Compute the braids associated to these  $\mathcal{L}$ -schemes :

$$\begin{aligned} b^{14} &= \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-2} \sigma_2^{-1} \sigma_1 \sigma_2^{-4} \sigma_1^{-1} \Delta_3^3, & b^{15} &= \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-5} \sigma_1^{-1} \Delta_3^3, \\ b^{16} &= \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_2^{-4} \sigma_1^{-1} \sigma_2^{-1} \Delta_3^3. \end{aligned}$$

These braids verify  $e(b) = 0$ , so they are quasipositive if and only if they are trivial. Computing their Garside normal form (see [Gar69] or [Jac90]), we find

$$b^{14} = \sigma_2^3 \sigma_1^2 \sigma_2^2 \sigma_1^2 \Delta_3^{-3}, \quad b^{15} = \sigma_1 \sigma_2^2 \sigma_1^2 \sigma_2^2 \sigma_1^2 \Delta_3^{-3}, \quad b^{16} = \sigma_1 \sigma_2^3 \sigma_1^2 \sigma_2^2 \sigma_1 \Delta_3^{-3}.$$

Thus, no one of these braids is quasipositive.  $\square$

**Lemma 5.2** *The  $\mathcal{L}$ -scheme depicted in Figure 24a) is not realizable by trigonal nonsingular real algebraic curves on  $\Sigma_3$ .*

*Proof.* The weighted comb associated to this  $\mathcal{L}$ -scheme is

$$w_1 = (g_3 g_6 g_1 g_4 g_1 g_6 g_5 g_2 g_3 g_6 g_1 g_4 g_1 g_6 (g_3 g_2)^3 g_3 g_6 g_1 g_4 g_1 g_6 g_5 g_2, 1, 2, 0).$$

We have  $\mu(w_1) = 0$  so, according to Proposition 3.13 the lemma is proved.

**Proposition 5.3** *The real scheme  $\langle JII8II1\langle 4 \rangle \rangle$  is not realizable by nonsingular symmetric dividing real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .*

*Proof.* Suppose that there exists a dividing symmetric curve which contradicts Proposition 5.3. Denote by  $X$  its quotient curve. Blow up  $\Sigma_2$  at the intersection point of  $X$  and  $E$  and then blow down the strict transform of the fiber. The obtained surface is  $\Sigma_3$  and the strict transform of  $X$  is a trigonal curve which has a double point with non-real tangents. Smooth this double point in order to obtain an oval. Then, according to Propositions 4.14 and 3.6, one should obtain one of the  $\mathcal{L}$ -schemes depicted in Figures 24a), b), c) and d). However, according to Lemmas 5.1 and 5.2, these  $\mathcal{L}$ -schemes are not algebraically realizable so there is a contradiction.  $\square$

**Lemma 5.4** *The  $\mathcal{L}$ -scheme depicted in Figures 25a) is not realizable by trigonal nonsingular real pseudoholomorphic curves on  $\Sigma_5$ .*

*Proof.* The braid associated to this  $\mathcal{L}$ -schemes and its determinant are :

$$b^{17} = \sigma_2^{-4} \sigma_1^{-5} \sigma_2^{-1} \sigma_1 \sigma_2^{-4} \sigma_1^{-1} \sigma_2 \Delta_3^5 \quad \text{and} \quad 301.$$

We have  $e(b^{17}) = 2$ , so according to Proposition 2.8, this braid is not quasipositive.

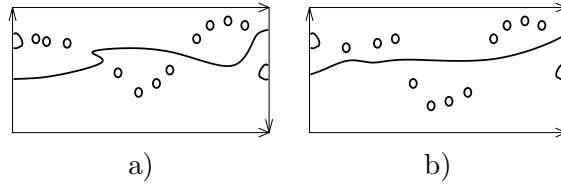


Figure 25:

**Lemma 5.5** *The  $\mathcal{L}$ -scheme depicted in Figures 25b) is not realizable by trigonal nonsingular real algebraic curves on  $\Sigma_5$ .*

*Proof.* The weighted comb associated to this  $\mathcal{L}$ -scheme is

$$w_2 = ((a^3 g_3 g_6 g_1 g_4 g_1 g_6)^3, 3, 6, 2) \quad \text{where} \quad a = g_3 g_2.$$

We have  $\mu(w_2) = 0$ , so according to Proposition 3.13, this  $\mathcal{L}$ -scheme is not realizable by trigonal nonsingular real algebraic curves on  $\Sigma_5$ .  $\square$

**Proposition 5.6** *The real scheme  $\langle J \amalg 4 \amalg 1 \langle 8 \rangle \rangle$  is not realizable by nonsingular symmetric dividing real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .*

*Proof.* Suppose that there exists a dividing symmetric curve which contradicts the lemma. Denote by  $X$  its quotient curve. Blow up  $\Sigma_2$  at the intersection point of  $X$  and  $E$  and then blow down the strict transform of the fiber. The obtained surface is  $\Sigma_3$  and the strict transform of  $X$ , still denoted by  $X$ , is a trigonal curve which has a double point with non-real tangents on the base  $\{y = 0\}$ . Let  $\tilde{X}$  be the cubic resolvent (see [OS03]) of the union of the base  $\{y = 0\}$  and  $X$ . The curve  $\tilde{X}$  is a trigonal real algebraic curve on  $\Sigma_6$  with a triple point coming from the triple point of  $X \cup \{y = 0\}$ . Blow up  $\Sigma_6$  at this triple point and then blow down the strict transform of the fiber. The obtained surface is  $\Sigma_5$  and the strict transform of  $\tilde{X}$  is a trigonal curve with ordinary double points. Smooth all the double points in order to obtain ovals. Then, according to Propositions 4.14 and 3.6, one should obtain one of the  $\mathcal{L}$ -schemes depicted in Figure 25. It has been proved in Lemmas 5.4 and 5.5 that these  $\mathcal{L}$ -schemes are not algebraically realizable, so there is a contradiction.  $\square$

## 5.2 Perturbation of a reducible symmetric curve

The standard method to construct a lot of different isotopy types of nonsingular algebraic curves is to perturb a singular curve in many ways. So the first idea to construct symmetric algebraic curves is to perturb in many symmetric ways a singular symmetric algebraic curve.

To perturb real algebraic singular curves, we use the Viro method. The unfamiliar reader can refer to [Vir84a], [Vir89], [Vir], [Ris92] and [IS03].

**Proposition 5.7** *All the real schemes listed in table 1 are realizable by nonsingular symmetric real algebraic curves of degree 7 on  $\mathbb{R}P^2$ . Moreover, those marked with a \* are realized by a dividing symmetric curve and those marked with a  $\circ$  are realized by a non-dividing curve.*

*Proof.* In order to apply the Viro method without change of coordinates, we consider here symmetry with respect to the line  $\{y - z = 0\}$ . Consider the union of the line  $\{x = 0\}$  and three symmetric conics on  $\mathbb{R}P^2$  tangent to each other in the two symmetric points  $[0 : 0 : 1]$  and  $[0 : 1 : 0]$ . Using the Viro method and the classification, up to isotopy, of the curves of degree 7 on  $\mathbb{R}P^2$  with the only singular point  $Z_{15}$  established by A. B. Korchagin in [Kor88], we perturb these reducible symmetric curves. In order to obtain nonsingular symmetric curves, we have to perturb symmetrically the two singular points. That is to say, if we perturb the singular point at  $[0 : 0 : 1]$  gluing the chart of a polynomial  $P(x, y)$ , we have to perturb the singular point at  $[0 : 1 : 0]$  gluing the chart of the polynomial  $y^7 P(\frac{x}{y}, \frac{1}{y})$ .  $\square$

**Remark.** Using this method, we constructed nonsingular symmetric algebraic curves of degree 7 on  $\mathbb{R}P^2$  realizing the complex schemes  $\langle J \amalg 4_+ \amalg 5_- \amalg 1_+ \langle 1_+ \rangle \rangle_I$  and  $\langle J \amalg 3_+ \amalg 6_- \amalg 1_- \langle 1_+ \rangle \rangle_I$ . So unlike in the  $M$ -curves case, the real scheme of a nonsingular symmetric curves of degree 7 on  $\mathbb{R}P^2$  does not determine its complex scheme.

**Proposition 5.8** *The complex schemes*

$$\langle J \amalg 1_- \langle 1_+ \amalg 3_- \rangle \rangle_I, \langle J \amalg 2_+ \amalg 3_- \amalg 1_- \langle 1_- \rangle \rangle_I \text{ and } \langle J \amalg 1_- \amalg 1_- \langle 2_+ \amalg 3_- \rangle \rangle_I$$

*are realizable by nonsingular symmetric real algebraic dividing curves of degree 7 on  $\mathbb{R}P^2$ .*

*Proof.* In [II01], symmetric sextics realizing the complex schemes  $\langle 1_- \langle 1_+ \amalg 3_- \rangle \rangle_I$ ,  $\langle 5 \amalg 1_- \langle 1_- \rangle \rangle_I$  and  $\langle 1 \amalg 1_- \langle 2_+ \amalg 3_- \rangle \rangle_I$  are constructed. Consider the union of each of these curves and a real line

$\langle J \rangle^\circ$	$\langle J \amalg 10 \amalg 1 \langle 1 \rangle \rangle^\circ$	$\langle J \amalg 8 \amalg 1 \langle 3 \rangle \rangle^\circ$	$\langle J \amalg 1 \langle 6 \rangle \rangle^\circ$	$\langle J \amalg 1 \langle 9 \rangle \rangle^\circ$
$\langle J \amalg 1 \rangle^\circ$	$\langle J \amalg 11 \amalg 1 \langle 1 \rangle \rangle^{\circ,*}$	$\langle J \amalg 9 \amalg 1 \langle 3 \rangle \rangle^{\circ,*}$	$\langle J \amalg 1 \amalg 1 \langle 6 \rangle \rangle^\circ$	$\langle J \amalg 1 \amalg 1 \langle 9 \rangle \rangle^{\circ,*}$
$\langle J \amalg 2 \rangle^\circ$	$\langle J \amalg 12 \amalg 1 \langle 1 \rangle \rangle^\circ$	$\langle J \amalg 10 \amalg 1 \langle 3 \rangle \rangle^\circ$	$\langle J \amalg 2 \amalg 1 \langle 6 \rangle \rangle^\circ$	$\langle J \amalg 2 \amalg 1 \langle 9 \rangle \rangle^\circ$
$\langle J \amalg 3 \rangle^\circ$	$\langle J \amalg 13 \amalg 1 \langle 1 \rangle \rangle^*$	$\langle J \amalg 11 \amalg 1 \langle 3 \rangle \rangle^*$	$\langle J \amalg 3 \amalg 1 \langle 6 \rangle \rangle^\circ$	$\langle J \amalg 3 \amalg 1 \langle 9 \rangle \rangle^{\circ,*}$
$\langle J \amalg 4 \rangle^\circ$			$\langle J \amalg 4 \amalg 1 \langle 6 \rangle \rangle^{\circ,*}$	
$\langle J \amalg 5 \rangle^\circ$	$\langle J \amalg 1 \langle 2 \rangle \rangle^\circ$	$\langle J \amalg 1 \langle 4 \rangle \rangle^\circ$		
$\langle J \amalg 6 \rangle^\circ$	$\langle J \amalg 1 \amalg 1 \langle 2 \rangle \rangle^\circ$	$\langle J \amalg 1 \amalg 1 \langle 4 \rangle \rangle^\circ$	$\langle J \amalg 6 \amalg 1 \langle 6 \rangle \rangle^\circ$	
$\langle J \amalg 7 \rangle^{\circ,*}$	$\langle J \amalg 2 \amalg 1 \langle 2 \rangle \rangle^{\circ,*}$	$\langle J \amalg 2 \amalg 1 \langle 4 \rangle \rangle^{\circ,*}$		$\langle J \amalg 1 \langle 10 \rangle \rangle^{\circ,*}$
$\langle J \amalg 8 \rangle^\circ$	$\langle J \amalg 3 \amalg 1 \langle 2 \rangle \rangle^\circ$	$\langle J \amalg 3 \amalg 1 \langle 4 \rangle \rangle^\circ$		$\langle J \amalg 1 \amalg 1 \langle 10 \rangle \rangle^\circ$
$\langle J \amalg 9 \rangle^{\circ,*}$	$\langle J \amalg 4 \amalg 1 \langle 2 \rangle \rangle^{\circ,*}$	$\langle J \amalg 4 \amalg 1 \langle 4 \rangle \rangle^\circ$		$\langle J \amalg 2 \amalg 1 \langle 10 \rangle \rangle^\circ$
$\langle J \amalg 10 \rangle^\circ$	$\langle J \amalg 5 \amalg 1 \langle 2 \rangle \rangle^\circ$	$\langle J \amalg 5 \amalg 1 \langle 4 \rangle \rangle^\circ$	$\langle J \amalg 1 \langle 7 \rangle \rangle^\circ$	$\langle J \amalg 3 \amalg 1 \langle 10 \rangle \rangle^\circ$
$\langle J \amalg 11 \rangle^{\circ,*}$	$\langle J \amalg 6 \amalg 1 \langle 2 \rangle \rangle^\circ$	$\langle J \amalg 6 \amalg 1 \langle 4 \rangle \rangle^{\circ,*}$	$\langle J \amalg 1 \amalg 1 \langle 7 \rangle \rangle^{\circ,*}$	$\langle J \amalg 4 \amalg 1 \langle 10 \rangle \rangle^*$
$\langle J \amalg 12 \rangle^\circ$	$\langle J \amalg 7 \amalg 1 \langle 2 \rangle \rangle^\circ$		$\langle J \amalg 2 \amalg 1 \langle 7 \rangle \rangle^\circ$	
$\langle J \amalg 13 \rangle^{\circ,*}$	$\langle J \amalg 8 \amalg 1 \langle 2 \rangle \rangle^{\circ,*}$	$\langle J \amalg 8 \amalg 1 \langle 4 \rangle \rangle^\circ$	$\langle J \amalg 3 \amalg 1 \langle 7 \rangle \rangle^{\circ,*}$	$\langle J \amalg 1 \langle 11 \rangle \rangle^\circ$
$\langle J \amalg 14 \rangle^\circ$	$\langle J \amalg 9 \amalg 1 \langle 2 \rangle \rangle^\circ$			$\langle J \amalg 1 \amalg 1 \langle 11 \rangle \rangle^{\circ,*}$
$\langle J \amalg 15 \rangle^*$	$\langle J \amalg 10 \amalg 1 \langle 2 \rangle \rangle^\circ$	$\langle J \amalg 10 \amalg 1 \langle 4 \rangle \rangle^*$	$\langle J \amalg 5 \amalg 1 \langle 7 \rangle \rangle^*$	$\langle J \amalg 2 \amalg 1 \langle 11 \rangle \rangle^\circ$
	$\langle J \amalg 11 \amalg 1 \langle 2 \rangle \rangle^\circ$			$\langle J \amalg 3 \amalg 1 \langle 11 \rangle \rangle^*$
$\langle J \amalg 1 \langle 1 \rangle \rangle^\circ$	$\langle J \amalg 12 \amalg 1 \langle 2 \rangle \rangle^*$	$\langle J \amalg 1 \langle 5 \rangle \rangle^\circ$		
$\langle J \amalg 1 \amalg 1 \langle 1 \rangle \rangle^\circ$		$\langle J \amalg 1 \amalg 1 \langle 5 \rangle \rangle^\circ$		$\langle J \amalg 1 \langle 12 \rangle \rangle^\circ$
$\langle J \amalg 2 \amalg 1 \langle 1 \rangle \rangle^\circ$	$\langle J \amalg 1 \langle 3 \rangle \rangle^\circ$	$\langle J \amalg 2 \amalg 1 \langle 5 \rangle \rangle^\circ$	$\langle J \amalg 1 \langle 8 \rangle \rangle^\circ$	
$\langle J \amalg 3 \amalg 1 \langle 1 \rangle \rangle^{\circ,*}$	$\langle J \amalg 1 \amalg 1 \langle 3 \rangle \rangle^\circ$	$\langle J \amalg 3 \amalg 1 \langle 5 \rangle \rangle^{\circ,*}$	$\langle J \amalg 1 \amalg 1 \langle 8 \rangle \rangle^\circ$	$\langle J \amalg 2 \amalg 1 \langle 12 \rangle \rangle^*$
$\langle J \amalg 4 \amalg 1 \langle 1 \rangle \rangle^\circ$	$\langle J \amalg 2 \amalg 1 \langle 3 \rangle \rangle^\circ$	$\langle J \amalg 4 \amalg 1 \langle 5 \rangle \rangle^\circ$	$\langle J \amalg 2 \amalg 1 \langle 8 \rangle \rangle^{\circ,*}$	
$\langle J \amalg 5 \amalg 1 \langle 1 \rangle \rangle^\circ$	$\langle J \amalg 3 \amalg 1 \langle 3 \rangle \rangle^{\circ,*}$	$\langle J \amalg 5 \amalg 1 \langle 5 \rangle \rangle^{\circ,*}$	$\langle J \amalg 3 \amalg 1 \langle 8 \rangle \rangle^\circ$	$\langle J \amalg 1 \langle 13 \rangle \rangle^\circ$
$\langle J \amalg 6 \amalg 1 \langle 1 \rangle \rangle^\circ$	$\langle J \amalg 4 \amalg 1 \langle 3 \rangle \rangle^\circ$	$\langle J \amalg 6 \amalg 1 \langle 5 \rangle \rangle^\circ$	$\langle J \amalg 4 \amalg 1 \langle 8 \rangle \rangle^\circ$	$\langle J \amalg 1 \amalg 1 \langle 13 \rangle \rangle^*$
$\langle J \amalg 7 \amalg 1 \langle 1 \rangle \rangle^{\circ,*}$	$\langle J \amalg 5 \amalg 1 \langle 3 \rangle \rangle^{\circ,*}$	$\langle J \amalg 7 \amalg 1 \langle 5 \rangle \rangle^{\circ,*}$		
$\langle J \amalg 8 \amalg 1 \langle 1 \rangle \rangle^\circ$	$\langle J \amalg 6 \amalg 1 \langle 3 \rangle \rangle^\circ$	$\langle J \amalg 8 \amalg 1 \langle 5 \rangle \rangle^\circ$		$\langle J \amalg 1 \amalg 1 \langle 1 \langle 1 \rangle \rangle \rangle^*$
$\langle J \amalg 9 \amalg 1 \langle 1 \rangle \rangle^{\circ,*}$	$\langle J \amalg 7 \amalg 1 \langle 3 \rangle \rangle^{\circ,*}$	$\langle J \amalg 9 \amalg 1 \langle 5 \rangle \rangle^*$		

Table 1:

oriented and disposed on  $\mathbb{R}P^2$  such that the (symmetric) perturbations according to the orientations satisfies the Rokhlin-Mischachev orientation formula. So, according to Theorem 4.8 in [Vir84b], the obtained real algebraic symmetric curves of degree 7 on  $\mathbb{R}P^2$  are of type  $I$  and realize the announced complex schemes.  $\square$

### 5.3 Parametrization of a rational curve

Here we apply the method used in [Ore98a] and [Ore98b]. Namely, we construct a singular rational curve and perturb it using Shustin's results on the independent perturbations of generalized semi-quasihomogeneous singular points of a curve keeping the same Newton polygon (see [Shu98] and [Shu99]).

**Proposition 5.9** *There exists a rational real algebraic curve of degree 4 on  $\mathbb{R}P^2$  situated with respect to the lines  $\{x = 0\}$ ,  $\{y = 0\}$ ,  $\{z = 0\}$  and  $\{y = -z\}$  as shown in Figure 26a), with a singular point of type  $A_4$  at  $[0 : 0 : 1]$ , a singular point of type  $A_1$  at  $p$  and a tangency point of order 2 with the line  $\{x = 0\}$  at  $[0 : 1 : 0]$ .*

*Proof.* Consider the map from  $\mathbb{C}$  to  $\mathbb{C}P^2$  given by  $t \mapsto [x(t) : y(t) : z(t)]$  where

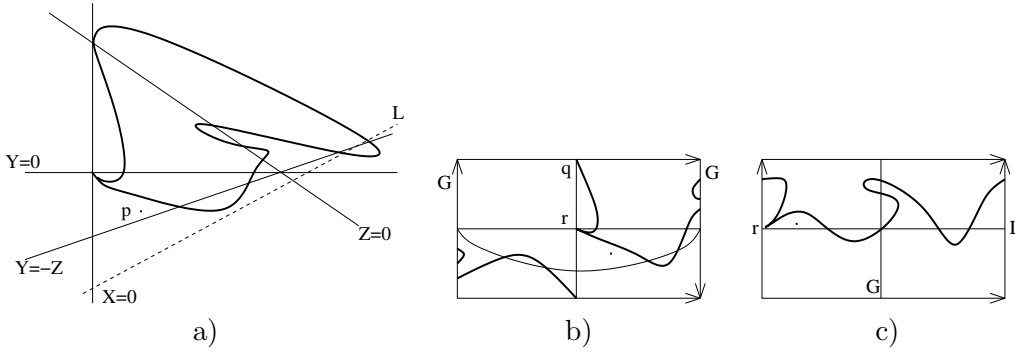


Figure 26:

$$\begin{cases} x(t) = t^2 \\ y(t) = \alpha t^2(t - \gamma)(t + \gamma) \\ z(t) = -((t - 1)(\epsilon t - 1)(\delta t - 1)) \end{cases} \quad \text{with } \alpha = \frac{1}{11}, \gamma = \delta = \frac{9}{10} \text{ and } \epsilon = \frac{99}{100}.$$

The curve defined by this map has a singular point of type  $A_4$  at  $[0 : 0 : 1]$ , as we can see using the following identity :

$$y(t)z(t) + \alpha\gamma^2x(t)z(t) - \alpha x(t)^2 = -\frac{81}{1000}t^7 + \frac{2781}{11000}t^6 - \frac{289}{1100}t^5.$$

Moreover, it is clear from the equations that the curve has a tangency point of order 2 with the line  $\{x = 0\}$  at  $[0 : 1 : 0]$ . This map define an algebraic curve of degree 4 on  $\mathbb{C}P^2$ ,  $C = \text{Res}_t(x(t)Y - y(t)X, x(t)Z - z(t)X)/X^3$ . Considering  $C$  on the affine plane  $\{Z = 1\}$ , we obtain

$$\begin{aligned} C = & -\frac{1}{11}Y^2 - \frac{13468021579}{1331000000000000}X^4 + \frac{1666467523}{1210000000000}YX^3 + \frac{275261}{3025000}X^2Y - \frac{65485017}{1100000000}X^2Y^2 \\ & + \frac{2781}{5500}XY^2 - \frac{81}{6050}XY + \frac{793881}{1000000}XY^3 + \frac{26346141}{6655000000}X^3 - \frac{6561}{13310000}X^2. \end{aligned}$$

Projecting this curve to the line  $\{y = 0\}$  and using the Budan Fourier theorem (see [BPR03]), one can check that  $C$  verifies the hypothesis of the proposition.  $\square$

**Corollary 5.10** *The complex scheme  $\langle J \amalg 5 \amalg 1 \langle 7 \rangle \rangle_{II}$  is realizable by nonsingular non-dividing symmetric real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .*

*Proof.* The strict transform of the curve constructed in Proposition 5.9 under the blow up of  $\mathbb{C}P^2$  at the point  $[0 : 1 : 0]$  is the rational real algebraic curve of bidegree  $(3, 1)$  on  $\Sigma_1$  depicted in Figure 26b). Blowing up the point  $q$  and blowing down the strict transform of the fiber, we obtain the rational real algebraic trigonal curve on  $\Sigma_2$  depicted on Figure 26c), with a singular point of type  $A_6$  at the point  $r$ . Then according to Shustin's results (see [Shu98] and [Shu99]), it is possible to smooth this curve as depicted in Figure 27a). Perturbing the union of this curve and the fiber  $G$ , we obtain the nonsingular curve of bidegree  $(3, 1)$  on  $\Sigma_2$  arranged with the base  $\{y = 0\}$  as shown in Figure 27b). The corresponding symmetric curve realizes the real scheme  $\langle J \amalg 5 \amalg 1 \langle 7 \rangle \rangle$  and according to Proposition 2.14 this is a non-dividing symmetric curve.  $\square$

**Corollary 5.11** *The real schemes  $\langle J \amalg 5 \amalg 1 \langle 8 \rangle \rangle$  and  $\langle J \amalg 4 \amalg 1 \langle 7 \rangle \rangle$  are realizable by nonsingular symmetric real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .*

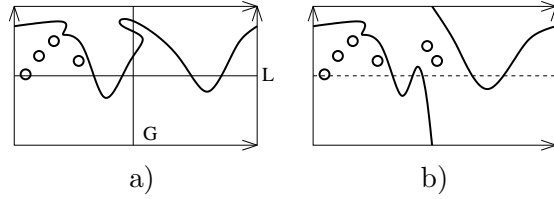


Figure 27:

*Proof.* One obtains these two curves modifying slightly the previous construction. To obtain the real scheme  $\langle J \text{ II } 5 \text{ II } 1 \langle 8 \rangle \rangle$ , one can keep all the ovals above the base depicted in Figure 27b). To obtain the real scheme  $\langle J \text{ II } 4 \text{ II } 1 \langle 7 \rangle \rangle$ , one can consider the line  $L$  instead of the line  $\{y + z = 0\}$  in Figure 26a).  $\square$

#### 5.4 Change of coordinates in $\Sigma_2$

**Proposition 5.12** *The real schemes  $\langle J \text{ II } 7 \text{ II } 1 \langle 4 \rangle \rangle$  and  $\langle J \text{ II } 5 \text{ II } 1 \langle 6 \rangle \rangle$  are realizable by nonsingular symmetric real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .*

*Proof.* In section 5.2, we constructed symmetric curves on  $\mathbb{R}P^2$  shown in Figure 28a). According to Lemma 4.1, their quotient curve  $X$  is depicted in Figure 28b).

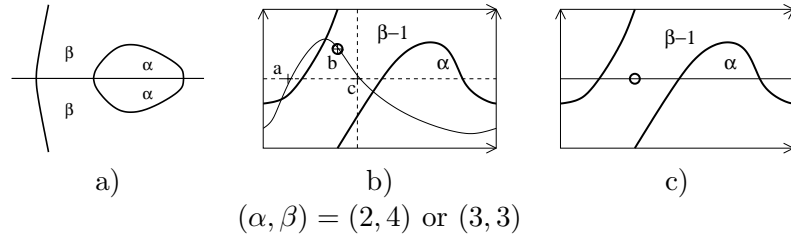


Figure 28:

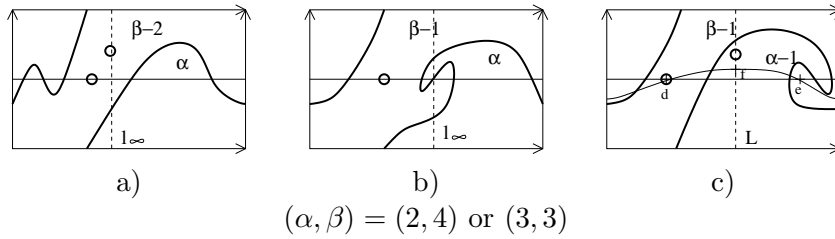


Figure 29:

Let  $H$  be the base which passes through the points  $a$ ,  $b$  and  $c$  as depicted in Figure 28b). All possible mutual arrangements for  $H$  and the quotient curves which do not contradict the Bezout theorem and Lemma 2.5 are depicted in Figures 28c) and 29).

First, we prohibit pseudoholomorphically the  $\mathcal{L}$ -schemes realized by the union of  $X$  and  $H$  in Figure 29a) and by  $X$  in Figure 29b). Choose  $l_\infty$  as depicted, and the braid corresponding to these  $\mathcal{L}$ -schemes are :

$$b_{\alpha,\beta}^{18} = \sigma_3^{-(\beta-1)} \sigma_1^{-1} \sigma_2^{-1} \sigma_3 \sigma_2^{-\alpha} \sigma_3^{-1} \sigma_2 \sigma_1^{-4} \sigma_2^{-1} \sigma_3^2 \sigma_2 \sigma_1 \sigma_2^{-2} \sigma_3^{-1} \Delta_4^2 \text{ with } (\alpha, \beta) = (3, 3), (2, 4),$$

$$b^{19} = \sigma_1^{-7} \sigma_2 \sigma_1 \Delta_3^2.$$



The braid  $b_{3,3}^{18}$  was already shown to be not quasipositive in section 4.2. The computation of the Alexander polynomials of the remaining braids gives

$$p^{19} = (-1 + t)(t^4 - t^3 + t^2 - t + 1), \quad p_{4,2}^{18} = (t^2 - t + 1)(-1 + t)^3.$$

Since  $e(b^{19}) = 1$  and  $e(b_{\alpha,\beta}^{18}) = 2$ , according to Proposition 2.8, none of these braids is quasipositive. Thus, the two remaining possibilities for the mutual arrangement of  $X$  and  $H$  are depicted in Figures 28c) and 29c).

In the first case, let  $H' = H$ .

In the second case, consider the base  $G$  passing through the points  $d$ ,  $e$  and  $f$  where  $f$  is some point on the fiber  $L$ . For some  $f$ , the base  $G$  passes through two ovals of  $X$ . Since  $G$  cannot have more than 7 common points with  $X$ , there exist  $f$  for which the mutual arrangement of  $G$  and  $X$  is as shown in Figure 28c). Let  $H'$  be the base corresponding to such an  $f$ .

The symmetric curves of degree 7 on  $\mathbb{R}P^2$  corresponding to the mutual arrangement of  $H'$  and  $X$  realize the real schemes  $\langle J \amalg 7 \amalg 1 \langle 4 \rangle \rangle$  and  $\langle J \amalg 5 \amalg 1 \langle 6 \rangle \rangle$ .  $\square$

**Proposition 5.13** *The real schemes  $\langle J \amalg 1 \amalg 1 \langle 12 \rangle \rangle$  and  $\langle J \amalg 9 \amalg 1 \langle 4 \rangle \rangle$  are realizable by nonsingular symmetric real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .*

*Proof.* In section 5.2, we constructed symmetric curves of degree 7 on  $\mathbb{R}P^2$  depicted in Figure 30a). These are  $M$ -curves, so according to Proposition 2.12, 2.11, the Bezout theorem, Lemmas 4.1 and 2.5, the  $\mathcal{L}$ -scheme realized by their quotient curve  $X$  is depicted in Figure 30b). Let  $H$  be the base

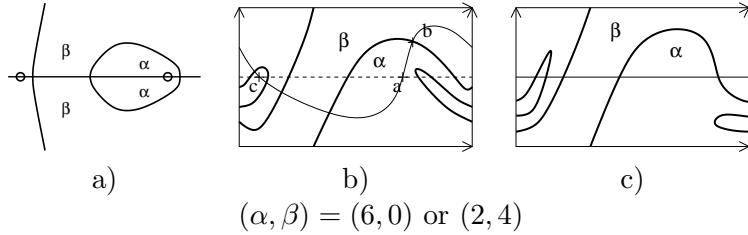


Figure 30:

which passes through the points  $a$ ,  $b$  and  $c$  as depicted in Figure 30b). The only possible mutual arrangement for  $H$  and  $X$  which does not contradict the Bezout theorem, Proposition 2.12 and Lemma 2.5 is depicted in Figure 30c). The corresponding symmetric curves realize the real schemes  $\langle J \amalg 1 \amalg 1 \langle 12 \rangle \rangle$  and  $\langle J \amalg 9 \amalg 1 \langle 4 \rangle \rangle$ .  $\square$

## 5.5 Construction of auxiliary curves

Here we construct charts of some symmetric curves. We will use these charts in section 5.6 to perturb symmetrically some singular symmetric real algebraic curves.

**Lemma 5.14** *For any real positive numbers  $\alpha, \beta, \gamma$ , there exist real curves of degree 3 on  $\mathbb{R}P^2$  having the charts and the arrangement with respect to the axis  $\{y = 0\}$  shown in Figures 31a) and b) with truncation on the segment  $[(0, 3), (3, 0)]$  equal to  $(x - \alpha y)(x - \beta y)(x - \gamma y)$ .*

*Proof.* Consider the points  $A = [\alpha : 1 : 0]$ ,  $B = [\beta : 1 : 0]$ ,  $C = [\gamma : 1 : 0]$  and four lines  $L_1, L_2, L_3, L_4$  as shown in Figure 31c). For  $t$  small enough and of suitable sign, the curve  $yzL_1 + tL_2L_3L_4$  is arranged with respect to the coordinate axis and the lines  $L_1, L_2, L_3, L_4$  as shown in Figure 31c).

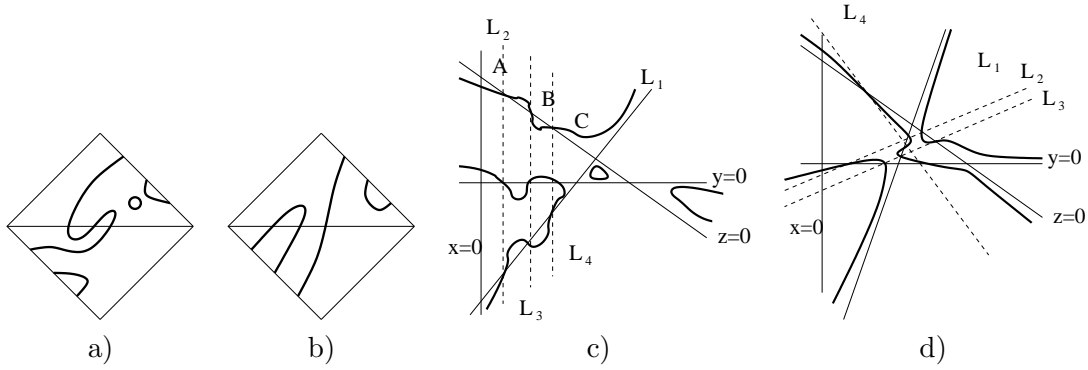


Figure 31:

To construct the curve with the chart depicted in Figure 31b), we perturb the third degree curve  $yzL_1$  as shown in Figure 31d).  $\square$

**Corollary 5.15** *For any real positive numbers  $\alpha, \beta, \gamma$ , there exist real symmetric dividing curves of degree 6 on  $\mathbb{R}P^2$  with a singular point of type  $J_{10}$  at  $[1 : 0 : 0]$  having the charts, the arrangement with respect to the axis  $\{y = 0\}$  and the complex orientations shown in Figures 32a), b) and c) with truncation on the segment  $[(0, 3), (6, 0)]$  equal to  $(x - \alpha y^2)(x - \beta y^2)(x - \gamma y^2)$ .*

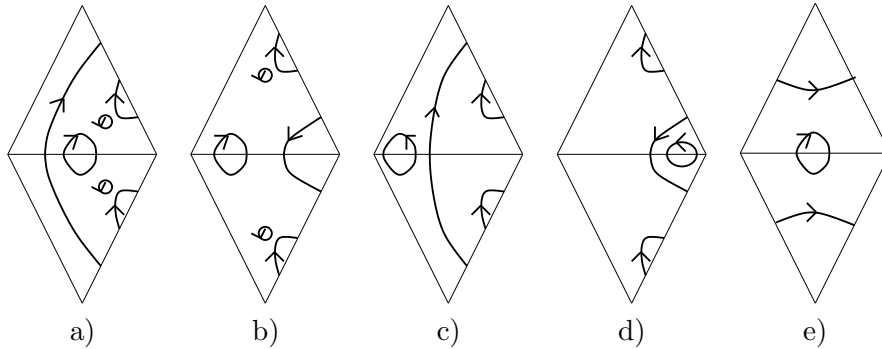


Figure 32:

*Proof.* The Newton polygon of the third degree curves constructed in Lemma 5.14 lies inside the triangle with vertices  $(0, 3)$ ,  $(0, 0)$  and  $(6, 0)$ , so these curves can be seen as a (singular) trigonal curve on  $\Sigma_2$ . The corresponding symmetric curves are of degree 6 and has the chart and the arrangement with respect to the axis  $\{y = 0\}$  shown in Figures 32a) and c). Moreover, it is well known that such curves are of type I, and we deduce their complex orientations from their quotient curve.

If we perform the coordinate changes  $(x, y) \mapsto (-x + \delta y^2, y)$  with  $\delta \in \mathbb{R}$  to the curves with chart depicted in Figure 32a) (resp. 32c)), we obtain curves with the chart depicted in Figure 32b) (resp. 32d)).  $\square$

The following lemma can be proved using the same technique.

**Lemma 5.16** *For any real positive numbers  $\alpha$  and  $\beta$ , there exist real symmetric dividing curves of degree 4 on  $\mathbb{R}P^2$  with a singular point of type  $A_3$  at  $[1 : 0 : 0]$  having the charts, the arrangement with respect to the axis  $\{y = 0\}$  and the complex orientations shown in Figure 32e) with truncation on the segment  $[(0, 2), (4, 0)]$  equal to  $(x - \alpha y^2)(x - \beta y^2)$ .*  $\square$

## 5.6 Perturbation of irreducible singular symmetric curves

**Proposition 5.17** *The complex schemes  $\langle J \amalg 2_- \amalg 1_+ \langle 4_+ \amalg 2_- \rangle \rangle_I$  and  $\langle J \amalg 2_+ \amalg 4_- \amalg 1_+ \langle 2_+ \rangle \rangle_I$  are realizable by nonsingular symmetric real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .*

*Proof.* First, we construct the symmetric singular dividing curve of degree 7 with two singular points  $J_{10}$  depicted in Figure 33d). To construct such a curve, we use the Hilbert method as in [Vir84a]. Let  $C_0$  be a symmetric conic. We symmetrically perturb the union of  $C_0$  and a disjoint symmetric line (Figure 33a)) keeping the tangency points with  $C_0$ . Next, we symmetrically perturb the union of the third degree curve obtained and  $C_0$  (Figure 33b)) keeping the tangency points of order 4 with  $C_0$ . Perturbing in a symmetric way the union of the curve of degree 5 obtained and  $C_0$  (Figure 33c)) keeping the tangency points of order 6 with  $C_0$ , we obtain a symmetric singular dividing curve  $C$  of degree 7 with two singular points  $J_{10}$  as depicted in Figure 33d).

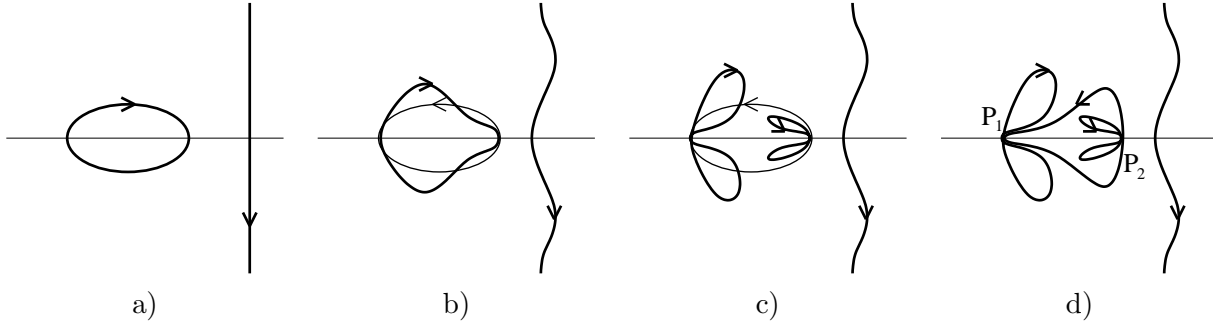


Figure 33:

Now we symmetrically perturb the singular points of  $C$  using the chart shown in Figure 32b) (resp. d)) in  $P_1$  and 32c) (resp. a)) in  $P_2$  and obtain the desired curves.  $\square$

**Proposition 5.18** *The complex scheme  $\langle J \amalg 1_+ \langle 6_+ \amalg 6_- \rangle \rangle_I$  is realizable by nonsingular symmetric real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .*

*Proof.* Consider the curve of degree 4 with a C-shaped oval constructed in [Kor88] and the coordinate system shown in Figure 34a). In this coordinate system, the Newton polygon of the curve is the trapeze with vertices  $(0, 0)$ ,  $(0, 3)$ ,  $(1, 3)$  and  $(4, 0)$ , and its chart is depicted in Figure 34b). This curve can be seen as a singular real algebraic curve of bidegree  $(3, 1)$  on the surface  $\Sigma_2$ . The corresponding symmetric curve of degree 7 has a singular point  $J_{10}$  at  $[1 : 0 : 0]$  and is depicted in Figure 34c). This curve is maximal according to Proposition 2.4, so of type I, and we can deduce its complex orientations from its quotient curve. Finally, we symmetrically smooth the singular point with the chart depicted in Figure 32a) and obtain the desired curve.  $\square$

Denote by  $f_P$  the real birational transform of  $\mathbb{C}P^2$  given by  $(x, y) \mapsto (x, y - P(x))$  in the affine coordinate  $\{z = 1\}$ , where  $P$  is a polynomial of degree 2.

**Proposition 5.19** *The complex scheme  $\langle J \amalg 6_+ \amalg 4_- \amalg 1_+ \langle 1_+ \amalg 1_- \rangle \rangle_I$  is realizable by nonsingular symmetric real algebraic curves of degree 7 on  $\mathbb{R}P^2$ .*

*Proof.* Consider the nodal curve of degree 3 depicted in Figure 35a) with a contact of order 3 at the point  $[0 : 1 : 0]$  with the line  $\{z = 0\}$ . Then there exists a unique polynomial  $P$  of degree 2 such that the image of the cubic under  $f_P$  is the curve of degree 4 depicted in Figure 35b), with a singular point

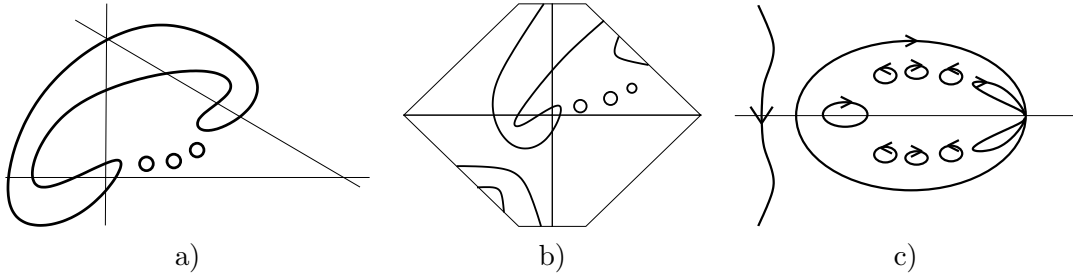


Figure 34:

of type  $A_4$  at  $[0 : 1 : 0]$  and a contact of order 4 at this point with the line  $\{z = 0\}$ . Moreover, the line  $\{y = 0\}$  intersects the quartic in two points, one of them is the node, and this line is tangent at one of the local branches at the node and at the second intersection point, a line  $\{y = az\}$  is tangent at the curve of degree 4. Perform the change of coordinates of  $\mathbb{C}P^2$   $[x : y : z] \mapsto [y : x : y - az]$ . For this new coordinate system, there exists a polynomial  $Q$  such that the image of the quartic under  $f_Q$  is the curve of degree 5 depicted in Figure 35c), with a singular point of type  $A_{10}$  at  $[0 : 1 : 0]$  and a contact of order 4 at this point with the line  $\{z = 0\}$ . Applying the change of coordinates of  $\mathbb{C}P^2$   $[x : y : z] \mapsto [y : z : x]$  and using Shustin's results on the independent perturbations of generalized semi-quasihomogeneous singular points of a curve keeping the same Newton polygon (see [Shu98] and [Shu99]), we can smooth the singular point  $A_{10}$  in order to obtain a curve with the chart shown in Figure 36a).

Hence, we can see this curve as a singular curve of bidegree  $(3, 1)$  on the surface  $\Sigma_2$ . The corresponding symmetric curve of degree 7 has a singular point  $A_3$  at  $[1 : 0 : 0]$  and is depicted in Figure 36b). This curve is maximal according to Proposition 2.4, so of type I, and we can deduce its complex orientations from its quotient curve. Finally, we symmetrically smooth the singular point with the chart depicted in Figure 32e) and obtain the desired curve.  $\square$

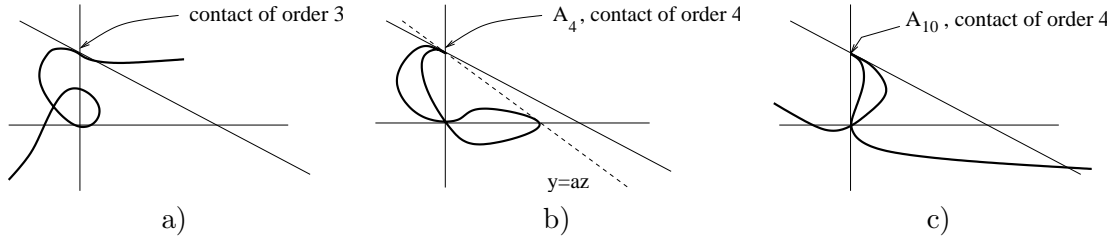


Figure 35:

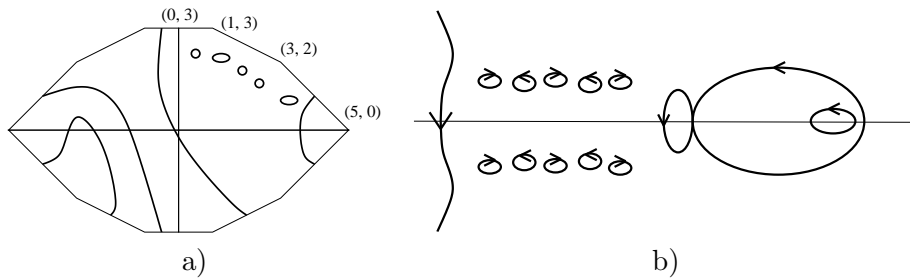


Figure 36:

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